31. 5. 2016

- **Application of Prop. 1:**

- Let $\Lambda = \mathbb{R}$, $d\lambda(x) = dx$ (Lebesgue measure) and B a Borel set of $\mathbb{R}$.

- Consider $\phi(x) := -\mathbb{1}_B(x)$.

$$\Rightarrow \boxed{\mathbb{P}(B \text{ is empty}) = \mathbb{E}\left(\prod_i (1 - \mathbb{1}_B(x_i))\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{B^n} dx_1 \dots dx_n \, \rho^{(n)}(x_1, \dots, x_n).}$$

Examples: ① For a PPP with density $\rho = \frac{d\nu}{dx}$, $\rho^{(n)}(x_1, \dots, x_n) = \rho(x_1) \cdots \rho(x_n)$.

② A non-trivial example from random matrix theory: GUE

- Let H be a $N \times N$ Hermitian matrix with

$$\begin{cases} H_{ii} \sim \mathcal{N}(0,1), & 1 \leq i \leq N, \\ \mathrm{Re}\, H_{ij} \sim \mathcal{N}(0, 1/2), & \\ \mathrm{Im}\, H_{ij} \sim \mathcal{N}(0, 1/2) & \end{cases} \Big\} \, 1 \leq i < j \leq N$$

all indep. r.v.

Lemma 3) $\mathbb{P}(H \in dH) = \frac{1}{\tilde{Z}_N} e^{-\frac{\mathrm{Tr}(H^2)}{2}} dH$

where $dH = \left(\prod_{i=1}^{N} dH_{ii}\right)\left(\prod_{1 \leq i < j \leq N} d\mathrm{Re}\, H_{ij} \, d\mathrm{Im}\, H_{ij}\right)$

and $\tilde{Z}_N$ a normalisation constant.

<u>Proof:</u> $\mathbb{P}(H\in dH) = \prod_{i=1}^{N}\left(\frac{e^{-\frac{H_{ii}^2}{2}}}{\sqrt{2\pi}}\, dH_{ii}\right)\cdot$

$$\cdot \prod_{1\leq i<j\leq N}\left(\frac{e^{-(\mathrm{Re}\,H_{ij})^2}}{\sqrt{\pi}}\, d\mathrm{Re}H_{ij}\cdot \frac{e^{-(\mathrm{Im}\,H_{ij})^2}}{\sqrt{\pi}}\, d\mathrm{Im}H_{ij}\right)$$

$$= \text{cst}.dH.\, e^{-\sum_{i=1}^{N}\frac{H_{ii}^2}{2}}\cdot e^{-\sum_{i<j}(\mathrm{Re}H_{ij})^2+(\mathrm{Im}H_{ij})^2}$$

. But $\frac{1}{2}\mathrm{Tr}(H^2) = \frac{1}{2}\sum_{i=1}^{N}\left(\sum_{j=1}^{N} H_{ij}H_{ji}\right)$

$$= \frac{1}{2}\sum_{i=1}^{N} H_{ii}^2 + \frac{1}{2}\sum_{i\neq j}(\mathrm{Re}H_{ij}+i\,\mathrm{Im}H_{ij})\cdot(\mathrm{Re}H_{ij}-i\,\mathrm{Im}H_{ij})$$

$$= \frac{1}{2}\sum_{i=1}^{N} H_{ii}^2 + \sum_{i<j}\left\{(\mathrm{Re}H_{ij})^2+(\mathrm{Im}H_{ij})^2\right\}.$$

#

<u>Remark:</u> • The proba. measure on Hermitian matrices of the form given by Lemma 3 is called Gaussian Unitary Ensemble (GUE).

Since the measure is invariant under Unitary transformations:

$$H \longmapsto UHU^{-1},\ U\in\mathcal{U}(N).$$

. By doing the change of variables

$$\underbrace{H}_{N^2 \text{ variables.}} \longmapsto (\underbrace{\Lambda}_{\text{eigenvalues of } H:\ N \text{ variables}}, U)$$

and computing the Jacobian one finally gets to the following result:

Prop 4: The distribution of the eigenvalues $\lambda = (\lambda_1, \dots, \lambda_N)$ of $H \sim GUE$ is given by

$$\mathbb{P}(\lambda \in d\lambda) = P_N(\lambda_1, \dots, \lambda_N)\, d\lambda_1 \dots d\lambda_N$$

where: $$P_N(\lambda_1, \dots, \lambda_N) = \frac{1}{Z_N} (\Delta_N(\lambda))^2 \prod_{i=1}^{N} \omega(\lambda_i)$$

with $\omega(\lambda) = e^{-\frac{\lambda^2}{2}}$ and

$$\Delta_N(\lambda) = \prod_{1 \le i < j \le N} (\lambda_j - \lambda_i) \equiv \det\left(\lambda_i^{j-1}\right)_{1 \le i,j \le N}$$

is the Vandermonde determinant.

~~For the calculation~~

Idea of the proof:

- Given H, $\exists\, U \in \mathcal{U}(N)$ s.t. $H = U \Lambda U^{-1}$, $\Lambda = [\lambda_i \delta_{ij}]_{1 \le i,j \le N}$.
- U consists of $M = N^2 - N$ indep. variables, call them $p_1, \dots, p_M$.
- $\{H_{ij}, 1 \le i \le j \le N\} \longrightarrow \{\lambda_1, \dots, \lambda_N, p_1, \dots, p_M\}$
- An infinitesimal transformation of H gives:

$$\delta H = \delta U \cdot \Lambda \cdot U^{-1} + U \cdot \delta\Lambda \cdot U^{-1} + U \cdot \Lambda \cdot \delta U^{-1}$$

and since: $UU^{-1} = \mathbb{1}$, $(\delta U) U^{-1} = -U \cdot (\delta U^{-1})$

$$\Rightarrow \delta H = U \left[U^{-1} \delta U \cdot \Lambda - \Lambda \cdot U^{-1} \delta U \right] U^{-1} + U \cdot \delta\Lambda \cdot U^{-1}$$

$$= U \cdot \delta\tilde{H} \cdot U^{-1} \quad \text{with} \quad \delta\tilde{H} = \delta\Lambda + [\underbrace{U^{-1}\delta U}_{\equiv \delta\Omega}, \Lambda]$$

(angular variables)

$$\underset{\text{Jacobian is } 1}{H \longrightarrow \tilde{H}} \underset{\text{Jacobian?}}{\longrightarrow (\Lambda, \Omega)}$$

In components: $$\delta \tilde{H}_{ij} = \delta \Lambda_{ij} + \sum_{k=1}^{N} \delta \Omega_{ik}\, \delta_{kj}\, \lambda_j - \sum_{k=1}^{N} \lambda_i\, \delta_{ik}\, \delta \Omega_{kj}$$

and $\delta \Lambda_{ij} = \delta_{ij} \cdot \delta \lambda_i$

$$\Rightarrow \delta \tilde{H}_{ij} = \delta_{ij} \cdot \delta \lambda_i + \delta \Omega_{ij} (\lambda_j - \lambda_i)$$

$$\Rightarrow J = \left| \det \frac{\partial(\tilde{H}_{11}, \dots, \tilde{H}_{N,N}, \mathrm{Re}\,\tilde{H}_{12}, \dots, \mathrm{Re}\,\tilde{H}_{N-1,N}, \mathrm{Im}\,\tilde{H}_{ij})}{\partial(\lambda_1, \dots, \lambda_N, \mathrm{Re}\,\Omega_{12}, \dots)} \right|$$

$$= \left| \det \left(\begin{array}{c|c|c} \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} & 0 & 0 \\ \hline 0 & \begin{matrix} \lambda_1 - \lambda_2 & & 0 \\ & \ddots & \\ 0 & & \lambda_{N-1} - \lambda_N \end{matrix} & 0 \\ \hline 0 & 0 & \begin{matrix} \lambda_1 - \lambda_2 & & 0 \\ & \ddots & \\ 0 & & \lambda_{N-1} - \lambda_N \end{matrix} \end{array} \right) \right|$$

$$= (\Delta_N(\lambda))^2 .$$

Rem.: To make the calculation rigorous one has to consider the fact that the change of variable is not smooth (it is up to zero-measure sets).
↳ the details can be found in the Random Matrix Book: http://cims.nyu.edu/~zeitouni/cupbook.pdf

Remark: • Due to $\Delta_N(\lambda)^2$, the eigenvalues "repels" each other: they are much less likely to be close to each other than a PPP with the same density :-).

• In particular, $\mathbb{P}(\lambda_i = \lambda_j \text{ for some } i \neq j) = 0$

i.e., $\xi = \sum_{i=1}^{N} \delta_{\lambda_i}$ is a simple point process.

• Goal: Determine $\rho^{(n)}$ for ξ.

- In particular, $\mathbb{P}(\lambda_{N,max} \leq s)$ given by Prop 1 with $\phi = -\mathbb{1}_{(s,\infty)}$

$\uparrow$ largest e.v.

• Also, if we have $\rho^{(n)}$, we can compute the moments:

• $$\mathbb{E}(\xi(A)) = \int_A dx\, \rho^{(1)}(x)$$

• $$\mathbb{E}\big(\xi(A)(\xi(A)-1)\big) = \int_{A^2} dx\,dy\, \rho^{(2)}(x,y)$$

$$\Rightarrow \mathrm{Var}(\xi(A)) = \int_{A^2} dx\,dy\, \rho^{(2)}(x,y) + \int_A dx\, \rho^{(1)}(x) - \left(\int_A dx\, \rho^{(1)}(x)\right)^2$$

Lemma 5) If $\mathbb{P}(\lambda \in d\lambda) = P_N(\lambda_1, \ldots, \lambda_N)\, d\lambda_1 \ldots d\lambda_N$
and $P_N(\lambda_1, \ldots, \lambda_N) = P_N(\lambda_{\sigma_1}, \ldots, \lambda_{\sigma_N})$,
$\forall \sigma \in S_N$ (permutations of $\{1, \ldots, N\}$)

$$\Rightarrow \rho^{(n)}(\lambda_1, \ldots, \lambda_n) = \frac{N!}{(N-n)!} \int_{\mathbb{R}^{N-n}} P_N(\lambda_1 \ldots \lambda_N)\, d\lambda_{n+1} \ldots d\lambda_N$$

Proof: (a) . Choose which of the N variables have to be set equal to $(\lambda_1, \ldots, \lambda_n)$.
There are $n! \binom{N}{n} = \frac{N!}{(N-n)!}$ such possibilities

. Since P_N is invariant under permutations, then each of the choices gives the same contribution to $\rho^{(n)}$, i.e.,

$$\int_{\mathbb{R}^{N-n}} P_N(\lambda_1 \ldots \lambda_N)\, d\lambda_{n+1} \ldots d\lambda_N.$$

#

. **Remark:** For any family of polynomials $\{q_k(x),\ k = 0 \ldots N-1\}$ with q_k a polynomial of degree k,

$$\Delta_N(\lambda) = \text{const} \cdot \det\left(q_{j-1}(\lambda_i)\right)_{1 \le i,j \le N}.$$

. Let us do a simple computation for GUE:

$$P_N(\lambda_1 \ldots \lambda_N) = \frac{1}{Z_N} \cdot \left\{\det\left(q_{j-1}(\lambda_i)\right)_{1 \le i,j \le N}\right\}^2 \prod_{i=1}^{N} \omega(\lambda_i).$$

$$= \frac{1}{Z_N} \cdot \det\left[K_N(\lambda_i, \lambda_j)\right]_{1 \le i,j \le N}$$

with $K_N(x,y) = \sqrt{w(x)\cdot w(y)}\cdot \sum_{k=0}^{N-1} q_k(x) q_k(y)$.

• Special choice of polynomials q_k's:

• Consider $q_k(x) = u_k \cdot x^k + \dots$ polynomials of degree k, with $u_k > 0$ and orthogonal:

$$\boxed{\int_{\mathbb{R}} dx\, w(x)\, q_k(x)\, q_e(x) = \delta_{k,e} \quad , 0 \le k,e \le N-1}$$

$\Rightarrow$ Lemma 6)

ⓐ $\int_{\mathbb{R}} dx\, K_N(x,x) = N$

ⓑ $\int_{\mathbb{R}} dy\, K_N(x,y) K_N(y,z) = K_N(x,z)$

Proof: ⓐ $\int_{\mathbb{R}} dx\, K_N(x,x) = \sum_{k=0}^{N-1} \underbrace{\int_{\mathbb{R}} dx\, w(x) q_k(x) q_k(x)}_{=1} =$

ⓑ $\int_{\mathbb{R}} dy\, K_N(x,y) K_N(y,z) =$

$= \sum_{k,e=0}^{N-1} \sqrt{w(x)}\sqrt{w(z)}\, q_k(x) q_e(z) \underbrace{\int_{\mathbb{R}} dy\, w(y) q_k(y) q_e(y)}_{=\delta_{k,e}}$

$= K_N(x,z)$ #.

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• We are ready to compute $\rho^{(n)}$ for GUE.

Prop. 7) For GUE:

$$\boxed{\rho^{(n)}(\lambda_1, \dots, \lambda_n) = \det\left[K_N(\lambda_i, \lambda_j)\right]_{1 \le i,j \le n}}$$

. We need one more identity:

The <u>Cauchy-Binet identity</u>:

$$\boxed{\det\left(\int_\Lambda \phi_i(x)\psi_j(x)\,d\lambda(x)\right)_{1\le i,j\le N} = \frac{1}{N!}\int_{\Lambda^N} \det\left(\phi_i(x_j)\right)_{1\le i,j\le N} \cdot \det\left(\psi_i(x_j)\right)_{1\le i,j\le N}\, d^N\lambda(x)}$$

. Proof can be found ~~e.g.~~ in the online notes of Johansson (Prop 2.10).

<u>Proof of Prop. 7:</u>

ⓐ $\int_{\mathbb{R}^N} \Rightarrow 1 = \int_{\mathbb{R}^N} d\lambda_1 \dots d\lambda_N\, P_N(\lambda_1 \dots \lambda_N)$

$$= \int_{\mathbb{R}^N} d\lambda_1 \dots d\lambda_N\, \frac{\omega(\lambda_1)\dots\omega(\lambda_N)}{\tilde{Z}_N}\left\{\det\left(q_{j-1}(\lambda_i)\right)\right\}^2$$

$$\overset{\text{Cauchy-Binet}}{=} \frac{N!}{\tilde{Z}_N}\,\underbrace{\det\Big(\underbrace{\int_{\mathbb{R}} d\lambda\,\omega(\lambda)\,q_{i-1}(\lambda)q_{j-1}(\lambda)}_{=\delta_{ij}}\Big)}_{=1}.$$

$\Rightarrow \underline{\tilde{Z}_N = N!}$.

(b) Using Lemma 5 we get:

$$\rho^{(n)}(\lambda_1,\dots,\lambda_n) = \frac{\cancel{N!}}{(N-n)!\,\cancel{N!}} \int_{\mathbb{R}^{N-n}} d\lambda_{n+1}\cdots d\lambda_N \, \det\left(K_N(\lambda_i,\lambda_j)\right)_{1\le i,j\le N}$$

$$\det\left\{K_N(\lambda_i,\lambda_j)\right\}_{1\le i,j\le m} =$$

$$= \det\left[\begin{array}{c|c} \left\{K_N(\lambda_i,\lambda_j)\right\}_{1\le i,j\le m-1} & \left[K_N(\lambda_i,\lambda_m)\right]_{1\le i\le m-1} \\ \hline \left[K_N(\lambda_m,\lambda_j)\right]_{1\le j\le m-1} & K_N(\lambda_m,\lambda_m) \end{array}\right]$$

$$= K_N(\lambda_m,\lambda_m)\cdot \det\left[K_N(\lambda_i,\lambda_j)\right]_{1\le i,j\le m-1}$$

$$+ \sum_{k=1}^{m-1} (-1)^{m-k} \cdot K_N(\lambda_k,\lambda_m) \cdot \det\left[\begin{array}{c} \left\{K_N(\lambda_i,\lambda_j)\right\}_{\substack{1\le j\le m-1 \\ 1\le i\le m,\, i\ne k}} \\ \hline \left\{K_N(\lambda_m,\lambda_j)\right\}_{1\le j\le m-1} \end{array}\right]$$

by linearity

$$\Rightarrow \int_{\mathbb{R}} d\lambda_m \det\left\{K_N(\lambda_i,\lambda_j)\right\}_{1\le i,j\le m} \overset{\text{Lemma 6}}{=\!=} N\cdot \det\left\{K_N(\lambda_i,\lambda_j)\right\}_{1\le i,j\le m-1}$$

$$+ \sum_{k=1}^{m-1} (-1)^{m-k} \det\left[\begin{array}{c} \left\{K_N(\lambda_i,\lambda_j)\right\} \\ \hline \left[K_N(\lambda_k,\lambda_j)\right] \end{array}\right]$$

$$= \left(N-(m-1)\right)\cdot \det\left\{K_N(\lambda_i,\lambda_j)\right\}_{1\le i,j\le m-1}$$

Plugging this iteratively gives the result. #

Small excursus:

. Another representation of K_N:

. Orthogonal polynomials with $q_k(x) = u_k\, q^k + \dots,\ u_k > 0$, satisfies a three-term recursion relation:

$$q_n(x) = (A_n \cdot x + B_n)\, q_{n-1}(x) - C_n \cdot q_{n-2}(x),\ n \geq 2$$

with $A_n > 0$, B_n, $C_n > 0$ some constants given by:

$$A_n = \frac{u_n}{u_{n-1}}\ ,\ B_n = \frac{A_n}{A_{n-1}} = \frac{u_n \cdot u_{n-2}}{(u_{n-1})^2}.$$

. Then: $\displaystyle\sum_{k=0}^{n-1} q_k(x)\, q_k(y) =$

$$= \begin{cases} \dfrac{u_{n-1}}{u_n} \cdot \dfrac{q_n(x) q_{n-1}(y) - q_{n-1}(x) q_n(y)}{x-y} & \text{for } x \neq y \\[2ex] \dfrac{u_{n-1}}{u_n} \cdot \left(q_n'(x)\, q_{n-1}(x) - q_{n-1}'(x)\, q_n(x) \right) & \text{for } x = y. \end{cases}$$

. These are proven at pages 48-50 of my online notes. (Berlin WS 2007/08)

. Application to GUE: The orthogonal polynomials q_k are given in terms of the classical orthogonal polynomials: $H_n(x)$.

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}\ ,\ n \geq 0,$$

satisfy: $\displaystyle\int_{\mathbb{R}} H_k(x) H_\ell(x)\, e^{-x^2} dx = \sqrt{\pi}\, 2^k k!\, \delta_{k\ell}$ with

$$H_n(x) = 2^n x^n + \dots$$

$$\Rightarrow q_k(y) = \frac{H_k\left(\frac{y}{\sqrt{2}}\right)}{(2\pi)^{1/4}\, 2^{k/2} \sqrt{k!}}.$$

- If we want to keep a finite density of e.v. as $N\to\infty$, we need to consider $\text{const}\cdot e^{-\frac{\mathrm{Tr}(H^2)}{2N}}\,dH$ instead of the previously given distribution on Hermitian matrices. Then by studying the polynomials in the $N\to\infty$ limit one gets results like the following:

(a) $$\rho^{(n)}(x) = K_N(x,x) \xrightarrow[x=\mu N]{N\to\infty} \frac{1}{\pi}\cdot\sqrt{1-\left(\frac{\mu}{2}\right)^2},\ \mu\in[-2,2]$$
$$\equiv \rho_\infty(\mu)$$

(b) For $\mu\in(-2,2)$,
$$\lim_{N\to\infty} K_N(\mu N+\xi_1, \mu N+\xi_2) = \frac{\sin(\pi\rho_\infty(\mu)(\xi_1-\xi_2))}{\pi(\xi_1-\xi_2)}.$$

Sine kernel with density $\rho_\infty(\mu)$

(See chapter 3.3 of my online notes Berlin WS 2007/08)

<u>Remark:</u> With more sophisticated methods one is able to prove such results for Hermitian random matrices with iid entries but not Gaussian distributed

↳ universality (one just need to have 4 finite moments and match the variance and mean to the Gaussian case).