

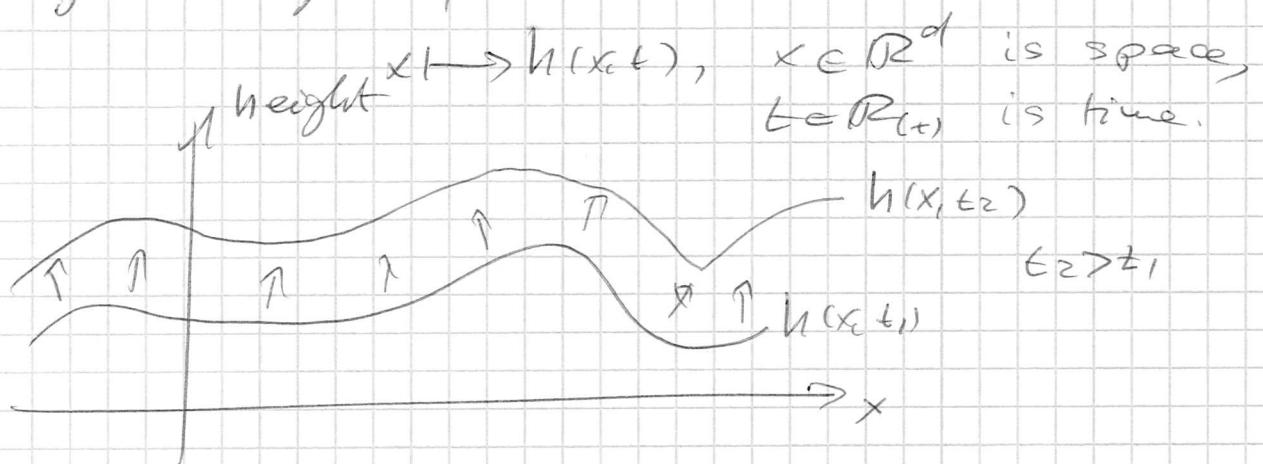
3) Interacting particle systems

In this chapter we are going to study the exclusion process, in particular the fluctuations of particles' positions using the mathematical tools introduced in Chapter 2. Before that, let us discuss in which framework the exclusion process fits.

3.1) Introduction to KPZ.

. KPZ stands for Kardar, Parisi-Zhang, who in 1986 wrote down an equation for the evolution of the growth of an interface.

. Consider a model of an interface described by a height function



. We consider models with:

① local growth rules (in space and time) and stochastic.

⇒ The macroscopic growth velocity v is (expected to be) a function of the slope

only: $\frac{\partial h}{\partial t} = N(\nabla h).$

- If one has a. (b) smoothing mechanism
 \Rightarrow the macroscopic evolution is deterministic,
 i.e., $h_{\text{mg}}(z, t) := \lim_{L \rightarrow \infty} \frac{1}{L} h(zL, tL)$ is deterministic.
- Kardar-Parisi-Zhang proposed as a model
 of irreversible growing interface PDE equation,

$$(KPZ) \quad \frac{\partial h}{\partial t} = \Delta h + \frac{1}{2} \lambda (\nabla h)^2 + \eta, \quad \lambda \neq 0.$$

\uparrow \uparrow
 smoothing Space-time
 (\Leftrightarrow surface tension) white noise
 \uparrow
 Non-linearity (from Taylor at $\nabla h = 0$)
 \Leftrightarrow irreversibility.

For a model to be in the KPZ class,
 it has to satisfy (a), (b) and

(c): let $N(u)$ be the speed (macro) of growth as a function of the slope u .
 $(u = \partial_z h_{\text{mg}}, N = \partial_t h_{\text{mg}})$

$$\Rightarrow \underline{N''(u) \neq 0}$$

This plays the role of $\lambda \neq 0$ in the KPZ eq.

The prediction of (KPZ) is that, even
 with $h(x, 0) \equiv 0, \forall z > 0$, the roughness of $h(x, t)$
 is like the one of Brownian motion

$\Rightarrow (\nabla h)^2$ is a "distribution squared" --- not
 easily definable mathematical object ;).

(98)
asymmetric

One model in the KPZ class is the simple exclusion process. In its totally asymmetric version, particles are on \mathbb{Z} , conditioned to be at most 1 at each site and each time, and otherwise jumping to their right site with jump rate 1.

Formally, $y \in \{0, 1\}^{\mathbb{Z}}$ is a configuration with $y_t(x) = \begin{cases} 1, & \text{if site } x \text{ at time } t \text{ is occupied} \\ 0, & \text{otherwise.} \end{cases}$

$$\text{and } (L\varphi)(q) = \sum_{x \in \mathbb{Z}} q(x) (1 - q(x+1)) \frac{(\varphi(y^{x,x+1}) - \varphi_y)}{\pi}$$

\equiv Configuration y with entries in $x, x+1$ exchanged.

The associated height function is given by:

$$H(x, t) \equiv 2 \cdot (\# \text{part. jumped from } 0 \rightarrow 1 \text{ in } [0, t])$$

$$+ \begin{cases} \sum_{y=1}^x (1 - 2y_t(x)), & x \geq 1, \\ 0, & x = 0, \\ - \sum_{y=x+1}^0 (1 - 2y_t(x)), & x \leq -1. \end{cases}$$

This ^(model) satisfies ②, ③, ④:

① clear.

② let $\frac{\partial}{\partial z} h(x, t) =: 4 - 2g(x, t)$

$\Rightarrow g$ satisfies free Burgers equation:
 $\partial_t g + \partial_z(g(1-g)) = 0$

③ $N^{-1}(u) + o \Rightarrow j''(s) + o$ and $j(s) = g(1-s)$.
 L'current

Large time scaling!

For large time t , one expects that:

- the height fluctuations $\sim L^{1/3}$
- the spatial correlation length $\sim L^{2/3}$

(notice the Brownian scaling between the two).

$$\Rightarrow h_T^{\text{resc}}(u) := \frac{h(\bar{s}t + uL^{2/3}t) - L \cdot h_{\text{av}}(\bar{s} + uL^{-1/3}, 1)}{L^{1/3}}$$

should have a non-trivial limit as $t \rightarrow \infty$.

Q: Is this limit model / geometry / initial conditions dependent?

A: One expects universality within subclasses
 (independence
 of the details
 of the model).

This is the reason why the study of a simplified model can give relevant results for non-solvable models.

⑨ If $h_{\text{av}}(\bar{s}) \neq 0 \Rightarrow$ One expect to see the

Flory process, A_2 , i.e.,

$$\lim_{t \rightarrow \infty} h_T^{\text{resc}}(u) = c_0 A_2(u/c_0)$$

for some model-dep. coefficients $c_0, c_1 \neq 0$.

The A_2 has the following properties:

- Stationary,
- $P(A_2(0) \leq s) = F_{A_2}(s)$,
- $\text{Cov}(A_2(s), A_2(u)) \sim \begin{cases} \text{Var}(A_2(0)) - |u|, & |u| \ll 1, \\ \frac{1}{u^2}, & |u| \gg 1 \end{cases}$
- Joint distributions are known.

(b) If $W_{\max}(\bar{s}) = 0$ & the initial condition is non-random

$$\Rightarrow \lim_{t \rightarrow \infty} h_t^{\text{rec}}(u) = c_W A_1(u/c_W)$$

where A_1 is the Ains_1 process (that we will discover in TASEP). It has the properties:

- Stationary,
- $P(A_1(0) \leq s) = F_{A_1}(2s)$,
- $\text{Cov}(A_1(s), A_1(u)) \sim \begin{cases} \text{Var}(A_1(0)) - |u|, & |u| \ll 1, \\ \sim c u^{3/2}, & |u| \gg 1. \end{cases}$
- Known joint distributions.

⑤ If $W_{\max}(\bar{s}) = 0$ & stationary initial conditions
the limit process is also known but we will not consider it further in this lecture.

3.2) The exclusion process

The state space is $S = \{0, 1\}^{\mathbb{Z}^d}$, a countable (usually \mathbb{Z}^d), $\sum_{x \in S} \gamma(x) = 1 \iff x \text{ is occupied by a particle}$
 $\gamma(x) = 0 \iff x \text{ is empty.}$

Denote by $\gamma^{x,y}(z) := \begin{cases} \gamma(x), & \text{if } z = y, \\ \gamma(y), & \text{if } z = x, \\ \gamma(z), & \text{if } z \notin \{x, y\}. \end{cases}$

and $\gamma^x(z) = \begin{cases} \gamma(z), & \text{if } z \neq x, \\ 1 - \gamma(x), & \text{if } z = x. \end{cases} := \Delta_x(\gamma)$

let $D = \{f \in C(\{0, 1\}^{\mathbb{Z}^d}) : \|f\|_1 := \sum_{x \in \mathbb{Z}^d} \sup_{y \in S} |\gamma^x(y) - f(y)| < \infty\}$.

For $f \in D$, let for a given Q-matrix q ,

$$\begin{aligned} (L f)(y) &= \sum_{\substack{x, y: \gamma(x)=1, \\ \gamma(y)=0}} q(x, y) (f(\gamma^{x,y}) - f(\gamma)) \\ &= \sum_{x, y} q(x, y) \gamma(x) (1 - \gamma(y)) (f(\gamma^{x,y}) - f(\gamma)) \end{aligned}$$

$q(x, y)$ is the jump rate of particles from x to y .

Definition 1) The exclusion process is the interacting particle system with generator $L = \bar{L}$ (provided the closure of L is a proba. op.).

The series in \otimes converges if

$$M := \sup_x \sum_{y \neq x} (q(x,y) + q(y,x)) < \infty.$$

Indeed:

$$\begin{aligned} & q(x,y) y(x) (1 - y(y)) |\mathcal{L}(y^{x,y}) - \mathcal{L}(y)| \\ & \leq q(x,y) y(x) (1 - y(y)) \{ |\mathcal{L}(y^{x,y}) - \mathcal{L}(y^x)| \\ & \quad + |\mathcal{L}(y^x) - \mathcal{L}(y)| \} \end{aligned}$$

$$\Rightarrow |(L\mathcal{L})(y)| \leq \sum_x \sum_{y \neq x} q(y,x) |\mathcal{L}(y^{x,y}) - \mathcal{L}(y^y)| = (y^y)^x \text{ if } y(x)=1, y(y)=0.$$

$$+ \sum_x \sum_{y \neq x} q(x,y) |\mathcal{L}(y^x) - \mathcal{L}(y)|$$

$$\leq III\mathcal{L}III \cdot \sum_x \sum_{y \neq x} (q(x,y) + q(y,x)).$$

The proof of existence of the process is similar to the one of the voter model, where one has an a-priori bound of the influence of spin flips at "u" and there are many them. The analogues for the exclusion process are:

Prop 2) Let $M = \sup_x \sum_{y \neq x} (q(x,y) + q(y,x))$.

Suppose $\mathcal{L}_g \in D$, $\lambda > 0$ and $\mathcal{L} - \lambda \mathcal{L}_g \in g$. Then, $D\mathcal{L}(u) \leq D\mathcal{L}_g(u) + \lambda M D\mathcal{L}_g(u)$

$$+ \lambda \sum_{v: v \neq u} (q(u,v) + q(v,u)) |\Delta\mathcal{L}(v)|.$$

In particular, $III\mathcal{L}III \leq III\mathcal{L}_gIII + 2\lambda M III\mathcal{L}_gIII$.

Thm 3) Suppose $M < \infty$. Then, $\mathcal{L} = \bar{\mathcal{L}}$ generates a probability semigroup $T(t)$ that satisfies

$$\|T(t)f\| \leq e^{2tM} \|f\|, \forall f \in \mathcal{L}.$$

⇒ In what follows we assume $M < \infty$ and also that the single particle motion is a irreducible Markov chain.

3.3) Product form stationary distributions.

Def 4) For a function α on Λ satisfying

$0 \leq \alpha(x) \leq 1, \forall x \in \Lambda$, let ν_α be the product measure on \mathbb{S}^1 with marginals

$$\nu_\alpha(\{y : y(x) = 1\}) = \alpha(x), \quad x \in \Lambda.$$

Q.: Under which conditions are ν_α stationary?

The following is a sufficient (and necessary) condition :

Thm 5) Suppose $0 < \alpha(x) < 1, \forall x$ and let

$$\pi(x) := \frac{\alpha(x)}{1 - \alpha(x)}.$$

Assume: @ $\sum_{Y: \alpha(y) = \alpha(x)} (q(x,y) - q(y,x)) = 0, \forall x \in \Lambda$

(b) $\forall x, y \in \Lambda$ s.t. $\alpha(x) \neq \alpha(y)$,

$$\pi(x)q(x,y) = \pi(y)q(y,x).$$

Then, $\nu_\alpha \in \mathcal{I}$.

Proof: Since D is a cone of \mathcal{L} , we need to show that $\forall f \in D$,

$$\int_S Lf d\sigma_2 = 0 \quad \textcircled{*}$$

\cdot $f \in D$, $\exists (f_i)_{i \in I}$ depending on finitely many coordinates s.t. $f_i \rightarrow f$ and $Lf_i \rightarrow Lf$.

\Rightarrow Need to check $\textcircled{*}$ only for these functions.

\cdot let f depending only on $\{y(x), x \in A\}$ for some finite $A \subset I$.

$$\begin{aligned} \int_S L(y^{x,y}) y(x) (1-y(y)) d\sigma_2(y) \\ \stackrel{\text{changes}}{=} \int_S L(y) y(y) (1-y(y)) \frac{\pi(x)}{\pi(y)} d\sigma_2(y). \end{aligned} \quad \textcircled{**}$$

$$\text{If } x, y \notin A, L(y^{x,y}) - L(y) = 0.$$

Thus, $\int_S Lf d\sigma_2 =$

$$= \sum_{x, y \in A} q(x, y) \int_S (L(y^{x,y}) - L(y)) y(x) (1-y(y)) d\sigma_2(y)$$

$$\textcircled{**} \sum_{x, y \in A} q(x, y) \int_S L(y) y(y) (1-y(x)) \frac{\pi(x)}{\pi(y)} d\sigma_2(y)$$

$$- \sum_{x, y \in A} q(x, y) \int_S L(y) y(x) (1-y(y)) d\sigma_2(y)$$

$$= \int_S f \left[\sum_{x \in A} \eta(x)(1-\eta(y)) \left[\frac{\pi(y)}{\pi(x)} q(y,x) - q(x,y) \right] \right] d\pi_2(y)$$

(****)

By assumption (b),

$$(****) = \sum_{\substack{x \in A \\ y \in A \\ \alpha(x)=\alpha(y)}} \eta(x)(1-\eta(y)) (q(y,x) - q(x,y))$$

$$= \sum_{\substack{x \in A \\ y \in A \\ \alpha(x)=\alpha(y)}} \eta(x) (q(y,x) - q(x,y))$$

$$- \sum_{\substack{x \in A \\ y \in A \\ \alpha(x)=\alpha(y)}} \eta(x) \eta(y) (q(y,x) - q(x,y))$$

= 0 by symmetry

Next, by assumption (a),

$$\sum_{\substack{x \in A \\ y \in A \\ \alpha(x)=\alpha(y)}} \eta(x) (q(y,x) - q(x,y)) = \sum_{x \in A} \eta(x) \sum_{\substack{y \in A \\ \alpha(x)=\alpha(y)}} (q(y,x) - q(x,y))$$

= 0

\Rightarrow To show:

$$\int_S f(y) \left[\sum_{\substack{x \notin A \\ y \in A \\ \alpha(x)=\alpha(y)}} \eta(x) (q(y,x) - q(x,y)) \right] d\pi_2(y) = 0$$

(****)

Since f depends only on $\eta(x)$, $x \in A$ and π_2 is a product measure, then

$$\tilde{f} = \left(\int_S f(y) d\pi_2(y) \right) \left(\sum_{\substack{x \notin A \\ y \in A \\ \alpha(x)=\alpha(y)}} \alpha(x) (q(y,x) - q(x,y)) \right)$$

(****)

$$\text{But } \sum_{\substack{x,y \\ \text{by symmetry}}} = \sum_{\substack{x \in A \\ y \in A \\ d(x,y)=d(y)}} \alpha(x) (q(y,x) - q(x,y))$$

$$= \sum_{y \in A} \alpha(y) \sum_{\substack{x : d(x,y)=d(y)}} (q(y,x) - q(x,y)) = 0$$

by assumption @.

~~#~~

Examples: ① $A = \mathbb{Z}$, $\begin{cases} q(x, x+1) = p \\ q(x, x-1) = q \end{cases}$

$$\begin{cases} q(x, y) = \alpha \text{ if } |x-y| > 1 \end{cases}$$

. $p=q$: Simple Symmetric Exclusion Process (SSEP).

. $p \neq q$: Asymmetric Simple Exclusion Process (ASEP)

. $p+q=0$: Totally Asymmetric Simple Exclusion Process (TASEP).

Applying Thm 5 we readily get:

$\forall i \in \{0,1\}, \gamma_i \in \mathbb{I}$

② Blocking measures:

Take ASEP, $\pi(x) = c \left(\frac{p}{q}\right)^x, c > 0$.

Then, Thm 5 implies that $\gamma_x \in \mathbb{I}$

with $\alpha(x) = \frac{c - (p/q)^x}{1 + c(p/q)^x}, \forall x \in \mathbb{A}$.

For $p > q$, the density of particles is not constant and goes exponentially fast to 0 (resp. 1) as $x \rightarrow -\infty$ (resp. ∞).

\Rightarrow Under proba. 1, y satisfies



$$\sum_{x \leq 0} y(x) < \infty \text{ and } \sum_{x \geq 0} (1-y(x)) < \infty.$$

These measures are called blocking measures and they are reversible also for $p \neq q$.

Remark: There are countable many configurations satisfying \star forming the disjoint union of $S_n := \{y : \sum_{x \leq n} y(x) = \sum_{x \geq n} (1-y(x)) < \infty\}$, $n \in \mathbb{Z}$.

It is easy to see that conditioned on S_n , $A_{\geq 0}$ is an irreducible Markov chain with unique stationary distribution $\mu_n := D_3(\cdot | S_n)$.

Wigget proved that

$$T_0 = \{y_S, S \in \mathcal{I}_0, \mathcal{J}_0^S \cup \{y_u, u \in \mathbb{Z}\}\}.$$

3.4) The graphical construction for ASEP on \mathbb{Z} .

We present here a graphical construction that can be used to see that the process is well-defined.

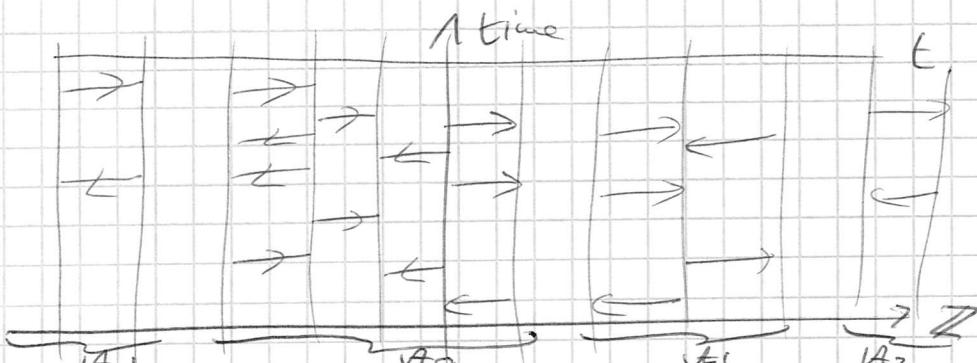
The problem is the following: with ω -many particles, the update of the first jump has happens with probability 1 at $t=0$.

For the finite number^(N) of particle case, one can update the configuration at each jump and it is a Markov chain on \mathbb{Z}^N , so no problem arises.

For the construction, we assign to each $T^{i,j}$, $i,j \in \mathbb{Z}^2$ with $|i-j|=1$ independent Poisson processes with intensity: $\begin{cases} \lambda \text{ for } T^{i,i+1}, \\ \mu \text{ for } T^{i,i-1}. \end{cases}$

$T^{i,j}$ are taken independently of the initial configuration y_0 .

At each occurrence of an event of the Poisson process $T^{i,j}$ we draw a line from i to j .



Then, we subdivide \mathbb{Z} into disjoint subsets $A_k, k \in \mathbb{Z}$, which are not connected by lines.

By Borel-Cantelli, each $\lim_{n \rightarrow \infty} \# T^{(n)} \cap A_k = 0$ almost surely.

\Rightarrow Within each random A_k , \exists a time ordering of lines (finitely many + time $\overset{\text{interval}}{\Delta t, t}$)

\Rightarrow We can construct the configurations $\eta_s, s \in \Delta t, t$, within each bloc.

This is the graphical construction.

Rem.: For higher dimensions, a similar argument holds but only for (deterministic) small times. This can be however iterated.

If $M = \infty$, such a construction does not work anymore.

3.5) Currents and conservation laws.

In this section we describe the macroscopic evolution of the particle density and their non-random limits.

Def. 6) let $\mu_t := \mu(T(t))$ be the distribution at time t . We call:

(a) $s(x, t) := \mu_t(\gamma(x)) \equiv \mathbb{E}_\mu(\gamma(x))$ the particle density at site x and time t .

(b) The average current of particles across the directed edge (x,y) is

$$j(x,y; t) = \mu_t (c(x,y_i y) - c(y, x_i y))$$

where $c(x,y_i y) = q(x,y) \eta(x) (1-\eta(y))$.

Now we focus on ASEP on $\Lambda = \mathbb{Z}$ or $\Lambda_L = \mathbb{Z}/L\mathbb{Z}$.

Prop. 7) A consequence of particles' conservations it holds

$$\frac{d}{dt} g(x,t) + \nabla_x j(x-1,x;t) = 0$$

where $\nabla_x j(x-1,x;t) = j(x,x+1;t) - j(x-1,x;t)$.

Proof: The forward equation is :

$$\frac{d}{dt} T(t) f = T(t) L f, \quad \forall f \in C(S).$$

Integrating w.r.t. μ gives:

$$\frac{d}{dt} \mu_t f = \mu_t L f \quad \text{where } \mu_t = \mu T(t).$$

Taking $f(y) = \eta(x)$, we get:

$$\frac{d}{dt} \mu_t f = \frac{d}{dt} g(x,t),$$

and $(Lf)(y) = \sum_{y \in \mathbb{Z}} \left\{ p \eta(y) (1-\eta(y+1)) + q \eta(y+1) (1-\eta(y)) \right\} \cdot [f(y+1) - f(y)]$

$$= p \eta(x) (1-\eta(x+1)) + q \eta(x+1) (1-\eta(x)) \\ - q \eta(x) (1-\eta(x-1)) + p \eta(x-1) (1-\eta(x)).$$

Notation:

$$\mu_\epsilon(\gamma(x)(1-\gamma(x+1))) = \mu_\epsilon(1_x O_{x+1}).$$

Taking expectation w.r.t μ_ϵ leads to:

$$\mathbb{E}_\epsilon L f = (P \mu_\epsilon(1_{x-1} O_x) + q \mu_\epsilon(O_x 1_{x+1}))$$

$$- (P \mu_\epsilon(1_x O_{x+1}) + q \mu_\epsilon(O_{x-1} 1_x))$$

For $A \in \mathcal{B}$, $j(x, y; \epsilon) \neq 0$ only if $y = x \pm 1$, where

$$j(x, x+1; \epsilon) = P \mu_\epsilon(1_x O_{x+1}) - q \mu_\epsilon(O_x 1_{x+1})$$

$$\Rightarrow \mu_\epsilon L f = - \{ j(x, x+1; \epsilon) - j(x-1, x; \epsilon) \}. \quad \#$$

Remark: The equation in Prop. 7 is the discrete space continuity equation.

3.5.1) SSEP, diffusive scaling.

. Take $p = q = 1/2$. Then it holds:

$$\text{Prop 8)} \quad \frac{d}{dt} S(x, \epsilon) = \frac{1}{2} \Delta_x f(x, t) = \frac{1}{2} (S(x+1, t) - 2S(x, t) + S(x-1, t)).$$

$$\begin{aligned} \text{Proof: } j(x, x+1; \epsilon) &= \frac{1}{2} \mu_\epsilon(1_x O_{x+1}) + \frac{1}{2} \mu_\epsilon(1_x 1_{x+1}) \\ &\quad - \frac{1}{2} \mu_\epsilon(1_x 1_{x+1}) - \frac{1}{2} \mu_\epsilon(O_x 1_{x+1}) \\ &= \frac{1}{2} \mu_\epsilon(1_x) - \frac{1}{2} \mu_\epsilon(1_{x+1}) \\ &= \frac{1}{2} S(x, t) - \frac{1}{2} S(x+1, t) = -\frac{1}{2} \nabla_x S(x, t). \end{aligned}$$

$$\Rightarrow \nabla_x j(x-1, x, t) = \Delta_x S(x, t) \quad \#$$

Remark: In general there is no closed formula for the one-point function (the density), but it is given in terms of higher (2-pt.) correlation functions.

The fact that for SSEP it is the case is a consequence of the following duality.

- let $H(y, A) = \mathbb{1}_{\{y=1 \text{ or } A\}}$ and the deal process $A_\epsilon := \{x \in \Lambda \mid y_\epsilon(x) = 1\}$. It is easy to verify (use the generators).

Thm 9) If $q(x, y) = q(y, x)$, then y_ϵ and A_ϵ are dual with respect to $H(y, A)$, i.e.,

$$\mathbb{P}^y(\eta_\epsilon = 1 \text{ or } A) = \mathbb{P}^A(\eta_\epsilon = 1 \text{ or } A_\epsilon).$$

- Now we want to describe the large scale behavior. Rescale space by $\frac{1}{L}$ and embed it in the continuum, i.e., consider $\frac{1}{L}\Lambda \subset \mathbb{R}$ or $\frac{1}{L}\Lambda_L \subset \mathbb{T} \equiv \mathbb{R}/\mathbb{Z}$ (the torus).

- let $X := \frac{x}{L} \in \mathbb{R}$ or \mathbb{T} the macroscopic spatial variable and set $\tilde{s}(X, t) := s([XL], t)$.

$$\begin{aligned} \text{Using Taylor} \Rightarrow s(x \pm \frac{1}{L}, t) &= \tilde{s}(X \pm \frac{1}{L}, t) = \tilde{s}(X, t) \pm \frac{1}{L} \partial_X \tilde{s}(X, t) \\ &\quad + \frac{1}{2L^2} \partial_X^2 \tilde{s}(X, t) + O(\frac{1}{L^3}). \end{aligned}$$

$$\Rightarrow \Delta_x s(x, t) = \frac{1}{L^2} \partial_x^2 \tilde{s}(x, t) + O\left(\frac{1}{L^3}\right).$$

From this we immediately see that to have a non-trivial limit of the eq. in Prop. 8 we need to consider the macroscopic time variable

T as $T = \frac{t}{L^2}$.

Define the macroscopic density field by

$$s_{\text{ma}}(x, T) := \lim_{L \rightarrow \infty} s([xL], TL^2). \quad \text{Then,}$$

$$\partial_T s_{\text{ma}}(x, T) = \frac{1}{2} \partial_x^2 s_{\text{ma}}(x, T) : \text{the diffusion heat equation.}$$

The scaling $(x, t) \rightarrow (X = \frac{x}{L}, T = \frac{t}{L^2})$ is called diffusive scaling.

3.5.2) ASEP; hydrodynamic scaling.

For $p \neq q$ a closed relation like in Prop. 8 does not exist, except in the stationary case.

The stationary case will give us the lattice version of what macroscopically holds for generic initial conditions.

Prop. 7 gives $\underset{\text{stationary}}{0} = \underset{\text{stationary}}{\frac{d}{dt}} \mu_e(1_x) = \delta(x-1, x; t) - \delta(x+1, x; t)$

$$\Rightarrow \text{the stationary current } j(x, x+1) = p \mu(1_x \alpha_{x+1}) - q \mu(\alpha_x 1_{x+1})$$

is site-independent.

For blocking measures, $\delta^{(x, x+1)} = 0$,
 while for ASEP with $\mu = \nu_3$, $s \in [0, 1]$ fixed,

$$\delta^{(x, x+1)} = (p - q) s(1-s) =: f(s).$$

Large scale behavior: let $X = \frac{x}{L}$ and

$$\tilde{\delta}^{(X, t)} := \delta^{([XL], t)}.$$

$$\Rightarrow \nabla_x \delta^{(x-1, x; t)} = \frac{1}{L} \partial_X \tilde{\delta}^{(X-1, X; t)} + O\left(\frac{1}{L^2}\right)$$

Thus, to have a non-trivial limit we have
 to take $T = \frac{t}{L}$. The scaling $(x, t) \rightarrow (X = \frac{x}{L}, T = \frac{t}{L})$

is known as hydrodynamic scaling.

Let $\tilde{J}(X, T) := \lim_{L \rightarrow \infty} \delta^{([XL]-1, [XL]; TL)}$. Then,
 $S_{\text{na}}(X, T) := \lim_{L \rightarrow \infty} S([XL], TL).$

$$\left[\partial_T S_{\text{na}}(X, T) + \partial_X \tilde{J}(X, T) = 0 \right].$$

Remark: It is plausible to assume that this PDE holds beyond equilibrium I-C, i.e., in general we expect that

$$J(X, T) = f(S(X, T)) \text{ with}$$

$$f(s) = (p - q)s(1-s).$$

This can be actually being proved.

3.6) TASEP

Now we consider $p=1, q=0$.

In this chapter we will analyze the fluctuations of particles' positions using the mathematical tools introduced in section 2.

3.6.1) Master equation and transition probability

- Consider TASEP with N particles. Let $x_k(t)$ be the position of particle with label k at time t . Ordering is conserved by TASEP dynamics.
- We consider the following ordering:

$$x_N(t) < \dots < x_2(t) < x_1(t).$$

For fixed (= non-random) initial conditions, $x_k(0)=y_k$, $k=1, \dots, N$, let

$$G(x_{N-}, x_1; t) := P\left(\{x_i(t) = x_i, 1 \leq i \leq N\} \mid \{x_k(0) = y_k\}_{1 \leq k \leq N}\right).$$

The evolution of G is given by the forward equation, also known as Master equation:

$$\begin{aligned} \frac{d}{dt} G(x_{N-}, x_1; t) &= - G(x_{N-}, x_1; t) \\ &\quad - \sum_{k=2}^N (1 - \delta_{x_{k-1}, x_k}) G(x_{N-}, x_1; t) \\ &\quad + G(x_{N-1}, x_{N-1}, \dots, x_1; t) \\ &\quad + \sum_{k=1}^{N-1} (1 - \delta_{x_{k+1}, x_k}) G(\dots, x_{k-1}, \dots; t). \end{aligned}$$

Thm (0) (Scheitz '97). Define

$$\textcircled{*} \quad \tilde{G}(x_{N,-}, x_i; \epsilon) := \det \left\{ F_{i,j}(x_{N+1-i} - y_{N+1-j}; \epsilon) \right\}_{1 \leq i, j \leq N}$$

$$\text{where } F_n(x, t) = \frac{1}{2\pi i} \oint_{\Gamma_{0,1}} dw \frac{(w-1)^{-n} e^{t(w-1)}}{w^{x-n+1}}.$$

Then, for $x_N < x_{N-1} < \dots < x_1$,

$$\tilde{G}(x_{N,-}, x_i; \epsilon) = G(x_{N,-}, x_i; \epsilon).$$

Proof.: First we verify the initial conditions:

$$\tilde{G}(x_{N,-}, x_i; 0) = \prod_{k=1}^N \delta_{x_k, y_k}.$$

$$F_n(x, 0) = \frac{1}{2\pi i} \oint_{\Gamma_{0,1}} dw \frac{1}{(w-1)^n w^{x-n+1}}.$$

- $F_n(x, 0) = 0$ for $x \geq 1$ since the pole at ∞ vanishes. $\textcircled{\times}$
- $F_n(x, 0) = 0$ for $n \leq 0$ and $x < n$, since the poles at $0, 1$ vanish. \square

Take $x_N < \dots < x_1$.

(a) If $x_N > y_N$, then $x_k > y_k \forall k$.

\Rightarrow The first column of $\textcircled{*}$ is given by

$$\underbrace{[F_0(x_{N,-}, y_N; 0) \dots F_{N,-}(x_1, y_N; 0)]}_{\text{by } \textcircled{\times}} \Rightarrow \tilde{G}(x_{N,-}, x_i; 0).$$

(b) If $x_n < y_n \Rightarrow x_n < y_k - \epsilon_{k+1}$
 $\Rightarrow x_n - y_{n+1-k} < 1 - \epsilon_k.$

\Rightarrow The first row of \tilde{G} is given by

$$\left[\underbrace{F_0(x_n - y_n; \alpha)}_{\text{by } \textcircled{I}} \quad \cdots \quad \underbrace{F_{N+1}(x_n - y_1; \alpha)}_{\text{by } \textcircled{II}} \right]^T$$

$$\Rightarrow \tilde{G}(x_n, \dots, x_1; \alpha) = 0.$$

(c) Finally, if $x_n = y_n$, $F_0(0; \alpha) = 1$ and otherwise
the first column contains 0's as in (b).

$$\Rightarrow \tilde{G}(x_n, \dots, x_1; \alpha) = \delta_{x_n, y_n} \tilde{G}(x_{n-1}, \dots, x_1; \alpha)$$

$$\stackrel{\text{iterate}}{=} \prod_{k=1}^N \delta_{x_k, y_k}.$$

The next task is to verify that \tilde{G} satisfies the master equation.

We have: $\frac{d}{dt} F_n(x, t) = -F_n(x, t) + F_{n-1}(x, t).$

$$\Rightarrow \tilde{G}(x_n, \dots, x_1; t) = \det \begin{pmatrix} F_0(x_n - y_n; t) & \cdots & F_{N+1}(x_n - y_1; t) \\ \vdots & & \vdots \\ F_{N+1}(x_1 - y_N; t) & \cdots & F_0(x_1 - y_1; t) \end{pmatrix}$$

Satisfies:

$$\begin{aligned} \frac{d}{dt} \tilde{G}(x_n, \dots, x_1; t) &= - \sum_{k=1}^N \tilde{G}(x_n, \dots, x_{k+1}; t) \\ &\quad + \sum_{k=1}^N \tilde{G}(\dots, x_{k-1}, \dots; t) \end{aligned}$$

= r.h.s. of the Master equation

$$+ \sum_{k=1}^{N-1} \tilde{G}(-, x_{k+1}, x_k = x_{k+1}, \dots; t) \quad \left. \right\}$$

$$- \sum_{k=2}^N \tilde{G}(-, x_k, x_{k-1} = x_{k+1}, \dots; t) \quad \left. \right\}$$

\Rightarrow To show: $\left. \right\} = 0$.

This holds true if $\tilde{G}(-, x_k, x_{k-1} = x_k, \dots; t)$
 $= \tilde{G}(-, x_k, x_{k-1} = x_{k+1}, \dots; t)$.

We use $F_n(x, t) = F_{n-1}(x, t) + F_n(x+1, t)$
 which follows from the identity

$$\left(\frac{w-1}{w}\right)^{-n} \frac{1}{w^x} = \left(\frac{w-1}{w}\right)^{-n+1} \frac{1}{w^x} + \left(\frac{w-1}{w}\right)^{-n} \frac{1}{w^{x+1}}$$

$$\Rightarrow \tilde{G}(-, x_0, x_{0-1} = x_0, \dots; t) = \det \left(F_{e-k} (x_{N+1-e-\tilde{y}_k}, t) \right) \quad \begin{matrix} \tilde{y}_k = y_{N+1-k} \\ 1 \leq k \leq n \end{matrix}$$

$$= \det \left(\dots \left(F_{N+1-\delta-k} (x_0 - \tilde{y}_k, t) \right) \underbrace{\dots}_{=} \left(F_{N+2-\delta-k} (x_0 - \tilde{y}_k, t) \right) \dots \right)$$

$$= F_{N+1-\delta-k} (x_0 - \tilde{y}_k, t)$$

$$+ F_{N+2-\delta-k} (x_0 + 1 - \tilde{y}_k, t)$$

$$= \tilde{G}(-, x_0, x_{0-1} = x_0 + 1, \dots; t). \quad \cancel{\#}$$

3.6.2) Determinantal correlations.

The key property that we use is:

$$F_{n+1}(x, \epsilon) = \sum_{y \geq x} F_n(y, \epsilon),$$

which comes from $F_{n+1}(x, \epsilon) = F_n(x, \epsilon) - F_n(x+1, \epsilon)$

and $\lim_{x \rightarrow \infty} F_n(x, \epsilon) = 0$ exponentially fast.

For later use, define

$$\Phi_k^n(x) = (-1)^k F_{-k}(x-y_{n-k}; \epsilon), \quad k \geq 0.$$

For $n+k < 0$, the pole at 1 vanishes

$$\Rightarrow F_{n+1}(x, \epsilon) = - \sum_{y < x} F_n(y, \epsilon) \Rightarrow \Phi_{n+1-k}^n(x) = \sum_{y < x} \Phi_{n-k}^n(y).$$

The integral representation for $\Phi_{n-k}^n(x)$ is then

$$\Phi_{n-k}^n(x) := \frac{1}{2\pi i} \int_{\Gamma_0} dw \frac{(1-w)^{N-k}}{w^{x+N-k-y_n+1}} e^{-\epsilon(w-1)}$$

and calling $\phi(x, y) := \mathbb{1}(x > y)$ it satisfies

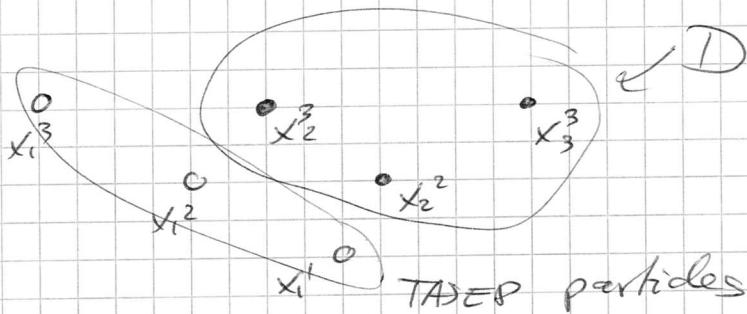
$$\Phi_{n+1-k}^n(x) = (\phi * \Phi_{n-k}^n)(x), \quad \forall k \geq 1.$$

The measure in Thm 10 is not determinantal, but after appropriate massaging of it, one recovers a determinantal measure like the GUE minor measure.

Lemma 11) Denote by $x_i^k = x_k$, $k=1, \dots, N$. Then,

$$G(x_N - x_i; t) = \sum_{\mathcal{D}} \det(F_{-\delta+1}(x_i^k - y_{N+1-\delta}; t))$$

where $\mathcal{D} = \{x_k^n, 2 \leq k \leq n \leq N \mid x_k^{n+1} < x_k^n \leq x_{k+1}^{n+1}\}$



Proof: To understand how it works, consider $N=3$.

Then, $G = \det \begin{bmatrix} F_0(x_1^3 - y_3, t) & F_{-1}(x_1^3 - y_2, t) & F_{-2}(x_1^3 - y_1, t) \\ F_1(x_1^2 - y_3, t) & F_0(x_1^2 - y_2, t) & F_{-1}(x_1^2 - y_1, t) \\ F_2(x_1^1 - y_3, t) & F_1(x_1^1 - y_2, t) & F_0(x_1^1 - y_1, t) \end{bmatrix}$

Substitute the last row by (using $F_{n+1}(x, t) = \sum_{y \geq x} F_n(y, t)$)

with $\sum_{x_3^3 > x_2^2} \sum_{x_2^2 > x_1^1} [F_0(x_3^3 - y_3, t) \quad F_{-1}(x_3^3 - y_2, t) \quad F_{-2}(x_3^3 - y_1, t)]$

and the second row by

$$\sum_{x_2^3 > x_1^2} [F_0(x_2^3 - y_3, t) \quad F_{-1}(x_2^3 - y_2, t) \quad F_{-2}(x_2^3 - y_1, t)].$$

$$\Rightarrow G = \sum_{\mathcal{D}} \det(F_{-\delta+1}(x_i^k - y_{N+1-\delta}; t))$$

$$\text{with } \tilde{\mathcal{D}} = \{x_2^3, x_1^2, x_3^3, x_2^2, \cancel{x_2^3}, x_1^1\}$$

But by antisymmetry of the determinant,
the sum

$$\sum_{\substack{x_3^3 > x_2^3 \\ x_3^3 > x_2^2}} \det(\dots) = 0$$

\Rightarrow It remains only the ^{now} symmetric part of
the domain \mathcal{D} , i.e., \mathcal{D}^+ .

The next lemma is:

Lemma 12) let $\phi(x, y) = \mathbb{I}(x > y)$, $x_n^{n-1} \equiv v_i \sqrt{t}$,

$$\phi(v_i \sqrt{t}, y) = 1.$$

let $x^n < x^{n+1}$ (interlace) \Leftrightarrow

$$x_1^{n+1} < x_1^n \leq x_2^{n+1} < x_2^n \leq \dots < x_y^n \leq x_{n+1}^{n+1}.$$

Then, $\prod_{n=2}^N \det(\phi(x_i^{n-1}, x_o^n)) = \prod_{1 \leq i < o} \mathbb{I}(x_1 < x_2 \dots < x_n)$.

Proof: The proof is trivial. \square

$\Rightarrow G$ is a marginal of a measure μ_N

$\alpha_N(x'_1, x'_2 = (x_1^2, x_2^2), \dots, x'^N = (x_1^N, \dots, x_N^N))$ given by:

$$\mu_N(x'_1, \dots, x'^N) = \text{const} \cdot \prod_{n=2}^N \det(\phi(x_i^{n-1}, x_o^n)), \det(\mathbb{I}_{ij}^{(N)}(x_i^o))$$

by Lemma 11 & Lemma 12.

Applying Thm 26 of Chapter 2 (compare with
Thm 38 too) we obtain the following.

Prop 13) For a measure μ_N (normalized to have total mass 1, not necessarily a proba. measure) with "const" to 0, the correlation functions are determined.

Let $\Phi_{n-k}^n(x) := (\phi^{(n,N)} * \Phi_{N-k}^n)(x)$, $k=1, \dots, N$

where $\phi^{(n,N)}(x,y) = \begin{cases} (\underbrace{\phi * \dots * \phi}_{n-\text{times}})(x,y), & \text{if } n > 1, \\ a, & \text{otherwise.} \end{cases}$

Let $\{\Phi_{n-e}^n(x), e=1, \dots, n\}$, $n=1, \dots, N$, families of functions s.t.

(a) They span $\{\phi(v_i t, x), \dots, \phi^{(n)}(v_i t, x)\}$

(b) $\sum_{x \in \mathbb{Z}} \Phi_{n-e}^n(x) \Phi_{n-k}^n(x) = \delta_{k,e}$, $1 \leq k, e \leq n$.

Then, if $\phi(v_i t, x) = \text{const} \cdot \Phi_0^n(x)$, the kernel has the particularly simple form:

$$K_t(n_1, x_1; n_2, x_2) = -\phi^{(n_1, n_2)}(x_1, x_2) \mathbb{I}(n_1 < n_2) + \sum_{k=1}^{n_2} \Phi_{n_1-k}^n(x_1) \Phi_{n_2-k}^n(x_2).$$

Remark: For TASEP with all particles having jump rates 1, we have to look for polynomials: $\Phi_k^n(x) \equiv$ polynomial of degree k , $k=0, \dots, n$. Then, $\phi(v_i t, x) = \text{const} \cdot \Phi_0^n(x)$ is also satisfied.

A simple verification leads to:

Lemma 14) For $n_1 < n_2$,

$$\phi^{(n_1, n_2)}(x_1, x_2) = \binom{x_1 - x_2 - 1}{n_2 - n_1 - 1} = \frac{1}{2\pi i} \oint_{\Gamma_0} dw \left(\frac{w}{1-w} \right)^{n_2 - n_1} \frac{1}{w^{x_1 - x_2 + 1}}.$$

3.6.3) Orthogonalisation.

Recall the formula for $\mathbb{E}_K^N(x)$ at page 117 and compare it with the one in Prop. 13.

(a) Step initial conditions.

First we consider $y_k = -k$, $k \geq 1$.

$$\Rightarrow \mathbb{E}_K^N(x) = \frac{1}{2\pi i} \oint_{\Gamma_0} dw \frac{(1-w)^{-k}}{w-x+N+1} e^{t(w-1)}.$$

Lemma 15) $\Phi_e^N(x) = \frac{-1}{2\pi i} \oint_{\Gamma_1} dz \frac{e^{-t(z-1)}}{(z-x+N+1)^{k+1}}$

Satisfies: (a) $\Phi_e^N(x)$ is a polynomial of degree R ,

$$(b) \sum_{x \in \mathbb{Z}} \mathbb{E}_K^N(x) \Phi_e^N(x) = \delta_{k,0}.$$

Proof: (a) Trivial. (Cauchy's residue Theorem).

$$(b) \sum_{x \in \mathbb{Z}} \mathbb{E}_K^N(x) \Phi_e^N(x) =$$

$$\begin{aligned} \mathbb{E}_K^N(x) &\stackrel{x \neq -n}{=} \sum_{x \geq -n} \frac{-1}{(2\pi i)^2} \oint_{\Gamma_1} dz \oint_{\Gamma_0} dw \frac{e^{-t(w-1)}}{e^{-t(z-1)}} \frac{(1-w)^k}{(1-z)^{k+1}} \frac{z^{x+y}}{w^{x+n+1}} \\ &\text{for } x \neq -n \end{aligned}$$

$$= \frac{-1}{(2\pi i)^2} \oint_{\Gamma_1} dz \oint_{\Gamma_0} dw \frac{e^{-t(w-1)}}{e^{-t(z-1)}} \frac{(1-w)^k}{(1-w)^{k+1}(w-z)}$$

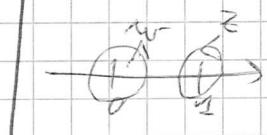
$$\sum_{x \geq -n} \frac{z^{x+y}}{w^{x+n+1}} = \frac{1}{w-z}$$

$$\text{if } z \neq w$$

$$= \frac{-1}{2\pi i} \oint_{\Gamma_1} dz \frac{1}{(1-z)^{k+1}} = \delta_{k,0}. \quad \#$$

With this result we can compute the Kernel for TASEP with step initial conditions.

Prop.16: For $Y_k = -k$, $k \geq 1$,

$$K_L(u_1, x_1; u_2, x_2) = -\frac{1}{2\pi i} \oint_{\Gamma_0} dw \left(\frac{w}{1-w} \right)^{u_2-u_1} \frac{1}{w^{x_1-x_2+1}} \Pi_{u_1, u_2}$$


$$+ \frac{1}{(2\pi i)^2} \oint_{\Gamma_0} dw \oint_{\Gamma_1} dz \frac{e^{tz}}{e^{tw}} \frac{(1-w)^{u_1}}{w^{x_1+u_1+1}} \cdot \frac{z^{x_2+u_2}}{(-z)^{u_2}} \frac{1}{w-z}$$

Proof: We need to compute $\sum_{k=1}^{u_2} \mathbb{P}_{u_1-k}(x_1) \mathbb{P}_{u_2-k}(x_2)$.

Since $\mathbb{P}_{u_2-k}(x) = 0$ for $k > u_2$, we can extend the sum to ∞ .

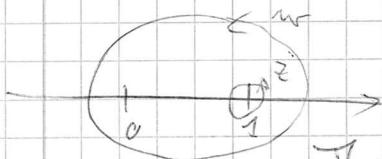
$$\Rightarrow \sum_{k=1}^{u_2} \mathbb{P}_{u_1-k}(x_1) \mathbb{P}_{u_2-k}(x_2) =$$

$$= \sum_{k=1}^{\infty} \frac{-1}{(2\pi i)^2} \oint_{\Gamma_0} dw \oint_{\Gamma_1} dz \frac{(1-w)^{u_1-k} e^{t(w-1)}}{w^{x_1+u_1+1}} \cdot \frac{z^{x_2+u_2}}{e^{t(z-1)} (-z)^{u_2}}$$

Take the sum

inside, provided $|1-z| < |1-w|$

$$= \frac{-1}{(2\pi i)^2} \oint_{\Gamma_0} dw \oint_{\Gamma_1} dz \frac{(1-w)^{u_1} e^{tw}}{w^{x_1+u_1+1}} \frac{z^{x_2+u_2}}{e^{t(z-1)} (-z)^{u_2}}$$



$$\cdot \underbrace{\sum_{k=1}^{\infty} \frac{(1-z)^{k-1}}{(1-w)^k}}$$

The pole at $w=z$ cancels the contribution of the pole in $\oint_{\Gamma_1} dz \frac{z^{x_2+u_2}}{e^{t(z-1)} (-z)^{u_2}}$ at $w=z$ of Lemma 14. $\#$

(b) Flat initial conditions.

We would like to get the kernel for the limiting situation $Y_k = -2k$, $k \in \mathbb{Z}$.

We start with $Y_k = -2k$, $k \geq 1$.

$$\Rightarrow \mathbb{P}_k^u(x) = \frac{1}{2\pi i} \oint_{\Gamma_0} dw \frac{((1-w)w)^k e^{t(w-1)}}{w^{x+2k+1}}$$

Lemma 17: $\Phi_e(x) = \frac{1}{2\pi i} \oint_{\Gamma_1} dz \frac{(1-2z) z^{x+2n}}{(z(1-z))^{e+1}} e^{-t(z-1)}$

[is a polynomial of degree e and

satisfying $\sum_{x \in \mathbb{Z}} \Phi_k^u(x) \Phi_e^u(x) = S_{k,e}$.

Proof: $\sum_{x \in \mathbb{Z}} \Phi_k^u(x) \Phi_e^u(x) =$

$$= \sum_{x \in \mathbb{Z}-2n} \frac{1}{(2\pi i)^2} \oint_{\Gamma_1} dz \oint_{\Gamma_0} dw \frac{(w(1-w))^k e^{tuw}}{w^{x+2n+1}} \cdot \frac{(1-2z) z^{x+2n} e^{-t(z-1)}}{(z(1-z))^{e+1}}$$

$$= \frac{1}{(2\pi i)^2} \oint_{\Gamma_1} dz \oint_{\Gamma_{0,B}} dw \frac{(w(1-w))^k e^{tuw}}{w^{x+2n+1}} \frac{(1-2z)}{e^{tz}(z(1-z))^{e+1}} \cdot \frac{1}{w^{x+2n}}$$

$$= \frac{1}{2\pi i} \oint_{\Gamma_1} dz \frac{(1-2z)}{(z(1-z))^{e+k+1}} = \frac{1}{2\pi i} \oint_{\Gamma_0} du \frac{1}{u^{e+k+1}} = S_{k,e}$$

$\begin{matrix} w(1-w) \\ \downarrow u \\ z(1-z) \end{matrix}$ $\begin{matrix} u \\ \downarrow \\ du = (1-2z)dz \end{matrix}$

*

Consequence of this together with
the change of variables $\begin{cases} n_i \rightarrow u_i + N \\ x_i \rightarrow x_i - 2N \end{cases}$ (in $N \rightarrow \infty$)

we get :

Prop 18: The convolution kernel per flat initial conditions, $y_k = -2k$, $k \in \mathbb{Z}$ is given by :

$$K_e(u_1, x_1; u_2, x_2) = \frac{-1}{2\pi i} \oint_{\Gamma_{0,1}} dw \left(\frac{w}{1-w} \right)^{u_1-u_2} \frac{1}{w^{x_1-x_2+1}} \frac{1}{(u_1+x_2)} + \frac{1}{2\pi i} \oint_{\Gamma_1} dz e^{t(1-2z)} \frac{z^{u_1+u_2+x_2}}{(1-z)^{u_1+u_2+x_1+1}}$$

Proof:

$$\begin{aligned}
 & \sum_{k=1}^{n_2} \Psi_{n_1-k}^{n_1}(x_1) \overline{\Phi_{n_2-k}^{n_2}(x_2)} \\
 &= \sum_{k=1}^{\infty} \overline{\Psi_{n_1-k}^{n_1}(x_1)} \overline{\Phi_{n_2-k}^{n_2}(x_2)} \\
 &= \sum_{k=1}^{\infty} \frac{1}{(2\pi i)^2} \oint_{\Gamma_0} dw \frac{(1-w)w}{w-x_1+2n_1+1} \int_{\Gamma_1} dz \frac{e^{t w} (1-2z) z^{x_2+2n_2-62}}{(z^{(1-z)})^{n_2-k+1}} \\
 &= \frac{1}{(2\pi i)^2} \oint_{\Gamma_0} dw \oint_{\Gamma_1} dz \frac{e^{t w} (1-w)w^{n_1}}{w-x_1+2n_1+1} \frac{(1-2z) z^{x_2+2n_2}}{e^{t z} (z^{(1-z)})^{n_2+1}}
 \end{aligned}$$

Provided $|z(1-z)| < |w(1-w)|$

$$\begin{aligned}
 & \cdot \sum_{k=1}^{\infty} \left(\frac{z(1-z)}{w(1-w)} \right)^k \\
 &= \frac{z(1-z)}{w(1-w) - z(1-z)} = \frac{z(1-z)}{(z-w)(z+w-1)}.
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{(2\pi i)^2} \oint_{\Gamma_0} dw \oint_{\Gamma_1} dz \frac{e^{t w} (w(1-w))^{n_1}}{w-x_1+2n_1+1} \frac{z^{x_2+2n_2} (1-2z)}{e^{t z} (z^{(1-z)})^{n_2} (z-w)(z+w-1)}
 \end{aligned}$$

$$\begin{aligned}
 &= -\frac{1}{(2\pi i)^2} \oint_{\Gamma_0} dw \oint_{\Gamma_1} dz \frac{e^{t w} (w(1-w))^{n_1}}{w-x_1+2n_1+1} \cdot \frac{z^{x_2+2n_2}}{e^{t z} (z^{(1-z)})^{n_2}} \left. \frac{1}{z-w} \right|_{z=w} \quad (A) \\
 &\quad - \frac{1}{(2\pi i)^2} \oint_{\Gamma_0} dw \oint_{\Gamma_1} dz \frac{e^{t w} (w(1-w))^{n_1}}{w-x_1+2n_1+1} \cdot \frac{z^{x_2+2n_2}}{e^{t z} (z^{(1-z)})^{n_2}} \left. \frac{1}{z-w} \right|_{z=-w} \quad (B)
 \end{aligned}$$

Do the change: $\begin{cases} x_i \rightarrow x_i - 2N \\ n_i \rightarrow n_i + N \end{cases} \Rightarrow$ The first term, (A), has a pole

in w only at $w=0$: $\frac{w^{-N+1}}{w^{x_1+2n_1+1}} \Rightarrow$ for $x_1+n_1 < N$, the pole vanishes

$$\Rightarrow \lim_{N \rightarrow \infty} (A) = 0.$$

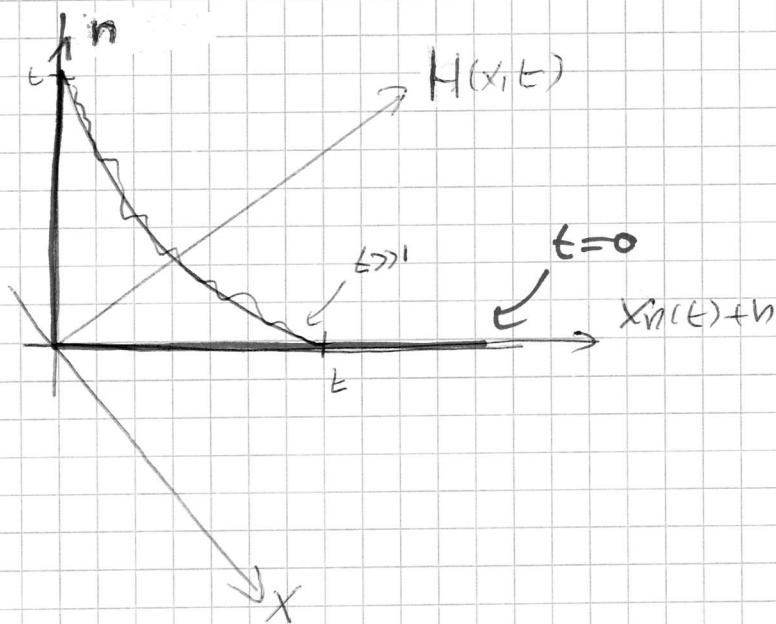
Similarly, the pole at $w=\alpha$ in (B) vanishes for N large enough and it remains the pole at $w=1-z$, namely, in the $N \rightarrow \infty$ limit

$$\begin{aligned} (B) &\xrightarrow[N \rightarrow \infty]{} \frac{1}{2\pi i} \oint_{\Gamma_1} dz \frac{e^{t(1-z)}}{(1-z)^{x_1+2n_1+1}} \frac{z^{x_2+2n_2}}{e^{tz}(z(1-z))^{n_2}} \\ &= \frac{1}{2\pi i} \oint_{\Gamma_1} dz \frac{e^{t(1-z)}}{(1-z)^{x_1+n_1+n_2+1}}. \quad \# \end{aligned}$$

3.6.4) Large times limit.

To describe the large time fluctuations we have to first know where to focus, i.e., the macroscopic behavior.

(a) Step initial conditions and the tiny process



We describe the system at fixed time.

For fixed $\alpha \in (0,1)$, TASEP particles with index $n = \lfloor \alpha t \rfloor$ will be around position $(1-2\sqrt{\alpha})t$:

$$\lim_{L \rightarrow \infty} \frac{x_{\lfloor \alpha t \rfloor}}{L} = 1 - 2\sqrt{\alpha}.$$

Equivalently, in terms of height function $H(x,t)$,

$$h_{\text{max}}(\beta) := \lim_{t \rightarrow \infty} \frac{H(\beta t, t)}{t} = \begin{cases} \frac{1}{2}(1 + \beta^2), & |\beta| \leq 1, \\ |\beta|, & |\beta| > 1. \end{cases}$$

From the results of Section 3.6.3 and the Gap probability formula (see Sect. 2.5, page 48)

We have, for any $1 \leq n_1 < n_2 < \dots < n_\ell$

$$\boxed{\mathbb{P}\left(\bigcap_{k=1}^{\ell} \{X_{n_k}(t) \geq a_k\}\right) = \det\left(\mathbb{I} - \chi_{a_k} K_t \chi_a\right)_{\ell \times \ell}, \quad \text{where } \chi_a(x) = \mathbb{1}_{(x < a)}}.$$

Similarly, in terms of height functions, we have the identity: $\{H(m-n, t) \geq m+n\} = \{X_n(t) \geq m-n\}$.

Thus:

$$\begin{aligned} & \mathbb{P}\left(\bigcap_{k=1}^{\ell} \{H(m_k - n_k, t) \geq m_k + n_k\}\right) \\ &= \mathbb{P}\left(\bigcap_{k=1}^{\ell} \{X_{n_k}(t) \geq m_k - n_k\}\right). \end{aligned}$$

By the KPZ scaling theory, one expects that the correlations are on $t^{2/3}$ scale and fluctuations in the $t^{1/3}$ scale.

Indeed, let $\begin{cases} h(u) = \alpha t + 2ut^{2/3} = \tilde{\alpha}t \\ x(u) = (1 - 2\sqrt{\tilde{\alpha}})t = (1 - 2\sqrt{\alpha})t - \frac{24}{\sqrt{\alpha}}t + \frac{u^2}{\alpha^{3/2}}t^{1/3} \end{cases}$

and the rescaled particle process:

$$X_t(u) := \frac{X_{n(u)}(t) - x(u)}{-t^{1/3}}$$

Then, define the correspondingly rescaled kernel by

$$K_t^{\text{resc}}(u_1, s_1; u_2, s_2) := t^{1/3} K_t^{(1)}(n(u_1), x(u_1) - s_1, t^{1/3}; n(u_2), x(u_2) - s_2 t^{1/3}).$$

Then, the asymptotic analysis of K_t^{resc} (similar to the one we made for the largest GUE eigenvalue) leads to:

$$\lim_{t \rightarrow \infty} K_t^{\text{resc}}(u_1, s_1; u_2, s_2) \stackrel{\text{caus}}{=} \frac{s_n^{-1}}{(2\pi i)^2} \int dW \int dZ \frac{1}{Z-W} \cdot \frac{e^{\frac{1}{3}Z^3 + u_2 \frac{Z^2}{s_n} - Z \left(\frac{s_2}{s_n} - \frac{u_2^2}{s_n^2} \right)}}{e^{\frac{1}{3}W^3 + u_1 \frac{W^2}{s_n} - W \left(\frac{s_1}{s_n} - \frac{u_1^2}{s_n^2} \right)}}$$

$$= \frac{1}{s_n} \cdot K_{\text{Airy}_2}\left(\frac{s_1}{s_n}, \frac{u_1}{s_n}; \frac{s_2}{s_n}, \frac{u_2}{s_n}\right) \text{ with}$$

$$s_n = \frac{1}{(1-\sqrt{\alpha})^{2/3} \sqrt{\alpha}}, \quad s_h = \alpha^{2/3} (1-\sqrt{\alpha})^{1/3}.$$

In terms of process we have,

$$\boxed{\lim_{t \rightarrow \infty} X_t(u) = s_n \text{Airy}_2(u/s_n)} \text{ where}$$

Airy_2 is the Airy₂ process of Definition 29, Sect. 2 (page 78).

In terms of height functions, say for $\alpha = 1/4$,

$$h_E^{\text{resc}}(u) := \frac{H(2u(t/2)^{2/3}, t) - (t^{1/2} + u^2(t/2)^{1/3})}{-(t/2)^{1/3}}$$

$\xrightarrow{t \rightarrow \infty} A_2(u)$ (in the sense of finite-dimensional distributions).

(b) Step initial conditions and the GUE minor process.

If we fix the number of levels and let time goes to infinity, under diffusion scaling limit, namely \sqrt{N}

$$\frac{x_k^u(\frac{1}{2}tL^2) - \frac{1}{2}tL^2}{L} \xrightarrow[L \rightarrow \infty]{} \xi_k^u(t)$$

where $\{\xi_k^u(t), 1 \leq k \leq N\}$ is distributed according to the GUE minor process measure, compare with Thm 35 and Corollary 39.

(c) Flat initial conditions.

Clearly the limit shape is flat.

Without loss of generality, we consider particles which are around the origin at time t , i.e., with particle number $\approx t^{1/4}$.

$$\text{let } X_t^{v_{sc}} := \frac{X_{[t^{1/4} + u t^{2/3}]}(t) + 2u t^{2/3}}{t^{1/3}}$$

Then, the asymptotic analysis of the rescaled kernel

$$K_t^{v_{sc}}(u_1, s_1; u_2, s_2) := 2^{x_2 - x_1} t^{1/3} K_t(n(u_1), x(u_1, s_1); n(u_2), x(u_2, s_2))$$

$$\text{where } \begin{cases} x(u_i, s_i) = [-2u_i t^{2/3} - s_i t^{1/3}] \\ n(u_i) = \left[\frac{t}{4} + u_i t^{2/3} \right] \end{cases}$$

gives:

$$\lim_{t \rightarrow \infty} K_t^{vesc}(u_1, s_1; u_2, s_2) = K_{A_1}(u_1, s_1; u_2, s_2)$$

where

$$K_{A_1}(u_1, s_1; u_2, s_2) = -\frac{1}{\sqrt{4\pi(u_2-u_1)}} \cdot e^{-\frac{(s_2-s_1)^2}{4(u_2-u_1)}} \mathbb{I}_{(u_2 > u_1)} \\ + A(s_1 + s_2 + (u_2 - u_1)^2) e^{(u_2 - u_1)(s_1 + s_2) + \frac{2}{3}(u_2 - u_1)^3}$$

Def. 19) The Aing₁ process, A_1 , is defined by its finite-dimensional distributions

$$P\left(\bigcap_{k=1}^n \{A_1(u_k) \leq s_k\}\right) = \det(1 - \gamma_k K_{A_1} x_k) \quad L^2(\{u_1, u_2, \dots, u_n\}; R)$$

where $x_k(u_n, x) = \mathbb{I}_{(x > s_k)}$

There are some known properties (compare with the ones of the Aing₂ process in Section 2.11, page 80):

- (a) A_1 is stationary,
- (b) $P(A_1(c) \leq s) = F_{\text{dist}}(2s)$,
- (c) $\text{Cov}(A_1(a), A_1(a)) = \begin{cases} \text{Var}(A_1(a)) - |a|, & \text{for small } |a|, \\ \text{Superexponentially decaying}, & \text{for large } |a|. \end{cases}$ See pages 132-133
- (d) There is a continuous version of A_1 ,
- (e) A_1 is not a Markov process.
- (f) let $H = -\frac{d^2}{dx^2}$ and $K(x, y) = A(x+y)$. Then,

$$K_{A_1}(u_1, s_1; u_2, s_2) = -\left(e^{-(u_2-u_1)H}\right)(s_1, s_2) \mathbb{I}_{(u_1 < u_2)} \\ + \left(e^{u_1 H} K e^{-u_2 H}\right)(s_1, s_2).$$

In particular, the asymptotic analysis shows:

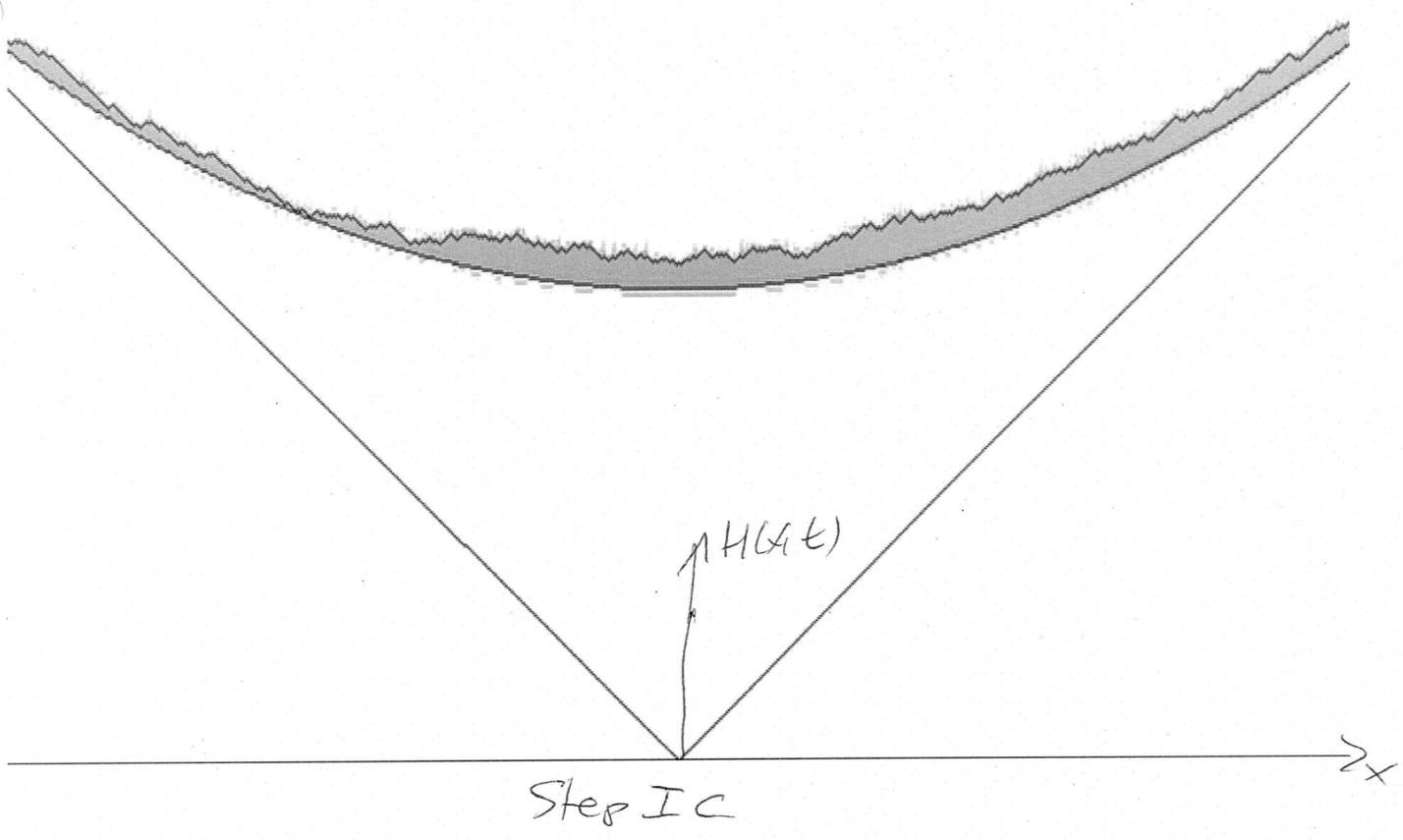
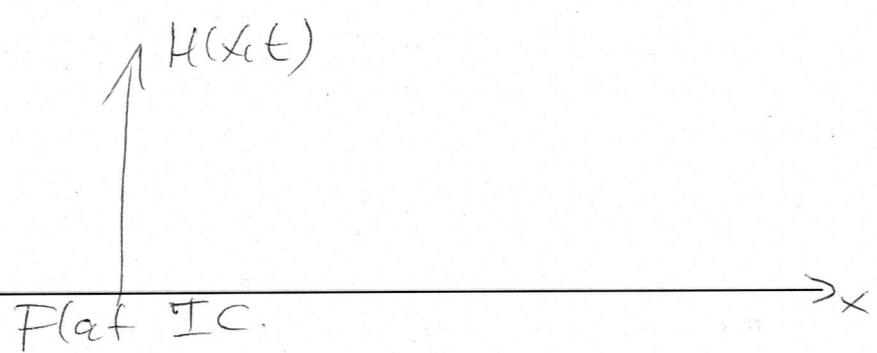
$\lim_{t \rightarrow \infty} X_t^{\text{desc}}(u) = \mathcal{F}_1(u)$ in the sense
of finite-dimensional distributions, by showing
that the Fredholm determinants converges.

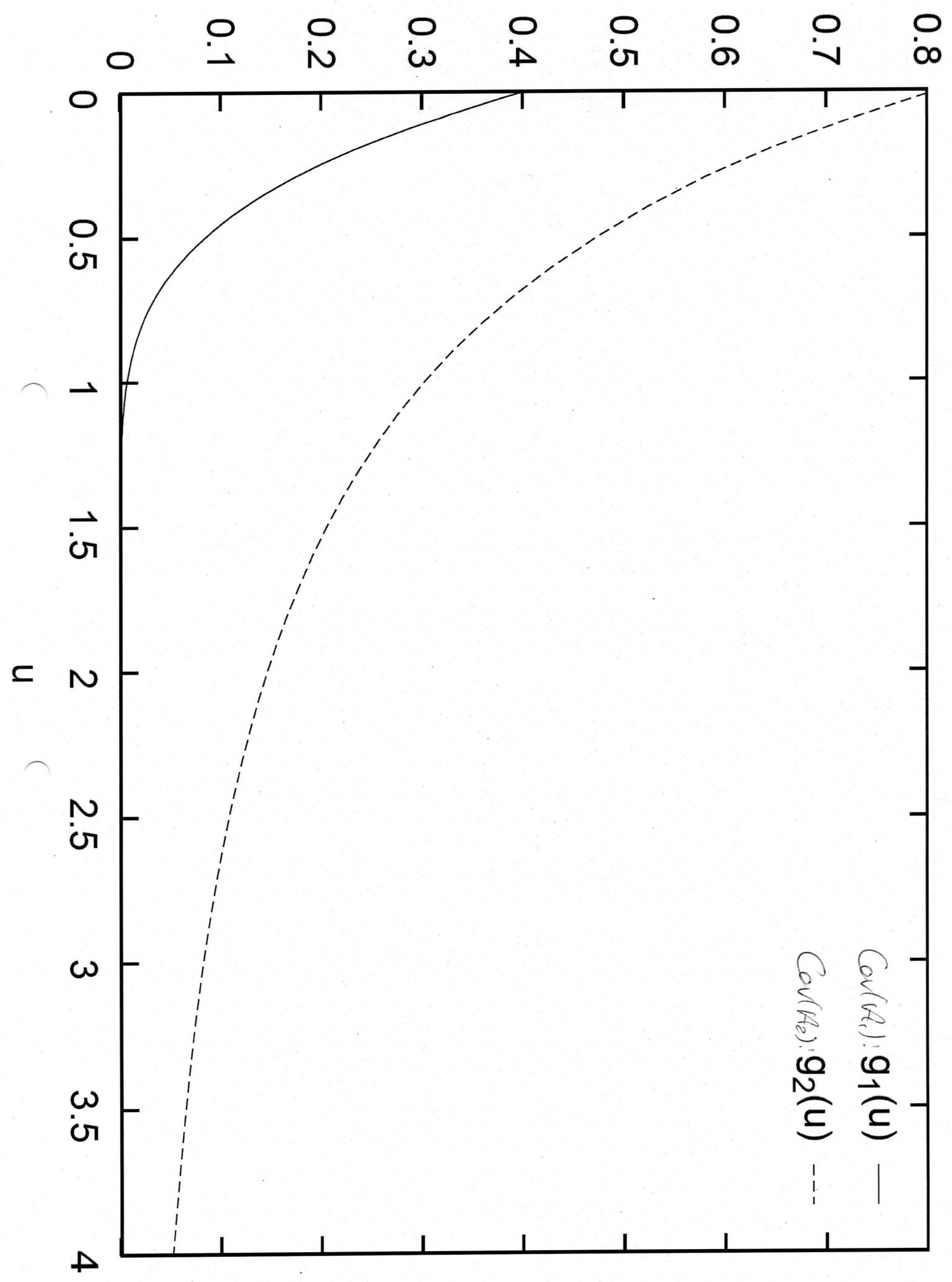
In terms of height function,

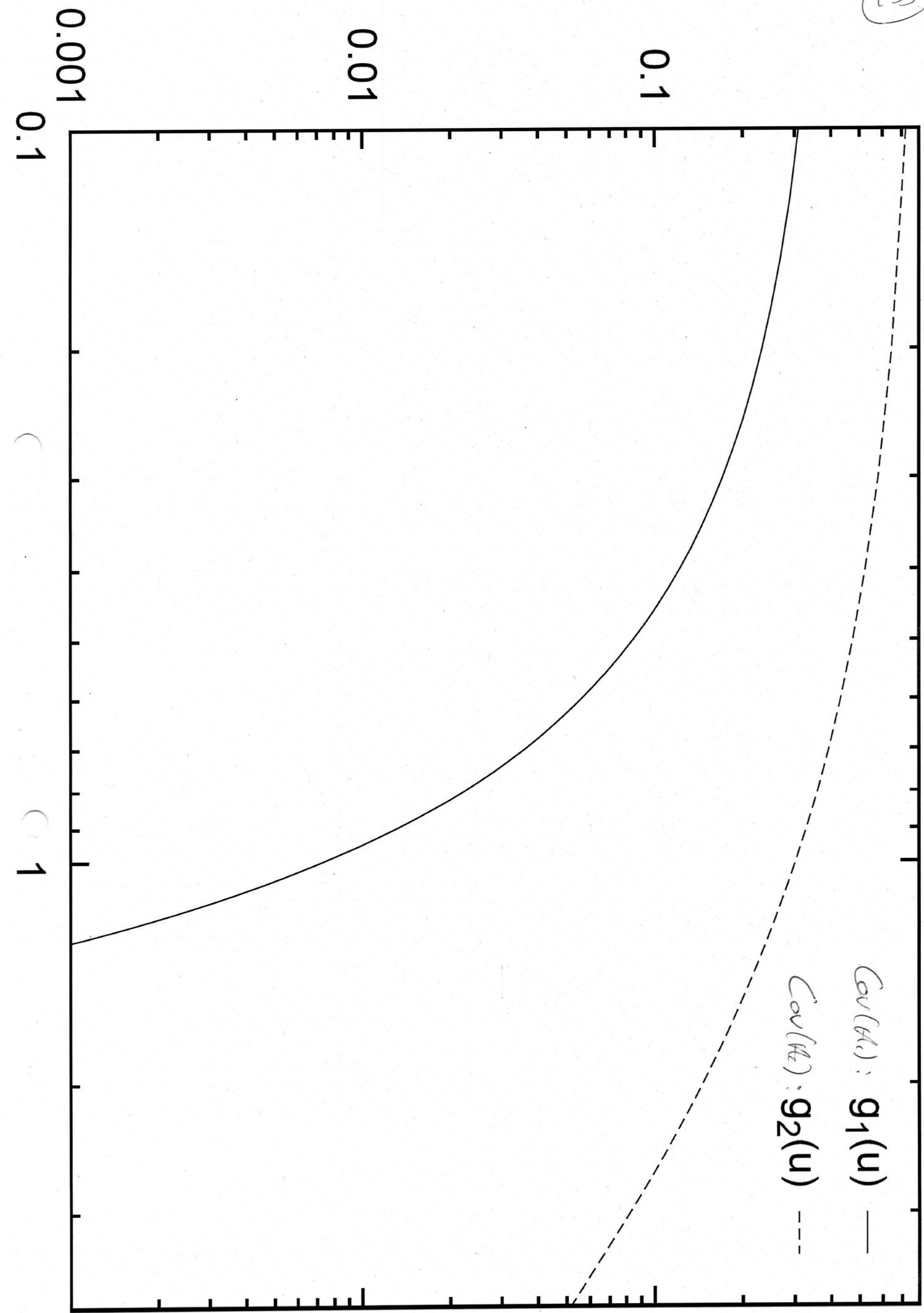
$$\lim_{t \rightarrow \infty} \frac{h(2ut^{2/3}, t) - t/2}{-t^{1/3}} = \mathcal{F}_1(u).$$

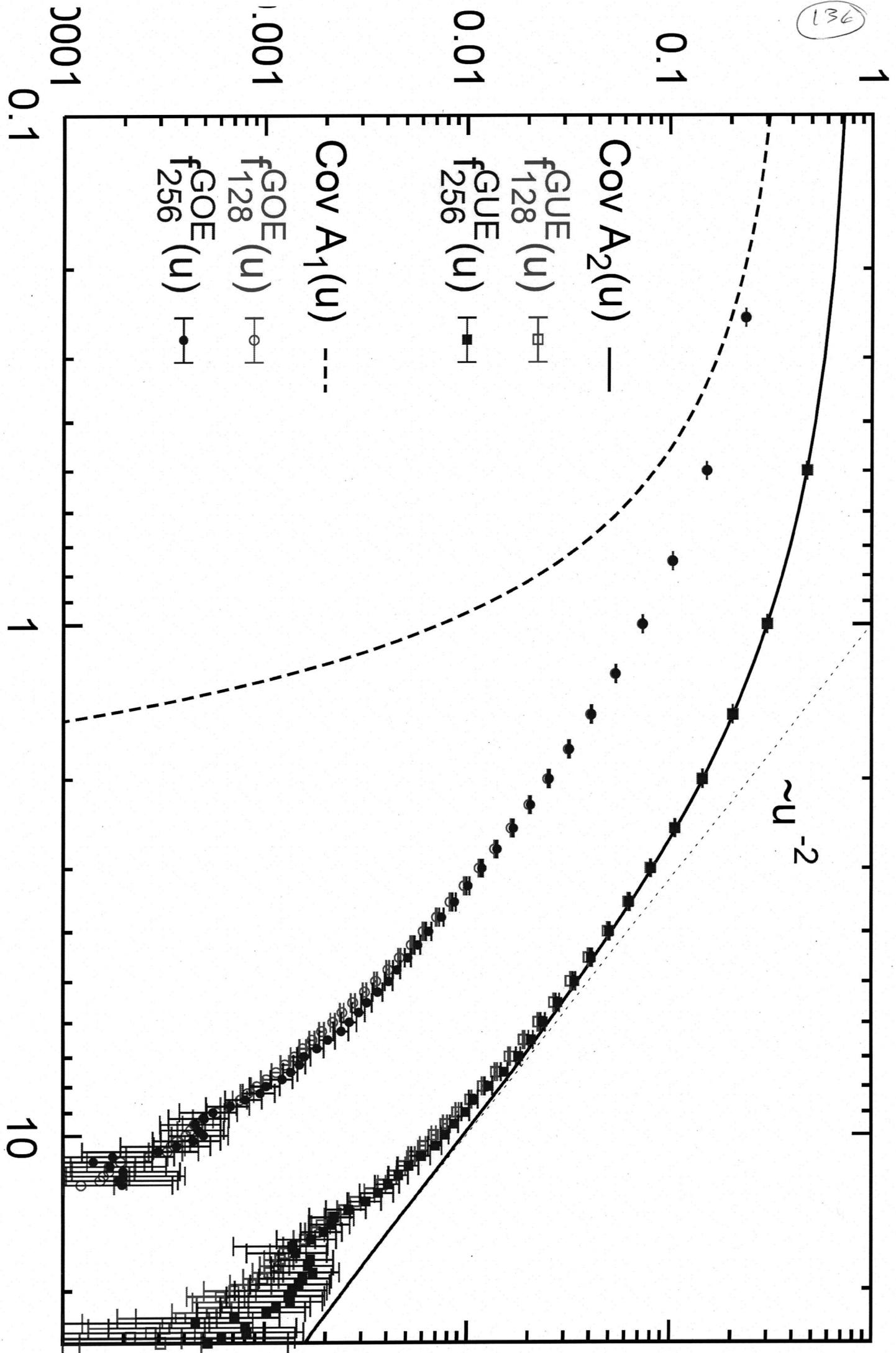
Remark: The ^{real} analogue of GUE Dyson's Brownian Motion leads to a process for the largest eigenvalue of a GUE matrix. This is however different from the R process.

See page 134









3.6.5) Slow-decomolutions

- Above we have obtained a result for the joint distributions at fixed time t and different particles. Here we want to show how, using a soft probabilistic argument, we can extend this result to cover for example the case of a fixed particle and different times.

- Consider ASEP ($p > q$). Denote by $H^{\text{step}}(x,t)$ the height function obtained using as initial condition $\eta(x,t=0)$.

Thm 20 (Slow decomposition).

Consider fixed $u, v \in \mathbb{R}$ and assume $\exists \ell, \tilde{\ell} \geq 0$, $\nu \in [0,1]$ and distributions D, \tilde{D} s.t:

$$\textcircled{a} \quad \frac{H(vt, t) - t\ell}{t^{1/3}} \xrightarrow[t \rightarrow \infty]{D} D.$$

$$\textcircled{b} \quad \frac{H(vt+ut^\alpha, t+t^\alpha) - t\ell - t^\alpha \tilde{\ell}}{t^{1/3}} \xrightarrow[t \rightarrow \infty]{D} \tilde{D}.$$

$$\textcircled{c} \quad \frac{H^{\text{step}}(vt, t) - t \cdot \tilde{\ell}}{t^{1/3}} \xrightarrow[\mathcal{D}]{t \rightarrow \infty} \tilde{D}.$$

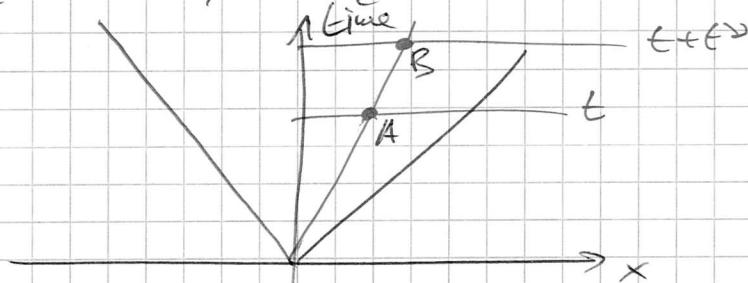
Then, $\forall \varepsilon > 0$,

$$\lim_{t \rightarrow \infty} \mathbb{P}(|H(vt+ut^\alpha, t+t^\alpha) - H(vt, t) - t^\alpha \tilde{\ell}| > \varepsilon) = 0.$$

Two examples: ① TASEP, Step I.c.

$v \in (-1, 1)$; D, \tilde{D} : GUE Tracy-Widom dist.
 $u = v$.

$$\ell = \frac{1}{2}(1+v^2), \tilde{\ell} = \frac{1}{2}(1+u^2).$$

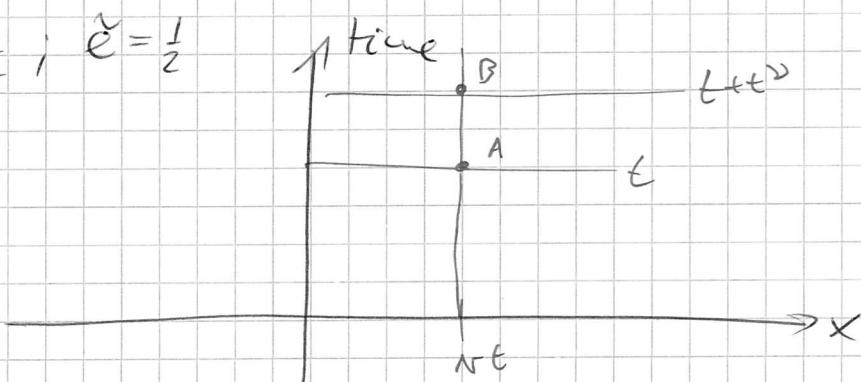


Thm 20 \Rightarrow fluctuation in A and B are identical up to $O(t^{1/3})$.

② TASEP, flat I.c.

$v \in \mathbb{R}, u=0$; D : Foe; \tilde{D} : Foe

$$\ell = \frac{1}{2}, \tilde{\ell} = \frac{1}{2}$$



To prove Thm 20 we use this elementary probability lemma.

Lemma 21: let $(X_n)_{n \geq 1}, (\tilde{X}_n)_{n \geq 1}$ random variables with X_n, \tilde{X}_n defined on the same prob. space $\mathcal{V}_{n \geq 1}$. Assume that $X_n \xrightarrow{\text{P}} \tilde{X}_n$. Then:

a) $(X_n \xrightarrow{\text{P}} D, \tilde{X}_n \xrightarrow{\text{P}} D)$ implies $(X_n - \tilde{X}_n \xrightarrow{\text{P}} 0)$

b) $(\tilde{X}_n \xrightarrow{\text{P}} D, X_n - \tilde{X}_n \xrightarrow{\text{P}} 0) \Rightarrow (X_n \xrightarrow{\text{P}} D)$.

Proof of Thm 20:

First consider the cement

$$I(x, t) := \frac{H(x, t) - x}{2}.$$

Then, $I(vt + ut^2, t + t^2) = I(vt, t)$

$$+ \underbrace{I_2(vt, vt + ut^2; t, t + t^2)}$$

(cement between time t and $t + t^2$).

- Consider the coupling obtained by using the same graphical construction but different initial conditions:

$$\begin{cases} \bar{\gamma}_T \text{ with } \bar{\gamma}_0 = y_T \\ \bar{\gamma}_T \text{ with } \bar{\gamma}_0(x) = H(x, vt). \end{cases}$$

By the coupling / graph const. we easily see that: $I_2(vt, vt + ut^2; t, t + t^2)$

is the cement of $\bar{\gamma}_T$ between $[0, t^2]$

is lower than the cement of $\bar{\gamma}_T$ between $[0, t^2]$.

Further, the cement of $\bar{\gamma}_T$ does not depend on $I(vt, t)$ and it is distributed as $I^{\text{step}}(ut^2, t^2)$.

$$\Rightarrow I(vt + ut^2, t + t^2) = I(vt, t) + I^{\text{step}}(ut^2, t^2) + X_t$$

for some random variables $X_t \leq 0$.

In terms of height functions:

$$H(vt+ut^\alpha, t+t^\alpha) = H(vt, t) + H^{\text{step}}(ut^\alpha, t^\alpha) + 2X_t.$$

$$\begin{aligned} \text{Define: } X_2(\epsilon) &:= \frac{H(vt+\epsilon t^\alpha, t+t^\alpha) - t\epsilon - t^\alpha \bar{e}}{\epsilon^{1/3}} \\ X_1(\epsilon) &:= \frac{H(vt, \epsilon) - t\epsilon}{\epsilon^{1/3}} \\ X_3(\epsilon) &:= \frac{H^{\text{step}}(ut^\alpha, t^\alpha) - t^\alpha \bar{e}}{\epsilon^{2/3}} \end{aligned}$$

$$\text{By assumption, } X_2(\epsilon) = X_1(\epsilon) + X_3(\epsilon). \frac{\epsilon^{2/3}}{\epsilon^{1/3}} + \frac{2X_t}{\epsilon^{1/3}}.$$

By Lemma 21@, $\frac{2X_t}{\epsilon^{1/3}} \xrightarrow{P} 0$; Since $\underline{\omega} < 1$,

also $X_2(\epsilon) - X_1(\epsilon) \xrightarrow{P} 0$, which is what we had to prove #

Now we apply this to obtain:

Thm 22: Consider TASEP with step I.C.

then, if fixed $\tilde{u}_1, \dots, \tilde{u}_n \in \mathbb{R}$,

$$\begin{aligned} \lim_{n \rightarrow \infty} P \left(\bigcap_{k=1}^n \{ S_{X_k}(4n + 4\tilde{u}_k(2n)^{2/3}) \geq 2\tilde{u}_k(2n)^{2/3} + \tilde{u}_k^2(2n)^{1/3} - s_k(2n)^{1/3} \} \right) \\ = P \left(\bigcap_{k=1}^n \{ \forall t \leq \tilde{u}_k \} \{ f_2(\tilde{u}_k) \leq s_k \} \right). \end{aligned}$$

Proof.: We have

$$\left\{ X_{\left[\frac{t}{4} + u \left(\frac{t}{2} \right)^{\frac{2}{13}} \right]} \geq -24 \left(\frac{t}{2} \right)^{\frac{2}{13}} + u^2 \left(\frac{t}{2} \right)^{\frac{1}{13}} - s \left(\frac{t}{2} \right)^{\frac{1}{13}} \right\}$$

$$\underset{\epsilon \gg 1}{\approx} \left\{ H_2(u) \leq s \right\} \quad \text{XXXX}$$

$$\text{and } \left\{ X_u (t \geq m-u) \right\} = \left\{ H(u-u, t) \geq u+u \right\}. \quad \text{X}$$

Denote $n=4t$ in the statement of the Thm. Then,

$$\left\{ X_{\frac{t}{4}} (t+4\tilde{u} \left(\frac{t}{2} \right)^{\frac{2}{13}}) \geq 2\tilde{u} \left(\frac{t}{2} \right)^{\frac{2}{13}} + \tilde{u}^2 \left(\frac{t}{2} \right)^{\frac{1}{13}} - s \left(\frac{t}{2} \right)^{\frac{1}{13}} \right\}$$

$$\stackrel{(*)}{=} \left\{ H \left(2\tilde{u} \left(\frac{t}{2} \right)^{\frac{2}{13}} + \tilde{u}^2 \left(\frac{t}{2} \right)^{\frac{1}{13}} - s \left(\frac{t}{2} \right)^{\frac{1}{13}}; t+4\tilde{u} \left(\frac{t}{2} \right)^{\frac{2}{13}} \right) \right. \\ \left. \geq \frac{t}{2} + 2\tilde{u} \left(\frac{t}{2} \right)^{\frac{2}{13}} + \tilde{u}^2 \left(\frac{t}{2} \right)^{\frac{1}{13}} - s \left(\frac{t}{2} \right)^{\frac{1}{13}} \right\}$$

$$= \left\{ H(n-t+u\epsilon^2, t+\epsilon^2) \geq \ell t + \tilde{\ell} \epsilon^2 - s \left(\frac{t}{2} \right)^{\frac{1}{13}} \right\} \quad \text{X}$$

$$\text{if we choose: } \begin{cases} \cdot \epsilon^2 \equiv 4\tilde{u} \left(\frac{t}{2} \right)^{\frac{2}{13}} \\ \cdot u = 0 \end{cases}$$

$$\begin{cases} \cdot u = 0 \rightarrow \tilde{\ell} = 12 \rightarrow \tilde{\ell} \epsilon^2 = 2\tilde{u} \left(\frac{t}{2} \right)^{\frac{2}{13}} \\ \cdot n-t = 2\tilde{u} \left(\frac{t}{2} \right)^{\frac{2}{13}} + \tilde{u}^2 \left(\frac{t}{2} \right)^{\frac{1}{13}} - s \left(\frac{t}{2} \right)^{\frac{1}{13}} \\ \quad \hookrightarrow \ell = \frac{t}{2}(1+n^2) \Rightarrow \ell t = \frac{t}{2} + \tilde{u}^2 \left(\frac{t}{2} \right)^{\frac{1}{13}} + o(1) \\ \cdot s = 5. \end{cases}$$

By slow-dec (Thm 20),

$$\underset{\epsilon \gg 1}{\approx} \left\{ H(n-t, t) \geq \ell t - s \left(\frac{t}{2} \right)^{\frac{1}{13}} \right\}$$

$$= \left\{ X \left[\frac{t}{z} - \tilde{u} \left(\frac{t}{z} \right) \right]^{2/3} \right\} \left(t \right) \geq 2 \tilde{u} \left(\frac{t}{z} \right)^{2/3} + \tilde{u}^2 \left(\frac{t}{z} \right)^{1/3} - s \left(\frac{t}{z} \right)^{1/3}$$

$\xrightarrow[t \gg z]{\approx}$ $\{ A_2(\tilde{u}) \leq s \} \stackrel{D}{=} \{ A_2(\tilde{u}) \leq s \}.$

by ~~xxx~~

Writing the joint distributions and letting $t \rightarrow \infty$
 we obtain the result easily. $\#$

Remark: What it is not yet known is for example the process along the space-time lines where slow-deceleration holds.