

1) Random Matrices

1.1) Introduction.

. Random matrices have been introduced in statistics by Wishart in the study of the covariance matrix in 1928 and later by physicists (Wigner '55, Dyson, Mather,...) to model the energy spectrum of heavy nuclei.

. As nuclei are described by Quantum Mechanics, its spectrum consists in the eigenvalues of an Hamiltonian H , i.e., $\alpha \lambda \in \mathbb{R}$ s.t. $\exists \psi \in L^2$ with $H\psi = \lambda\psi$.

. For heavy nuclei, H is not precisely known and even if it was, to get its spectrum analytically was (is) out of reach.

. They observed experimentally that certain properties like the eigenvalues' spacing statistics did not depend on the chosen nucleus, i.e., there was some universal behavior.

. The idea was then to replace a real Hamiltonian of heavy nuclei by a random Hamiltonian with a large number of bound states
 \Rightarrow approximation by a large random matrix.

. Depending on the intrinsic symmetries of the system (e.g., time reversal or rotation invariance), physicists naturally divided the class of matrices

(2)

that they introduced as:

- (a) real symmetric matrices (for time-reversal with rotation invariance or integer magnetic momenta),
- (b) real quaternionic matrices (for time-reversal with half-integer magnetic momenta),
- (c) complex hermitian matrices (for non time-reversal systems, like when an external magnetic momentum is present).

In this lecture we will consider mainly (c) and (a).

1.2) Wigner matrices

Def 1) (Real Wigner matrices)

• let $\{Z_{ij}\}_{\substack{i,j \geq 1}}^{\infty}$ and $\{Y_i\}_{i \geq 1}^{\infty}$ two families of (real) iid random variables with mean zero and $E(Z_{1,2}^2) = 1$.

(a) The (symmetric) $N \times N$ matrix $H^{(w)}$ with entries

$$H_{ij}^{(w)} = H_{ji}^{(w)} = \begin{cases} \frac{1}{\sqrt{N}} \cdot Z_{ij}, & \text{if } i < j, \\ \frac{1}{\sqrt{N}} \cdot Y_i, & \text{if } i = j, \end{cases}$$

is called (real) Wigner Matrix.

(b) If, further, $Z_{ij} \sim \mathcal{U}(0, 1)$ and $Y_i \sim \mathcal{N}(0, 1)$, then $H^{(w)}$ is a GOE random matrix.

Rem.: GOE means Gaussian Orthogonal Ensemble and the reason of the name is that the probability distribution on $H^{(w)}$ is of Gaussian form and it is invariant under orthogonal transformations,
See next Lemma.

(3)

Lemma 2) The probability measure \mathbb{P} on GOE random matrices is given by

$$\mathbb{P}(H \in dH) = \text{const. } e^{-\frac{N \text{Tr}(H^2)}{4}} dH,$$

where $dH = \prod_{1 \leq i < j \leq N} dH_{ij}$ and const is a

$$\text{normalisation constant} \left(= \left(\frac{N}{2\pi} \right)^{\frac{N(N+1)/4}{2}} \cdot \frac{1}{Z^{N/2}} \right).$$

Proof.:

$$\mathbb{P}(H \in dH) = \prod_{\substack{\text{indep. of } 1 \leq i < j \leq N \\ \text{entries}}} \mathbb{P}(H_{ij} \in dH_{ij})$$

$\prod_{1 \leq i < j \leq N}$

$$= \prod_{1 \leq i < j \leq N} \frac{e^{-\frac{H_{ij}^2}{2/N}}}{\sqrt{2\pi/N}} \prod_{i=1}^N \frac{e^{-\frac{H_{ii}^2}{2 \cdot 2/N}}}{\sqrt{2\pi \cdot 2/N}}$$

$$= \underbrace{\left(\frac{N}{2\pi} \right)^{\frac{N(N+1)/4}{2}}}_{H_{ij} \neq H_{ji}} \underbrace{\left(\prod_{1 \leq i < j \leq N} dH_{ij} \right) \prod_{i=1}^N}_{i \neq j} \underbrace{e^{-\frac{H_{ij} H_{ji}}{4/N}}}_{\#}$$

$$= \text{const} \left(\prod_{1 \leq i < j \leq N} dH_{ij} \right) \cdot e^{-\frac{N \text{Tr}(H^2)}{4}}$$

Remark: One can actually prove (see e.g. Mehta's RM book) that for real symmetric matrices the only measures satisfying: \rightarrow independent entries \rightarrow invariance under the rotations (i.e., the group $O(N)$, are of the form

$$\exp(-a \text{Tr}(H^2) + b \text{Tr}(H) + c), \quad a > 0,$$

i.e., up to a global shift of the spectrum the (and scaling)
only proba. measures with these 2 requirements is
the one of Lemma 2.

In a similar way one defines complex Wigner matrices (4) and GUE (U=Unitary) matrices.

Def. 3) (Complex Wigner Matrices)

- Consider two families of iid random variables, $\{Z_{ij}\}_{j \geq 1}$ and $\{Y_i\}_{i \geq 1}$. real-valued, with mean zero, $\mathbb{E}(Z_{1j}^2) = 0$, $\mathbb{E}(|Z_{1j}|^2) = 1$.

(a) The (hermitian) $N \times N$ matrix $H^{(w)}$ with entries

$$\overline{H_{ij}^{(w)}} = H_{ji}^{(w)} = \begin{cases} \frac{1}{\sqrt{N}} Z_{ij}, & \text{if } i < j \\ \frac{1}{\sqrt{N}} Y_i, & \text{if } i = j \end{cases}$$

is called hermitian Wigner matrix.

- (b) If, further, $\operatorname{Re}(Z_{ij}) \sim N(0, 1/2)$, $\operatorname{Im}(Z_{ij}) \sim N(0, 1/2)$, $Y_i \sim N(0, 1)$, then $H^{(w)}$ is a GUE random matrix ($\operatorname{Re} Z_{ij}$ indep from $\operatorname{Im} Z_{ij}$).

Rem.: GUE stands for Gaussian Unitary Ensemble.

Lemma 4: The probability measure \mathbb{P} on GUE

random matrices is given by

$$\mathbb{P}(H \in dH) = \text{const. } e^{-\frac{N}{2} \operatorname{Tr}(H^2)} dH,$$

$$\text{where } dH = \prod_{i=1}^N d\operatorname{Re} H_{ii} \prod_{1 \leq i < j \leq N} d\operatorname{Re} H_{ij} d\operatorname{Im} H_{ij}$$

and const is a normalisation constant.

Proof.: The proof is almost identical to the one of Lemma 2:

$$\begin{aligned} \mathbb{P}(H \in dH) &= \prod_{1 \leq i < j \leq N} \text{const. } e^{-\frac{(\operatorname{Re} H_{ij})^2 + (\operatorname{Im} H_{ij})^2}{2N}} \prod_{i=1}^N e^{-\frac{H_{ii}^2}{2N}} dH \\ &= \text{const. } e^{-\frac{N}{2} \operatorname{Tr}(H^2)} dH. \end{aligned}$$

1.3) Wigner semicircle law

- One first natural question is to know the spectral properties of Wigner matrices.

The first result, due to Wigner, is a kind of law of large number for the eigenvalues density.

Def. 5) Denote by $\lambda_1^N \leq \lambda_2^N \leq \dots \leq \lambda_N^N$ the N (real) eigenvalues of a ^(random) matrix. Then we define by

$$L_N := \frac{1}{N} \sum_{i=1}^N \delta_{\lambda_i^N}$$

the empirical distribution of the eigenvalues.

Def. 6) We define the semicircle distribution as the probability distribution $\mu_{sc}^T(x) dx$, $x \in \mathbb{R}$ with density $T(x) = \frac{1}{2\pi} \sqrt{4-x^2} \mathbb{I}_{|x| \leq 2}$.

Thm. 7) (Wigner semicircle law). Assume that the y_{ij}, z_{ij} have all finite moments.

For a Wigner matrix (both real and complex) the empirical distribution L_N converges weakly, in probability, to the semicircle distribution.

Explicitly: $\forall f \in C_c(\mathbb{R})$, $\forall \varepsilon > 0$,

$$\lim_{N \rightarrow \infty} \mathbb{P}(|L_N(f) - \mu_{sc}^T(f)| > \varepsilon) = 0,$$

where $L_N(f) = \frac{1}{N} \sum_{i=1}^N f(\lambda_i^N)$ and $\mu_{sc}^T(f) = \int_{-\infty}^{\infty} dx T(x) f(x)$.



Remarks: Wigner proved it using a method of moments for which the finite moment assumptions were needed. However, one can nowadays prove it assuming three moments only.

Further, one can prove the convergence also as an almost sure convergence (see Biblio notes 2.7 in the book by Anderson, Guionnet, and Zeitouni "An Introduction to Random Matrices", currently available at Zeitouni's homepage).

To the proof of Wigner's result we come back in the next section after having introduced some useful mathematical tools.

Before that let us just note that a similar result holds for other classes of random matrices. Here is one example:

Def. 8) (Wishart matrices / sample covariance matrices)

- Consider M independent identically distributed mean zero samples $\vec{Y}_1, \dots, \vec{Y}_M$, with \vec{Y}_i is a $N \times 1$ vector.
- Construct $Y^{(N)} = (\vec{Y}_1, \dots, \vec{Y}_M)$ a $N \times M$ matrix. Then one defines the $N \times N$ Wishart matrix $W^{(N)} := \frac{1}{N} \cdot Y^{(N)} \cdot (Y^{(N)})^T$.

Theorem) Assume that all moments of Y_i 's are finite.

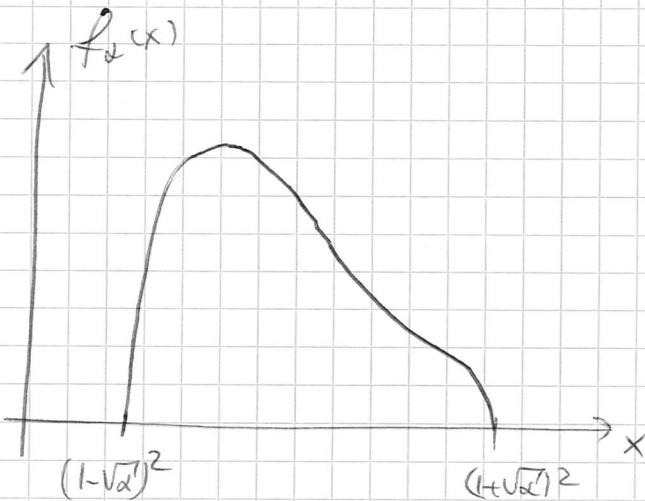
Then, if $\alpha := \lim_{N \rightarrow \infty} \frac{M}{N} \in [1, \infty)$, then

$$L_N \Rightarrow f_\alpha(x) dx, \text{ where } f_\alpha(x) = \frac{\sqrt{(x-a)(b-x)}}{2\pi x} \prod_{[a \leq x \leq b]}$$

with $a = (1 - \sqrt{\alpha})^2$, $b = (1 + \sqrt{\alpha})^2$.

$f_\alpha(x) dx$ is known as

Marchenko-Pastur distribution.



(7)

1.4) Stieltjes' transform of a probability measure.

We now introduce the Stieltjes' transform, whose usefulness in RMT is analogue to the use of characteristic functions in proving CLT.

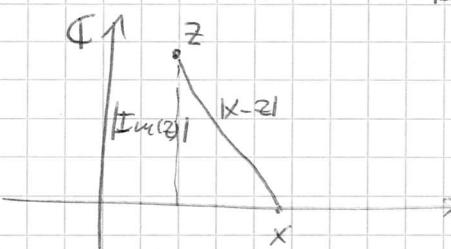
Def. 10) [let $\mu \in \mathcal{M}_1(\mathbb{R})$ a probability measure on \mathbb{R} .
Its Stieltjes' transform is defined by

$$S_\mu(z) := \int_{\mathbb{R}} \frac{1}{x-z} \mu(dx), \quad z \in \mathbb{C} \setminus \mathbb{R}$$

Rem.: ① For $z \in \mathbb{C} \setminus \mathbb{R}$, $\operatorname{Re}\left(\frac{1}{x-z}\right)$ and $\operatorname{Im}\left(\frac{1}{x-z}\right)$ are continuous and bounded functions of $x \in \mathbb{R}$.

② $|S_\mu(z)| \leq \frac{1}{|\operatorname{Im}(z)|}.$

This simply follows from $\frac{1}{|x-z|} \leq \frac{1}{|\operatorname{Im}(z)|}$.



③ S_μ is analytic in $\mathbb{C} \setminus \operatorname{support}(\mu)$.

(d) If a random variable X is distributed according to μ , then

$$S_\mu(z) = \mathbb{E}\left(\frac{1}{X-z}\right).$$

The main property of Stieltjes' transform is that one can recover the measure μ from it.

Prop 11) If $a < b$ with neither endpoints on an atom of μ ,

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \frac{1}{\pi} \int_a^b \frac{S_\mu(\lambda+i\varepsilon) - S_\mu(\lambda-i\varepsilon)}{2i} d\lambda \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{\pi} \int_a^b \text{Im}(S_\mu(\lambda+i\varepsilon)) d\lambda \stackrel{\oplus}{=} \mu((a,b)). \end{aligned}$$

Rem: Generically, replace $\mu((a,b))$ with $\mu((a,b)) + \frac{1}{\varepsilon}\mu(\{a\}) + \frac{1}{\varepsilon}\mu(\{b\})$.

Proof: The first equality just follows from

$$\begin{aligned} & \frac{1}{2i} \left(\frac{1}{x-(\lambda+i\varepsilon)} - \frac{1}{x-(\lambda-i\varepsilon)} \right) = \frac{1}{2i} \frac{x-\lambda+i\varepsilon-x+\lambda+i\varepsilon}{(x-\lambda)^2+\varepsilon^2} = \frac{a}{(x-\lambda)^2+\varepsilon^2} \\ &= \text{Im} \left(\frac{1}{x-(\lambda+i\varepsilon)} \right). \end{aligned}$$

let $X \sim \mu$ and $C_\varepsilon \sim \text{Cauchy}(\varepsilon)$, X and C_ε independent. ($\mathbb{P}(C_\varepsilon dx) = \frac{\varepsilon}{\pi(\varepsilon^2+x^2)} dx \xrightarrow[\varepsilon \rightarrow 0]{} \delta_0$)

$$\Rightarrow \text{Im}\left(\frac{1}{\pi} S_\mu(\lambda+i\varepsilon)\right) = \frac{1}{\pi} \int_{\mathbb{R}} \mu(dx) \frac{\varepsilon}{(x-\lambda)^2+\varepsilon^2}$$

is the density w.r.t. Lebesgue of $X+C_\varepsilon$ at position $\lambda \in \mathbb{R}$.

Then, \oplus follows from weak convergence of $X+C_\varepsilon$ to X as $\varepsilon \rightarrow 0$.

The next issue regards whether convergence of Stieltjes' transform imply convergence of the measures.

Prop. 12) let $\mu_n \in M_1(\mathbb{R})$ be a sequence of probability measures.

(a) If μ_n converges weakly to a probability measure μ , then $S_{\mu_n}(z)$ converges to $S_\mu(z)$ for each $z \in \mathbb{C} \setminus \mathbb{R}$.

(b) If μ_n are random and $\forall z \in \mathbb{C} \setminus \mathbb{R}$, $S_{\mu_n}(z)$ converges in probability to a deterministic limit $S(z)$ that is the Stieltjes' transform of a probability measure μ , then μ_n converges weakly in probability to μ .

Proof: (a) If $\mu_n \Rightarrow \mu \Rightarrow f$ bounded continuous functions,

$$\int f d\mu_n \rightarrow \int f d\mu.$$

Since for fixed $z \in \mathbb{C} \setminus \mathbb{R}$, $x \mapsto \frac{1}{x-z}$ is bounded and continuous, $S_{\mu_n}(z) \rightarrow S_\mu(z)$.

(b) Let n_k be a subsequence on which μ_{n_k} converges vaguely (i.e., $\int f d\mu_{n_k} \rightarrow \int f d\mu$ $\forall f$ continuous and vanishing at ∞) to a (sub-)probability measure μ (i.e., $\mu(\mathbb{R}) \leq 1$).

Since $x \mapsto \frac{1}{x-z}$ is continuous and vanishes at ∞ ,

$$S_{\mu_k}(z) \rightarrow S_\mu(z), \quad \forall z \in \mathbb{C} \setminus \mathbb{R}.$$

$\Rightarrow S_\mu = S'$. By uniqueness of Stieltjes transform, all subsequential limits are the same $\Rightarrow \mu_n \rightarrow \mu$.

More details: $(z_i)_{i \in \mathbb{N}} \rightarrow z_0 \in \mathbb{C} \setminus R$, $z_i \neq z_0$.

For $\nu_1, \nu_2 \in M_1(\mathbb{R})$,

$$S(\nu_1, \nu_2) = \sum_{i \geq 1} \frac{1}{2^i} |S_{\nu_1}(z_i) - S_{\nu_2}(z_i)|$$

Then, $S(\nu_n, \nu) \rightarrow 0 \Rightarrow \nu_n \xrightarrow{\text{weakly}} \nu$.

Indeed, \exists subseq. s.t. $\nu_n \xrightarrow{\text{weakly}} \nu$.
 $\Rightarrow S_{\nu_n}(z_i) \rightarrow S_\nu(z_i)$ f.c.

But $S_{\nu_n}(z_i)$ is uniformly b.d. ($i \in \mathbb{N}, i$)

and $S(\nu_n, \nu) \rightarrow 0 \Rightarrow S_{\nu_n}(z_i) \rightarrow S_\nu(z_i)$.

$$\Rightarrow S_\nu(z) = S_\nu(z) \quad \forall z = z_i.$$

\Rightarrow by analyticity, $S_\nu(z) = S_\theta(z)$, $\forall z \in \mathbb{C} \setminus R$.

By inversion formula, $\theta = \theta \Rightarrow \theta \in M_1(\mathbb{R})$ f.c.

and so $\nu_n \xrightarrow{\text{weakly}} \theta = \nu$.

By last assumption, $S(\mu_n, \mu) \rightarrow 0$ in Prob.,

$\Rightarrow \mu_n \xrightarrow{\text{weakly}} \mu$, in proba.

We want to analyze $L_N = \frac{1}{N} \sum_{i=1}^N S_{\lambda_i}$.

We will study the convergence of $S_{L_N}(z) = \frac{1}{N} \sum_{i=1}^N \frac{z}{\lambda_i - z}$.

Some notations: For a symmetric matrix X_N , with ev. λ_i^N 's,

i.e., $X_N = O \cdot \text{diag}(\lambda_1^N, \dots, \lambda_N^N) \cdot O^T$, for some $O \in O(N)$,

we define the matrix

$$\mathcal{F}(X_N) := O \cdot \text{diag}(\mathcal{F}(\lambda_1^N), \dots, \mathcal{F}(\lambda_N^N)) \cdot O^T.$$

In the particular case of $\mathcal{F}(x) = (x-z)^{-1}$ (which is Lipschitz $\forall z \in \mathbb{C} \setminus \mathbb{R}$), denote

$$R_{X^N}(z) := (X^N - z\mathbb{I})^{-1}.$$

Then, $\boxed{S_{L_N}(z) = \frac{1}{N} \cdot \text{Tr}(R_{X^N}(z))}$

Three preliminary results:

Lemma 13) It holds: $R_{X^N}(z) = z^{-1} (X^N \cdot R_{X^N}(z) - \mathbb{I})$, $z \in \mathbb{C} \setminus \mathbb{R}$.

Proof: easy computation. E.g., multiply both sides by $X^N - z\mathbb{I}$ to the right. $\#$

Lemma 14) If $\Im \sim \mathcal{N}(0, \sigma^2)$, $\mathcal{F} \in C^1(\mathbb{R})$ with $\mathcal{F}, \mathcal{F}'$ polynomial growth, then

$$\mathbb{E}(\Im \mathcal{F}(\Im)) = \mathbb{E}(\mathcal{F}'(\Im)) \mathbb{E}(\Im^2).$$

Proof: a straightforward computation (integrate by parts). $\#$

Prop 15) For Wigner matrices, if \mathcal{F} is Lipschitz,

then $\forall \delta > 0$, $\exists c = c(\mathcal{F}) < \infty$ s.t.

$$c \cdot N^2 S^2$$

$$\mathbb{P}(|\text{Tr}(\mathcal{F}(X^N)) - \mathbb{E}(\text{Tr}(\mathcal{F}(X^N)))| \geq \delta \cdot N) \leq 2e^{-c \cdot N^2 S^2}.$$

Prop 15 is a concentration inequality result and we will come back to it after the proof of the Wigner semicircle law.

Proof of Thm 7 (=Wigner semicircle law) for GOE.

We prove it for the case of GOE.

The general case needs some slightly different approaches as we will not be able to use Lemma 16.

For GUE the proof is similar.

By Prop 15 we need only to see that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \mathbb{E}(\text{Tr}(R_{X^N}(z))) = S_{\text{Lsc}}(z).$$

$$\begin{aligned} \overline{S}_{L_N}(z) &:= \frac{1}{N} \mathbb{E}(\text{Tr}(R_{X^N}(z))) = -\frac{1}{2} + \frac{1}{2N} \cdot \mathbb{E}(\text{Tr}(X^N \cdot R_{X^N}(z))) \\ &\stackrel{\text{Lemma 13}}{=} -\frac{1}{2} + \frac{1}{2N} \cdot \sum_{i,k=1}^N \mathbb{E}\left((X^N)_{ik} \cdot (R_{X^N}(z))_{k,i}\right) \\ &= -\frac{1}{2} + \frac{1}{2N} \cdot \sum_{i,k=1}^N \mathbb{E}\left(\frac{\partial}{\partial X_{ik}^N} (R_{X^N}(z))_{k,i}\right) \mathbb{E}(X_{ik}^N)^2 \quad (*) \end{aligned}$$

Introduce the matrix:

$$\Delta_N^{i,j}(i,e) = \begin{cases} 1 & , (i,k) = (j,e) \text{ or } (i,k) = (e,j) \\ 0 & , \text{ otherwise} \end{cases}$$

For X^N symmetric matrices,

$$\frac{\partial}{\partial X_{ik}^N} R_{X^N}(z) = -R_{X^N}(z) \Delta_N^{i,j} R_{X^N}(z).$$

$$\text{Since: } \frac{\partial}{\partial X_{ik}^N} [(X^N - z\mathbb{I})^{-1} (X^N - z\mathbb{I})] = 0$$

$$\mathbb{E} \left[\frac{\partial}{\partial X_{ik}^N} (X^N - z\mathbb{I})^{-1} \right] (X^N - z\mathbb{I}) + (X^N - z\mathbb{I})^{-1} \frac{\partial}{\partial X_{ik}^N} (X^N - z\mathbb{I}).$$

(12)

$$\begin{aligned}
\Rightarrow \Theta &= -\frac{1}{z} - \frac{1}{2N} \cdot \sum_{i,k=1}^N \mathbb{E} \left(\left(R_{xN}(z) \Delta_N^{i,k} R_{xN}(z) \right)_{k,i} \right) \cdot \mathbb{E} \left((X_{ik}^N)^2 \right) \\
&= -\frac{1}{z} - \frac{1}{2N^2} \cdot \sum_{\substack{i,k=1 \\ i \neq k}}^N \mathbb{E} \left((R_{xN}^{(z)})_{k,i}^2 + (R_{xN}^{(z)})_{kk} (R_{xN}^{(z)})_{ii} \right) \\
&\quad - \frac{2}{2N^2} \sum_{i=1}^N \mathbb{E} \left((R_{xN}^{(z)})_{ii}^2 \right) \\
&= -\frac{1}{z} - \frac{1}{2N^2} \cdot \sum_{i,k=1}^N \left[\mathbb{E} \left((R_{xN}^{(z)})_{ki}^2 \right) + \mathbb{E} \left((R_{xN}^{(z)})_{kk} \cdot (R_{xN}^{(z)})_{ii} \right) \right] \\
&= -\frac{1}{z} - \frac{1}{2N} \cdot \underbrace{\mathbb{E} \left(\frac{1}{N} \text{Tr} \left((R_{xN}(z))^2 \right) \right)}_{\substack{=\frac{1}{N} \sum_{i=1}^N \frac{1}{(X_i^N - z)^2} \leq \frac{1}{(\text{Tr}(z))^2} \\ \rightarrow 0 \text{ as } N \rightarrow \infty}} - \frac{1}{z} \cdot \underbrace{\mathbb{E} \left[\text{Tr} \left(\frac{1}{N} R_{xN}(z) \right) \right]^2}_{=\frac{1}{N} \sum_{i=1}^N \frac{1}{X_i^N - z}}
\end{aligned}$$

Prop. 15 implies that

$$\text{Var} \left(\text{Tr} \left(\frac{1}{N} R_{xN}(z) \right) \right) = \mathbb{E} \left[\left(\text{Tr} \left(\frac{1}{N} R_{xN}(z) \right) \right)^2 \right] - \left[\mathbb{E} \left(\text{Tr} \left(\frac{1}{N} R_{xN}(z) \right) \right) \right]^2 \xrightarrow[N \rightarrow \infty]{} 0.$$

\Rightarrow We have obtained that

$$\bar{S}_N(z) = -\frac{1}{z} - \frac{1}{z} (\bar{S}_N(z))^2 + \varepsilon_N(z)$$

with $\varepsilon_N(z) \rightarrow 0$ as $N \rightarrow \infty$.

Therefore any limit point $s(z)$ of $\bar{S}_N(z)$

satisfies :

$$s(z) = -\frac{1}{z} - \frac{1}{z} (s(z))^2$$

$$\Leftrightarrow s(z)(z+s(z))_+ + 1 = 0. \quad \text{(K*)}$$

For $z \in \mathbb{C}_+ = \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$, $\text{Im}(z) > 0$

(13)

and for $z \in \mathbb{C}_- = \{z \in \mathbb{C} \mid \text{Im}(z) < 0\}$, $\text{Im}(z) < 0$.

The solution of $\star\star$ is:

$$S(z) = \frac{\sqrt{z^2 - 4} - z}{2} \quad (\text{with } \sqrt{\cdot} \text{ branch chosen to satisfy } \text{Re } z \leq 0).$$

Then, by Prop 11 we easily verify that $S(z)$ is the Stieltjes transform of μ_{sc} . #

1.5) On the concentration inequality (Prop 15).

• For a function $G: \mathbb{R}^M \rightarrow \mathbb{R}$,

$$\|G\|_L := \sup_{x \neq y \in \mathbb{R}^M} \frac{|G(x) - G(y)|}{\|x - y\|_2} : \begin{array}{l} \text{the Lipschitz} \\ \text{constant} \\ \text{(if } L < \infty\text{)}. \end{array}$$

Lemma 16) let $g: \mathbb{R}^N \rightarrow \mathbb{R}$ be Lipschitz and X be a $N \times N$ matrix (hermitian).

① $F: X \rightarrow g(\lambda_1(X), \dots, \lambda_N(X))$ is Lipschitz on \mathbb{R}^{N^2} with $\|F\|_L \leq \sqrt{2} \|g\|_L$

② If φ is Lipschitz on \mathbb{R} ,

$\tilde{F}: X \rightarrow \text{Tr}(\varphi(X))$ is Lipschitz on $\mathbb{R}^{N(N+1)/2}$ with $\|\tilde{F}\|_L \leq \sqrt{2N} \|\varphi\|_L$.

The proof is based on this result:

Lemma 17 (See Lemma 2.1.19 in the RM book of Anderson, Guionnet, Zeitouni).

Let $\lambda_1(A) \leq \dots \leq \lambda_N(A)$ and $\lambda_1(B) \leq \dots \leq \lambda_N(B)$ be the e.v. of symmetric (or hermitian) matrices. Then,

$$\sum_{i=1}^N |\lambda_i(A) - \lambda_i(B)|^2 \leq \text{Tr}(A - B)^2.$$

Proof of Lemma(6). For symmetric matrices,

$$\cdot \|X-Y\|_2^2 = \sum_{i,j=1}^N |X_{ij}-Y_{ij}|^2 = \text{Tr}\{(X-Y)^2\}$$

and for hermitian matrices one have

$$\begin{aligned} \cdot \|X-Y\|_2^2 &= \sum_{i=1}^N |X_{ii}-Y_{ii}|^2 + \sum_{i>j} \text{Re}(X_{ij}-Y_{ij})^2 + \sum_{i>j} \text{Im}(X_{ij}-Y_{ij})^2 \\ &\geq \frac{1}{2} \cdot \text{Tr}\{(X-Y)^2\}. \end{aligned}$$

(a) $\frac{|F(X)-F(Y)|}{\|X-Y\|_2} \leq \frac{|g(\lambda(x)) - g(\lambda(y))|}{\|\lambda(x) - \lambda(y)\|_2} \cdot \sqrt{\frac{1}{2} \text{Tr}\{(X-Y)^2\}}$

$\lambda(x) = (\lambda_1(x), \dots, \lambda_N(x))$ $\leq \|g\|_2 \leq \sqrt{2}$

Lemma 17

(b) $\frac{|\tilde{F}(X)-\tilde{F}(Y)|}{\|X-Y\|_2} = \sum_{i=1}^N \frac{|\ell(\lambda_i(x)) - \ell(\lambda_i(y))|}{|\lambda_i(x) - \lambda_i(y)|} \frac{\|\lambda_i(x) - \lambda_i(y)\|_2}{\|X-Y\|_2}$

$\leq \frac{1}{\sqrt{N}} \cdot \frac{\|\lambda(x) - \lambda(y)\|_2}{\|X-Y\|_2} \leq \sqrt{2N} \frac{1}{\sqrt{N}} \leq \sqrt{2N} \frac{1}{\sqrt{N}} \leq \sqrt{2N} \frac{1}{\sqrt{N}} \leq \sqrt{2N} \frac{1}{\sqrt{N}}$

Def. 18) We say that $P \in M_1(\mathbb{R}^n)$ satisfy the

Log-Sobolev inequality (LSI) with constant c if $f \in C^1(\mathbb{R}^n) \cap L^2(P)$,

$$\int f^2 \ln\left(\frac{f^2}{Sf^2 dP}\right) dP \leq 2c \int |f'|^2 dP.$$

A general condition for P to satisfy LSI is the following: Lemma 19: (See Lemma 2.3.2 of RM book)

$V(dx) = \text{const } e^{-V(x)} dx$ with $V(x) - \frac{\|x\|_2^2}{2c}$ convex

\Rightarrow Satisfy LSI with constant c .

Lemma 20 (See Lemma 2.33 of RM book)

- If \mathbb{P} satisfy the LSI, ^{with constant c} and G is a Lipschitz function on \mathbb{R}^M , then $\forall \lambda \in \mathbb{R}$,

$$\mathbb{E}(e^{\lambda(G - \mathbb{E}(G))}) \leq e^{\frac{c}{2}\lambda^2 \|G\|_2^2}$$

$$\Rightarrow \forall s > 0, \mathbb{P}(|G - \mathbb{E}(G)| \geq s) \leq 2e^{-\frac{s^2}{2c\|G\|_2^2}}.$$

Proposition 15 then follows directly from Lemma 18 with $G = \text{Tr}(\mathcal{F}(x^\mu))$ and Lemma 20 with $M = N(N+1)/2$. For more details, see RM book, section 2.3.

1.6) Eigenvalues distributions for GOE and GUE.

Given a ^{Wigner} random matrix we have that at a "global scale" (macroscopic) the randomness is not anymore visible as $L_n \Rightarrow \mu_{sc}$ and the latter is a non-random object.

Now we want to focus on scales where the randomness is still visible. For that purpose we first compute the joint distributions of eigenvalues for GOE and GUE.

Prop. 21: let $\lambda_1, \dots, \lambda_N$ be the eigenvalues of a GOE matrix (i.e., under the measure of Lemma 2). Then,

$$\mathbb{P}(\lambda \in d\lambda) = \text{const. } \prod_{i \in \{1, \dots, N\}} |\lambda_i - \lambda| \prod_{i=1}^N e^{-\frac{N\lambda_i^2}{4}} d\lambda_i.$$

(Here we consider the distribution on unordered eigenvalues).

(16)

Proof: Given a GOE matrix H with e.v. $\lambda = (\lambda_1 \leq \dots \leq \lambda_N)$,

$\exists g \in O(N)$ s.t. $H = g \cdot \mathbf{1} \cdot g^{-1}$, where

$$\mathbf{1}_{ij} = \lambda_i \delta_{ij}, \quad 1 \leq i, j \leq N.$$

. Since H has $M = N(N+1)/2$ indep. entries,

there are $M-N$ "angular" variables in g , call them $\theta_{1,-}, \theta_{2,-}, \dots, \theta_{M-N,-}$.

To compute is the Jacobian of the change of variables $\{\tilde{H}_{ij}, 1 \leq i, j \leq N\} \rightarrow \{\lambda_1, \dots, \lambda_N, \theta_{1,-}, \dots, \theta_{M-N,-}\}$.

We have: $dH = d(g \mathbf{1} g^{-1})$

$$= (dg) \mathbf{1} g^{-1} + g (d\mathbf{1}) g^{-1} + g \mathbf{1} (dg^{-1}).$$

Now, since $d(g g^{-1}) = 0 \Rightarrow (dg^{-1}) = -g^{-1}(dg)g^{-1}$.

$$\Rightarrow dH = g [g'(dg) \mathbf{1} - \mathbf{1} g' dg] g^{-1} + g (d\mathbf{1}) g^{-1}.$$

Therefore $dH = g d\tilde{H} g^{-1}$ with $d\tilde{H} = d\mathbf{1} + \underbrace{[g' dg, \mathbf{1}]}_{:= d\mathcal{L}}.$

The Jacobian $H \rightarrow \tilde{H}$ is one (just a rotation). (Haar measure)

Q.: Jacobian $\tilde{H} \rightarrow (\lambda, \mathcal{L}) = ?$

We have: $d\tilde{H}_{ij} = \delta_{ij} d\lambda_i + \sum_{k=1}^N d\mathcal{L}_{ik} \delta_{ki} \lambda_k - \sum_{k=1}^N \lambda_i \delta_{ik} d\mathcal{L}_{kj}$

$$= \delta_{ij} d\lambda_i + d\mathcal{L}_{ij} (\lambda_j - \lambda_i).$$

$$\Rightarrow J = \left| \det \left(\frac{\partial (\tilde{H}_{1,1}, \dots, \tilde{H}_{N,N}; \tilde{H}_{1,2}, \dots, \tilde{H}_{1,N}, \dots, \tilde{H}_{N,1}, \dots)}{\partial (\lambda_1, \dots, \lambda_N; \mathcal{L}_{1,2}, \dots, \mathcal{L}_{1,N}, \dots, \mathcal{L}_{N-1,N})} \right) \right|$$

(17)

$$= \det \begin{vmatrix} 1 & 0 & & & \\ 0 & 1 & & & \\ & & \ddots & & \\ & & & 1 & 0 \\ 0 & & & & \ddots \end{vmatrix} \cdot \prod_{i=1}^n \lambda_i - \lambda_{i+1}$$

$$= \prod_{1 \leq i < j \leq n} |\lambda_j - \lambda_i|.$$

Therefore, $dH = \prod_{1 \leq i < j \leq n} |\lambda_j - \lambda_i| \prod_{i=1}^n d\lambda_i \cdot d\Omega$ ⊗
(This is the Haar measure on $\mathrm{SO}(N)$).

By Lemma 2,

$$\mathbb{P}(H \in dH) = \text{const. } e^{-\frac{n}{4} \text{Tr}(H^2)} dH$$

$$\stackrel{\text{⊗}}{=} \text{const. } e^{-\frac{n}{4} \cdot \text{Tr}(A^2)} \prod_{1 \leq i < j \leq n} |\lambda_j - \lambda_i| \prod_{i=1}^n d\lambda_i d\Omega$$

After integrating out the "angular" variables in Ω , it results in the statement to be proven. #

Remark: Actually, to make the proof rigorous we also need to verify that the change of variable is differentiable. This is "fortunately" the case up to sets of measure zero. For details see Section 2.5.2 of the RM book.

- A similar computation works for GUE matrices too.
We get :

Prop. 22: let $\lambda_1, \dots, \lambda_N$ be the eigenvalues of a GUE matrix. Then,

$$\mathbb{P}(\lambda \in d\lambda) = \text{const. } \prod_{1 \leq i < j \leq N} (\lambda_i - \lambda_j)^2 \prod_{i=1}^N e^{-\frac{N\lambda_i^2}{2}} d\lambda_i.$$

(here too, we consider the measure on unordered eigenvalues).

- An important consequence of Prop 21 and Prop 22 is that the eigenvalues and eigenvectors are independent.

Let v_1, \dots, v_N denote the eigenvectors with eigenvalues $\lambda_1, \dots, \lambda_N$ of a GOE / GUE matrix, with their first non-zero entry positive w.r.t.

Cor. 23: The collection (v_1, \dots, v_N) is independent of the eigenvalues $(\lambda_1, \dots, \lambda_N)$. Each of the eigenvectors v_1, \dots, v_N is distributed uniformly on

$$S_+^{N-1} = \{x = (x_1, \dots, x_N) \mid x_i \in \mathbb{R}, \|x\|_2 = 1, x_1 > 0\} \text{ for GOE}$$

and

$$S_{\mathbb{C},+}^{N-1} = \{x = (x_1, \dots, x_N) \mid x_i \in \mathbb{R}, x_i \in \mathbb{C}, i \geq 2, \|x\|_2 = 1, x_1 > 0\} \text{ for GUE.}$$

Further, (v_1, \dots, v_N) is distributed like a sample of Haar measure on $O(N)$ (for GOE) or $U(N)$ (for GUE), with each column multiplied by a scalar s.t. all columns belongs to S_+^{N-1} (for GOE) or $S_{\mathbb{C},+}^{N-1}$ (for GUE).

(19)

Proof.: let H be diagonalized as $H = U \Lambda U^*$.

. $T \Lambda T^*$ has the same eigenvalues as Λ
and for GUE/GUE has the same distribution
for any orthogonal/unitary T independent of Λ .

. Choose T uniformly according to the Haar measure and independently of U

$\Rightarrow TU \sim$ Haar distributed and indep. of U

\Rightarrow The ev. and. ev. are independent.

. The rest follows from the fact that if U is Haar

\Rightarrow up to multiplication by a constant s.t. first entry > 0 ,
it is uniformly on $S_+^{N-1} / S_{\alpha,+}^{N-1}$. **

. Remark: $\prod_{1 \leq i < j \leq N} (\lambda_j - \lambda_i) = \det_{1 \leq i, j \leq N} (\lambda_i^{j-1})$ and it

is called Vandermonde determinant.

(To see it, either prove it by recursion or
notice that both sides are polynomials that
vanishes at $\lambda_i = \lambda_j$ for any $i \neq j$ and the
leading prefactor matches).

1.7) Correlation functions for GUE eigenvalues.

. For GUE, we have $P(\lambda \in d\lambda) = P_N(\lambda_1, \dots, \lambda_N) d\lambda_1 \dots d\lambda_N$

$$\text{where } P_N(\lambda_1, \dots, \lambda_N) = \frac{1}{Z_N} \cdot \left(\Delta_N(\lambda) \right)^2 \prod_{i=1}^N e^{-\frac{N}{2} \lambda_i^2}.$$

. We want to determine the probability density of finding
eigenvalues at $\lambda_1, \dots, \lambda_n$ with $\lambda_i \in \mathbb{R}, 1 \leq i \leq n$ (and $n \leq N$ of course),
and denote it by $s^{(n)}(\lambda_1, \dots, \lambda_n)$.

Def. 22: The n -point correlation function, $\mathcal{G}^{(n)}(\lambda_1, \dots, \lambda_n)$, is the probability density of finding an eigenvalue at each of the λ_i , $1 \leq i \leq n$.

Lemma 23: For a symmetric measure on \mathbb{R}^N with density P_N ,

$$\mathcal{G}^{(n)}(\lambda_1, \dots, \lambda_n) = \frac{n!}{(N-n)!} \int_{\mathbb{R}^{N-n}} d\lambda_{n+1} \dots d\lambda_N P_N(\lambda_1, \dots, \lambda_n).$$

△ Normalization.

Proof.: There are $\frac{n!}{n!(N-n)!}$ possibility of choosing

n out of the N eigenvalues to be the ones at $(\lambda_1, \dots, \lambda_n)$ and further $n!$ to decide which one is at which of the λ_i 's $\Rightarrow \frac{n!}{(N-n)!}$.

Each has a contribution given by

$$\int_{\mathbb{R}^{N-n}} d\lambda_{n+1} \dots d\lambda_N P_N(\lambda_1, \dots, \lambda_n) \quad *$$

The next goal is to get a formula for $\mathcal{G}^{(n)}(\lambda_1, \dots, \lambda_n)$ for GUE. For this, orthogonal polynomials plays a central role. We define them for the case of polynomials on \mathbb{R} but formulas can be straightforwardly adapted for the case of \mathbb{Z} by replacing Lebesgue by counting measure on \mathbb{Z} .

Def. 24: Given a weight function $w: \mathbb{R} \rightarrow \mathbb{R}_+$,

the orthogonal polynomials $\{q_k(x), k \geq 0\}$

are defined by the following requirements:

(a) $q_k(x)$ is a polynomial of degree k with

$$q_k(x) = u_k x^k + \dots, \quad u_k > 0,$$

(b) $\forall k, e$, $\langle q_k, q_e \rangle_w := \int_{\mathbb{R}} dx w(x) q_k(x) q_e(x) = S_{k,e}$.

Consider $w(x) := e^{-\frac{N}{2}x^2}$ and let $q_k(x)$ be the orthogonal polynomials with respect $w(x)$.

Lemma 25: For GUE,

$$P_N(\lambda_1, \dots, \lambda_N) = \text{const. } \det \left(\sum_{k=0}^{N-1} q_k(\lambda_i) q_k(\lambda_j) w(\lambda_i) \right)_{1 \leq i, j \leq N}$$

Proof: Clearly: $D_N(\lambda) = \det(\lambda_i^{j-1})_{1 \leq i, j \leq N}$

$$= \text{const. } \det(q_{j-1}(\lambda_i))_{1 \leq i, j \leq N}$$

$$\Rightarrow P_N(\lambda_1, \dots, \lambda_N) = \text{const. } \det(q_{k-1}(\lambda_i)) \det(q_{m-k}(\lambda_i))_{1 \leq i \leq N} \cdot \prod_{i=1}^N w(\lambda_i)$$

$$= \text{const. } \det \left(\sum_{k=1}^N q_{k-1}(\lambda_i) q_{k-1}(\lambda_j) w(\lambda_i) \right)_{1 \leq i, j \leq N}$$

Lemma 26: Denote by

$$K_N(x, y) = \sqrt{w(x)} \sqrt{w(y)} \cdot \sum_{k=0}^{N-1} q_k(x) q_k(y).$$

Then, (a) $\int_{\mathbb{R}} dx K_N(x, x) = 1$ and

$$(b) \int_{\mathbb{R}} dz K_N(x, z) K_N(z, y) = K_N(x, y).$$

Proof.: (a) $\int_{\mathbb{R}} dx w(x) \sum_{k=0}^{N-1} q_k(x) q_k(x) = \sum_{k=0}^{N-1} 1 = N.$

(b) $\int_{\mathbb{R}} dz \sqrt{w(x)} \sqrt{w(y)} \sum_{k, l=0}^{N-1} q_k(x) q_k(z) q_l(z) q_l(y) w(z)$

$$= \sqrt{w(x)} \sqrt{w(y)} \sum_{k, l=0}^{N-1} q_k(x) q_l(y) \int_{\mathbb{R}} dz w(z) q_k(z) q_l(z) = K_N(x, y)$$

$= \delta_{x,y}$

With these two Lemmas we can prove the following:

Prop. 27: For GUE,

$$S^{(n)}(\lambda_1, \dots, \lambda_n) = \det_{1 \leq i, j \leq n} (K_N(\lambda_i, \lambda_j)).$$

Proof.: By Lemma 23 and 25 we have, for $n \leq N$,

$$S^{(n)}(\lambda_1, \dots, \lambda_n) = \frac{N!}{(N-n)!} \int_R d\lambda_{n+1} \dots d\lambda_N \det_{1 \leq i, j \leq n} (K_N(\lambda_i, \lambda_j)).$$

We need to integrate $N-n$ times. Each step is similar.

Assume that we have already reduced the size of the determinant to a $m \times m$ (i.e., $\lambda_{m+1}, \dots, \lambda_N$ has been already integrated out).

To compute:

$$\int_R \det_{1 \leq i, j \leq m} (K_N(\lambda_i, \lambda_j)) d\lambda_m \quad (*)$$

We develop the det. w.r.t. the last column:

$$\begin{aligned} \det_{1 \leq i, j \leq m} (K_N(\lambda_i, \lambda_j)) &= K_N(\lambda_m, \lambda_m) \cdot \det_{1 \leq i, j \leq m-1} (K_N(\lambda_i, \lambda_j)) \\ &\quad + \sum_{k=1}^{m-1} (-1)^{m-k} K_N(\lambda_m, \lambda_k) \det \left(\begin{array}{c|cc} K_N(\lambda_i, \lambda_j) & 1 \leq i \leq m-1 \\ \hline K_N(\lambda_m, \lambda_j) & 2 \leq j \leq m-1 \end{array} \right) \end{aligned}$$

We take $K_N(\lambda_m, \lambda_m)$ in the last row (by linearity) and then integrate over λ_m using Lemma 26

$$\Rightarrow (*) = N \cdot \det_{1 \leq i, j \leq m-1} (K_N(\lambda_i, \lambda_j)) + \sum_{k=1}^{m-1} (-1)^{m-k} \cdot \det \left(\begin{array}{c|cc} K_N(\lambda_i, \lambda_j) & 1 \leq i \leq m-1 \\ \hline K_N(\lambda_{m+1}, \lambda_j) & 1 \leq j \leq m-1 \end{array} \right)$$

$$= (N - (m-1)) \det \left(K_N(\lambda_i, \lambda_j) \right)_{1 \leq i, j \leq m-1}.$$

This computation applied for $m = N, N-1, \dots, n+1$ reads:

$$S^{(n)}(\lambda_1, \dots, \lambda_n) = \text{const} \cdot N! \det \left(K_N(\lambda_i, \lambda_j) \right)_{1 \leq i, j \leq n}.$$

To determine the constant, notice that it does not depend on n and for $n=1$,

$$S^{(1)}(\lambda_1) = \text{const} \cdot N! K_N(\lambda_1, \lambda_1).$$

$$\text{But: } \int_{\mathbb{R}} S^{(1)}(\lambda_1) d\lambda_1 = N = \underbrace{\text{const} \cdot N! \int_{\mathbb{R}} K_N(\lambda_1, \lambda_1) d\lambda_1}_{= N \text{ by Lemma 26.}}$$

$$\Rightarrow \text{const} = \frac{1}{N!}.$$

//

Remark: The form in Lemma 25 is generic for any set of polynomials, while Prop 26/27 crucially depends on the fact that the polynomials q_N are the orthogonal polynomials wrt. $w(x)$.

Remark: For orthogonal polynomials there is a three term recurrence from which one can deduce the Christoffel-Darboux representation:

$$K_N(x, y) = \sqrt{w(x)w(y)} \frac{c_{N-1}}{c_N} \frac{q_N(x)q_{N-1}(y) - q_{N-1}(x)q_N(y)}{x-y}$$

(for $x \neq y$) and for $x=y$, just take the limit.

Remark: $S^{(n)}(x) = \text{density of eigenvalues at } x$.
 $\Rightarrow \int_{\mathbb{R}} S^{(n)}(x) dx = N \neq 1$.

Now we want to determine an explicit expression for K_N , which is also useful for studying the $N \rightarrow \infty$ limit at the edge/bulk of the spectrum. We can write

$$K_N(x, y) = \sum_{k=0}^{N-1} q_k(x) q_k(y) \sqrt{w(x) w(y)}$$

but also use an equivalent form of the kernel,

namely $\tilde{K}_N(x, y) = \sum_{k=0}^N q_k(x) q_k(y) w(x)$. Indeed:

Remark: $\det(K_N(x_i, y_j)) = \det(K_N(x_i, x_j) \frac{e^{g(x_i)}}{e^{g(x_j)}})$

for any function g (s.t. $g(x_i) \notin \{0, \infty\}$).

Notation: $K_N(x, y) \stackrel{\text{def}}{=} K_N(x, y) \frac{e^{g(x)}}{e^{g(y)}}$ (conjugation).

\Rightarrow We need to find - functions $\{\Phi_k^n(x), k=0, \dots, N-1\}$
and functions $\{\tilde{\Phi}_k^n(x), k=0, \dots, N-1\}$ s.t.:

$$\left\{ \begin{array}{l} \Phi_k^n(x) = \text{Poly}_k(x) \\ \tilde{\Phi}_k^n(x) = \text{Poly}_k(x) \cdot w(x) \\ \int_R dx \tilde{\Phi}_k^n(x) \tilde{\Phi}_l^n(x) = S_{k,l} e. \end{array} \right.$$

Their, $\tilde{\Phi}_k^n(x) = \alpha \cdot q_k(x)$, $\tilde{\Phi}_k^n(x) = \frac{1}{\alpha} \cdot q_k(x) w(x)$ for some constant $\alpha \neq 0$.

Consequently, $K_N(x, y) = \sum_{k=0}^{N-1} \tilde{\Phi}_k^n(x) \tilde{\Phi}_k^n(y)$.

Lemma 28: Consider first $w(x) := e^{-x^2/2}$. Then,

$$\tilde{\Phi}_k^n(x) = \frac{1}{2\pi i} \int_{\gamma+iR} \int_{\gamma-iR} dz e^{z^2/2 - 2xz} z^k \quad (\forall s \in R \text{ fixed})$$

and $\tilde{\Phi}_k^n(x) = \frac{1}{2\pi i} \int_0^{8+iR} dw e^{-w^2/2 + 2wx} \frac{1}{w^{k+1}}$

satisfy \circledast above.

Proof: By Cauchy residue's theorem we clearly have that $\Phi_k^n(x) = \text{Poly}_k(x)$.

Further, $\Phi_k^n(x) = (-1)^k \frac{d^k}{dz^k} \bar{\Phi}_0^n(x)$ and

$$\bar{\Phi}_0^n(x) = \frac{1}{2\pi i} \int_{\gamma+iR} \int_{\gamma-iR} dz e^{z^{q_2}-zx}$$

$$\begin{aligned} z = iy &= \frac{1}{2\pi i} \int_R^{\infty} dy e^{-\left(\frac{y^2}{2} + iyx\right)} \\ &= \frac{1}{2\pi} \int_R^{\infty} dy e^{-\frac{1}{2}(y+ix)^2} - e^{-x^2/2} \\ &= \frac{e^{-x^2/2}}{\sqrt{2\pi}} = \text{Poly}_0(x) \cdot \omega(x). \end{aligned}$$

$\Rightarrow \Phi_k^n(x) = \text{Poly}_k(x) \omega(x)$ by differentiating by parts.

Orthogonal conditions:

$$(\Phi_k^n, \bar{\Phi}_e^n) = \int_{\mathbb{R}} dx \frac{1}{2\pi i} \int_{\gamma+iR} \int_{\gamma-iR} dz dw e^{z^{q_2}} \frac{1}{2\pi i} \int_{\mathbb{R}} dw e^{-w^{q_2}} \frac{z^k}{w^{e+1}} \frac{e^{-(z-w)x}}{2\pi i} \quad \textcircled{*}$$

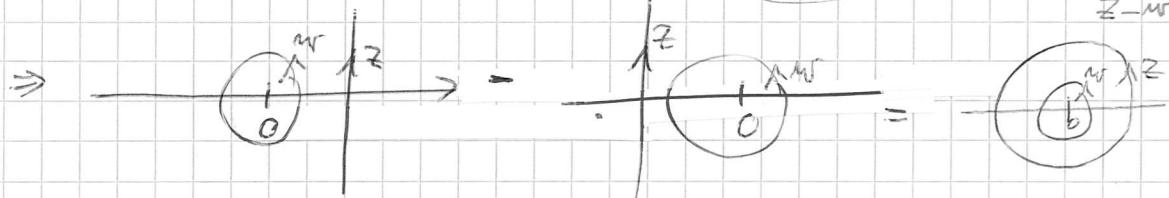
We would like to take $\int dz$ inside the integrals.

We can do it as follows:

\rightarrow for $x \geq 0$, choose $\delta > 0$ s.t. $\operatorname{Re}(z-w) > 0$

\rightarrow for $x \leq 0$, choose $\delta < 0$ s.t. $\operatorname{Re}(z-w) < 0$.

$$\begin{aligned} \textcircled{*} &= \frac{1}{(2\pi i)^2} \int_{\gamma+iR} \int_{\gamma-iR} dz dw \frac{e^{z^{q_2}}}{e^{w^{q_2}}} \frac{z^k}{w^{e+1}} \cdot \int_{\mathbb{R}^+} dx e^{-(z-w)x} \\ &\quad + \frac{1}{(2\pi i)^2} \int_{\gamma+iR} \int_{\gamma-iR} dz dw \frac{e^{z^{q_2}}}{e^{w^{q_2}}} \frac{z^k}{w^{e+1}} \cdot \int_{\mathbb{R}^-} dx e^{-(z-w)x} \quad \text{X} \\ &= \frac{1}{z-w} \end{aligned}$$



$$\Rightarrow \text{Res. at } z=w = \frac{1}{(2\pi i)^2} \oint_{\Gamma_0} dw \oint_{\Gamma_N} dz \frac{e^{z^2/2}}{e^{w^2/2}} \frac{z^k}{w^{k+1}} \cdot \frac{1}{z-w}$$

Res. at $z=w$

$$= \frac{1}{2\pi i} \oint_{\Gamma_0} dw \frac{z^k}{w^{k+1}} = S_{k, e} \quad \#$$

Lemma 29: $\tilde{K}_N(x, y) := \sum_{k=0}^{N-1} \bar{\Phi}_k^N(x) \Phi_k^N(y)$

$$= -\frac{1}{(2\pi i)^2} \int dz \int dw \frac{e^{z^2/2 - zx}}{e^{w^2/2 - wy}} \cdot \frac{z^k}{w^k} \cdot \frac{1}{w-z}$$

Sei $R \gg 1$

$\forall \delta > 0$ and the contours s.t. they do not cross.

Proof: $\tilde{K}_N(x, y) = \sum_{k=0}^{N-1} \bar{\Phi}_k^N(x) \Phi_k^N(y)$

$$= \frac{1}{(2\pi i)^2} \int dz \int dw \frac{e^{z^2/2 - zx}}{e^{w^2/2 - wy}} \cdot \underbrace{\sum_{k=0}^{N-1} \frac{z^k}{w^{k+1}}}_{= 1 - z^N/w^N}$$

Sei $R \gg 1$

The term $\frac{1}{w-z}$ disappears

since at $w=0$ there is not anymore a pole.

What remains is what we had to prove. $\#$

The kernel $\tilde{K}_N(x, y)$ is not yet the one we wanted, since we have chosen a weight $w(x) = e^{-x^2/2}$ instead of $w(x) = e^{-Nx^2/2}$. The effect is just a scaling by a factor \sqrt{N} in space.

As S^a are sort of densities, also the kernel has to be properly rescaled. Here is how it goes.

Prop 30: Let $S^{(n)}(x_1, \dots, x_n) = \det_{1 \leq i, j \leq n} (K(x_i, x_j))$

and consider the scaling:

$$x_i = \mu + \tau y_i$$

$$\text{Then, } \tilde{S}^{(n)}(y_1, \dots, y_n) = \det_{1 \leq i, j \leq n} (\tilde{K}(y_i, y_j))$$

$$\text{with } \tilde{K}(y, y') = \tau \cdot K(\mu + \tau y, \mu + \tau y').$$

$$\begin{aligned} \text{Proof: We have: } S^{(n)}(x_1, \dots, x_n) dx_1 \dots dx_n &= \tilde{S}^{(n)}(y_1, \dots, y_n) dy_1 \dots dy_n \\ &\leftarrow \det_{1 \leq i, j \leq n} (K(x_i, x_j)) \tau^n dy_1 \dots dy_n \\ &= \det_{1 \leq i, j \leq n} (\tau \cdot K(\mu + \tau y_i, \mu + \tau y_j)) dy_1 \dots dy_n. \end{aligned}$$

Thm 31: The correlation Kernel K_N for GUE given as in Prop. 27 (with $w(x) = e^{-\frac{|x|^2}{2}}$) is given by:

$$K_N(x, y) = -\frac{N}{(2\pi)^2} \int_{\mathbb{R}^2} dz \int_{\mathbb{R}^2} dw \frac{e^{-(z^2/2 - zx + bwz)}}{e^{N(w^2/2 - wy + bw)}} \frac{1}{|w-z|}$$

Proof: By Prop 30,

$$K_N(x, y) = \sqrt{N} \cdot \tilde{K}_N(\sqrt{N}x, \sqrt{N}y)$$

where \tilde{K}_N is the Kernel of Lemma 29.

By the change of variables $z = \sqrt{N} \cdot Z$ and $w = \sqrt{N} \cdot W$ one arrives then to Thm. 31.

1.8) Universal scaling limits

- By the Wigner semicircle law we have that the density of eigenvalues is roughly

$$N \cdot S_{\text{sc}}(E) \quad \text{at } E, \text{ where } S_{\text{sc}}(E) = \frac{\sqrt{4-E^2}}{\pi}, |E| \leq 2.$$

- We call "bulk" the region where the empirical measure converges to a positive density and "edge" the boundary of that set.

a) Bulk: Consider now a fixed $E \in (-2, 2)$

and the scaling s.t. the average spacing between eigenvalues is 1, i.e.,

$$\lambda_i = E + \frac{m_i}{NS_{\text{sc}}(E)}.$$

According to Prop. 30 we define the kernel for the rescaled eigenvalues as:

$$K_N^{\text{bulk}}(\xi, \eta) := \frac{1}{NS_{\text{sc}}(E)} \cdot K_N\left(E + \frac{\xi}{NS_{\text{sc}}(E)}, E + \frac{\eta}{NS_{\text{sc}}(E)}\right).$$

Then the following holds:

Thm. 32: Uniformly for ξ, η in a bounded set,

$$\lim_{N \rightarrow \infty} K_N^{\text{bulk}}(\xi, \eta) \stackrel{\text{con.}}{=} \frac{\sin(\pi(\xi - \eta))}{\pi(\xi - \eta)}, \quad \textcircled{2}$$

Rew.: $\textcircled{2}$ means that $\exists g_N(\xi)$ s.t.

$$\lim_{N \rightarrow \infty} \frac{e^{g_N(\xi)}}{e^{g_N(\eta)}} K_N^{\text{bulk}}(\xi, \eta) = \frac{\sin(\pi(\xi - \eta))}{\pi(\xi - \eta)}.$$

Rec.:

$\frac{\sin(\pi(\beta - \gamma))}{\pi(\beta - \gamma)}$ is known as Sinc Kernel.

b) Edge: By symmetry we consider only the upper edge of the spectrum, i.e., $E=2$.

In order to see a non-trivial limit, i.e., to see the individual eigenvalues on a scale of order 1, we need to consider a precise scale.

e.v. density

Heuristics:



$$\Rightarrow \# \text{e.v. in } [2-\varepsilon, 2] \approx N \int_0^\varepsilon \sqrt{x} dx \approx N \varepsilon^{3/2}$$

which is $O(1)$ exactly when $\varepsilon = O(N^{-2/3})$.

⇒ The scaling at the edge is:

$$\lambda_i = 2 + \mu_i N^{-2/3}$$

By Prop 30, the accordingly rescaled kernel is given by:

$$K_N^{\text{edge}}(\beta, \gamma) := N^{-2/3} K_N(2 + \beta N^{-2/3}, 2 + \gamma N^{-2/3})$$

with K_N given in Thm 31.

In the $N \rightarrow \infty$ limit we obtain:

Theorem 33: Uniformly in ξ, ζ in a bounded set,

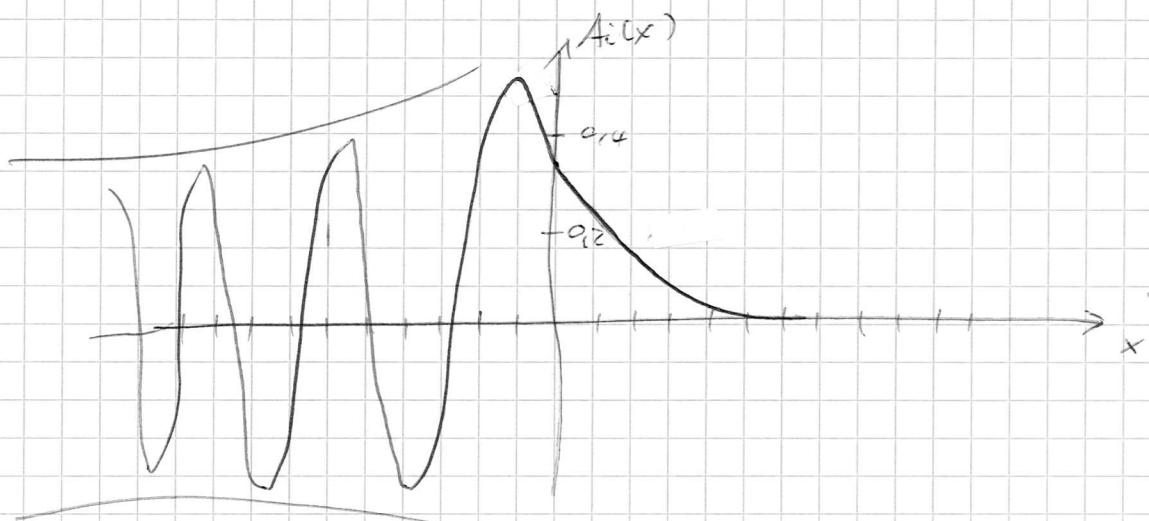
$$\lim_{N \rightarrow \infty} K_N^{\text{edge}}(\xi, \zeta) \stackrel{\text{cais.}}{=} \frac{-1}{(2\pi i)^2} \int_{\text{StiR}} dw \int_{\text{StiR}} dz e^{\frac{z^{3/3} - z\xi}{w^{3/3} - w\zeta}} \cdot \frac{1}{w^2} \\ =: K_{\text{Airy}}(\xi, \zeta). \quad (\forall s > 0)$$

Remark: The kernel K_{Airy} is called the Airy Kernel because it can be written in terms of Airy functions.

Def. 34: The Airy function $x \mapsto \text{Ai}(x)$ is the unique solution of the equation

$$\frac{d^2 y(x)}{dx^2} = x y(x)$$

with $y(x) \sim \exp\left(-\frac{2}{3}x^{3/2}\right)$ as $x \rightarrow \infty$.



Lemma 35: The Airy function has the following integral representations:

$$\left. \begin{aligned} \text{1. } \text{Ai}(x) &= \frac{1}{2\pi i} \int_{\text{StiR}} dz e^{\frac{z^{3/3} - zx}{w^{3/3}}} , \quad s > 0 \\ \end{aligned} \right\}$$

$$\left. \begin{aligned} \text{2. } \text{Ai}(x) &= \frac{1}{2\pi i} \int_{\text{StiR}} dw e^{-\frac{w^{3/3} + wx}{z^{3/3}}} , \quad s > 0 \end{aligned} \right\}$$

Proof: The second representation comes from the change of variables $z = -w$ from the first one.

$$\begin{aligned} \frac{d^2}{dx^2} \left(\frac{1}{2\pi i} \int_{\text{SiR}} dz e^{z^{3/3} - zx} \right) &= \frac{1}{2\pi i} \int_{\text{SiR}} dz e^{z^{3/3} - zx} \cdot z^2 \\ &= e^{-zx} \cdot e^{z^{3/3}} \Big|_{\text{SiR}} + \frac{1}{2\pi i} \int_{\text{SiR}} dz e^{z^{3/3} - zx} \cdot x \\ \text{S by parts} \quad &\quad \text{S-iR} \quad \text{SiR} \\ &= 0 \\ &= x \cdot \text{Ai}(x). \end{aligned}$$

Asymptotics ($x \rightarrow \infty$):

By choosing δ s.t. $\frac{\delta^3}{3} - \delta x$ is minimal, i.e., take $\delta = \sqrt{x}$ and by setting $z = \sqrt{x} + iw$ we get:

$$\begin{aligned} \text{Ai}(x) &= \frac{i}{2\pi x} \int_{-\infty}^{\infty} dw e^{(\sqrt{x}+iw)^{3/3} - (\sqrt{x}+iw)x} \\ &= \frac{i}{2\pi} \int_{-\infty}^{\infty} dw e^{\frac{x^{3/2}}{3} + ixw - \sqrt{x}w^2 - iw^3} e^{-\sqrt{x}x - iw\bar{x}} \\ &= \frac{e^{-\frac{2}{3}x^{3/2}}}{2\pi} \int_{\text{IR}} dw e^{\frac{-\sqrt{x}w^2}{3} + iw^{3/3}} \cdot e^{\text{irrelevant for } x \rightarrow \infty \text{ as compared with } w^3} \\ &\approx \frac{e^{-\frac{2}{3}x^{3/2}}}{2\pi} \cdot \frac{\sqrt{\pi}}{x^{1/4}} \cdot \# \end{aligned}$$

With the representations in Lemma 35 we can easily rewrite $K_{\text{Airy}}(z, s)$ as follows.

$$\begin{aligned}\text{Lemma 36: } K_{\text{Airy}}(z, s) &= \int_0^\infty d\lambda A_i(\bar{z}+\lambda) A_i(s+\lambda) \\ &= \frac{A_i(\bar{z}) A_i'(s) - A_i'(\bar{z}) A_i(s)}{z-s}.\end{aligned}$$

Proof: Use $\frac{1}{z-w} = \int_0^\infty d\lambda e^{-(z-w)\lambda}$ for $\operatorname{Re}(z-w) > 0$.

$$\begin{aligned}\Rightarrow K_{\text{Airy}}(z, s) &= \int_0^\infty d\lambda \left(\int_{-\infty+i\mathbb{R}}^s \frac{dw}{2\pi i} e^{-w^3/3 + w(z+\lambda)} \right) \\ &\quad \cdot \left(\int_{S+i\mathbb{R}}^\infty \frac{dz}{2\pi i} e^{z^3/3 - z(z+\lambda)} \right) \\ &\stackrel{\text{Lemma 35}}{=} \int_0^\infty d\lambda A_i(\bar{z}+\lambda) A_i(s+\lambda).\end{aligned}$$

To see the second equality; compute:

$$\begin{aligned}&(z-s) \int_0^\infty d\lambda A_i(\bar{z}+\lambda) A_i(s+\lambda) \\ &\times A_i''(x) = A_i''(x) \\ &\stackrel{x}{=} \int_0^\infty d\lambda A_i''(\bar{z}+\lambda) A_i(s+\lambda) - \int_0^\infty d\lambda A_i(\bar{z}+\lambda) A_i''(s+\lambda) \\ &\stackrel{\text{by parts}}{=} \left. A_i'(\bar{z}+\lambda) A_i(s+\lambda) \right|_0^\infty - \int_0^\infty d\lambda \cancel{A_i'(\bar{z}+\lambda) A_i'(s+\lambda)} \\ &\quad - \left. A_i(\bar{z}+\lambda) A_i'(s+\lambda) \right|_0^\infty + \int_0^\infty d\lambda \cancel{A_i'(\bar{z}+\lambda) A_i'(s+\lambda)} \\ &= A_i(\bar{z}) A_i'(s) - A_i'(\bar{z}) A_i(s).\end{aligned}$$

Here are a few other properties of the Airy Kernel:

(a) $K_{\text{Airy}} = K_{\text{Airy}}^2$ (reproducing kernel)

(b) For $H = -\frac{d^2}{dx^2} + x$, the Airy operator, the generalized eigenfunctions are $f_\lambda(x) := \text{Ai}(x-\lambda)$:

$$H f_\lambda = \lambda f_\lambda.$$

\Rightarrow The Airy Kernel is the spectral projection onto $\{H \leq 0\}$, i.e., $K_{\text{Airy}}(z, s) = \int_{\mathbb{R}} d\lambda f_\lambda(z) f_\lambda(s)$.

(c) K_{Airy} is locally trace-class and $\|P_s K_{\text{Airy}} P_s\| < 1$ for $s > -\infty$, where $P_s = \mathbf{1}_{(s, \infty)}$ (Projection).

Now let us go back to Thm. 33 and prove it.

Proof of Thm 33:

• Define $f_0(z) = \frac{z^2}{2} - 2z + \ln z$ and

$$f_2(z, \bar{z}) = -z \cdot \bar{z}.$$

$$\text{Then, } K_N(\bar{z}, s) = -\frac{N^{1/3}}{(2\pi i)^2} \int dz dw \frac{e^{-Nf_0(w)}}{e^{Nf_0(w) + N^{1/3}f_2(w, \bar{z})}} \frac{1}{w - z}$$

For z, \bar{z} in a bounded region, the $N \rightarrow \infty$ asymptotics is governed by $e^{N(f_0(z) - f_0(w))}$. Thus, let us start by studying f_0 closer.

• Critical points of f_0 : $\frac{d}{dz} f_0(z) = z - 2 + \frac{1}{z} = 0 \Leftrightarrow z = 1$

$$\cdot \frac{d^2}{dz^2} f_0(z)|_{z=1} = 1 - \frac{1}{z^2}|_{z=1} = 0$$

$$\cdot \frac{d^3}{dz^3} f_0(z)|_{z=1} = \frac{2}{z^3}|_{z=1} = 2.$$

and $\frac{d^4}{dz^4} f_0(z)$ is bounded for z away from 0.

$$\xrightarrow{\text{Taylor}} f_0(z) = f_0(1) + \frac{1}{3!} 2(z-1)^3 + O((z-1)^4)$$

$$= f_0(1) + \frac{(z-1)^3}{3} + O((z-1)^4)$$

$$\text{and } f_2(z, 3) = f_2(1, 3) - \frac{5}{3}(z-1).$$

Heuristics: Assumption 1: The contribution to the integral comes from a Z, W δ -neighborhood of 1.

Under this Assumption (to be proven later), if we do the change of variables

$$Z = 1 + zN^{1/3}, \quad W = 1 + wN^{1/3},$$

then $K_n^{\text{edge}}(z, y) \approx -\frac{1}{(2\pi i)^2} \int dz \int dw e^{\frac{Nf_0(1) + z^{1/3} N f'_2(1, 3) - zy}{Nf_0(1) + w^{1/3} N f'_2(1, 3) - wy} \cdot e^{\frac{1}{w-z} \cdot e^{O(N^{1/3} w^4)}}}$

Next under Assumption 2: The contributions of the error terms can be neglected in the $N \rightarrow \infty$ limit, we get

$$K_n^{\text{edge}}(z, y) \underset{N \rightarrow \infty}{\underset{\text{const.}}{\approx}} K_{\text{Ave}}(z, y).$$

So, what remains is to make Assumptions 1 and 2 precise and prove them.

Consider: $Z = R + i\omega$, $\omega \in \mathbb{R}$ and $R \geq 1$

$$W = S e^{i\varphi}, \quad \varphi \in [-\pi, \pi) \text{ and } 0 < S \leq 1.$$

Set $U(\omega) := \operatorname{Re}(f_0(R+i\omega))$ and $N(\varphi) := \operatorname{Re}(-f_0(S e^{i\varphi}))$.

$$\text{Then, } U(\omega) = \operatorname{Re}\left(\frac{(R+i\omega)^2 - 2(R+i\omega) + \ln(R+i\omega)}{2}\right)$$

$$= \frac{R^2}{2} - \frac{\omega^2}{2} - 2R + \frac{1}{2} \ln(R^2 + \omega^2)$$

$$\text{and } N(\varphi) = \operatorname{Re}\left(-\frac{S^2}{2} e^{2i\varphi} + 2Se^{i\varphi} - \ln(Se^{i\varphi})\right)$$

$$= -\frac{S^2}{2} \cos(2\varphi) + 2S \cos(\varphi) - \ln S.$$

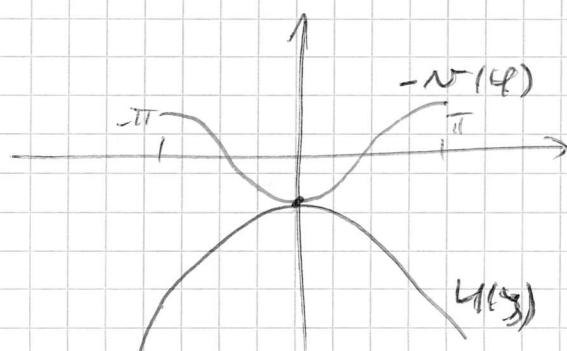
The function $U(\omega)$ (resp. $N(\varphi)$) reaches its maximum at $\omega=0$ (resp. $\varphi=0$) and it is otherwise monotone. Indeed,

$$\frac{dU(\omega)}{d\omega} = -\omega + \frac{1}{2} \frac{2\omega}{R^2 + \omega^2} = -\omega \left(1 - \frac{1}{R^2 + \omega^2}\right) \begin{cases} < 0, \omega > 0 \\ > 0, \omega < 0 \end{cases}$$

$\stackrel{?}{>} 0$
 $(R \geq 1)$

$$\begin{aligned} \frac{dN(\varphi)}{d\varphi} &= \frac{S^2}{2} \sin(2\varphi) \stackrel{?}{>} -2S \sin(\varphi) \\ &= 2S \sin(\varphi) \left(1 - S \cos \varphi\right) \end{aligned} \begin{cases} < 0, \varphi \in (0, \pi), \\ > 0, \varphi \in (-\pi, 0). \end{cases}$$

$\stackrel{?}{>} 0$
 $(S \leq 1)$



If we consider ^{only} the integral over $\{Z = R + i\omega, \omega \in \mathbb{R} \setminus [S, S]\}$, we get $\int_{I_S} dz e^{Nf_0(Z) + N^{1/3}f_2(Z, 3)} h(z)$.
 $\underbrace{h(z)}_{\text{smooth, bounded fct.}}$
 $= e^{Nf_0(R) + N^{1/3}f_2(R, 3)} O(e^{-C(S)N})$ for some $C(S) > 0$.

Similarly, let $C_S := \{W = 3e^{i\varphi}, \varphi \in [-\pi, -S] \cup [S, \pi]\}$,

then

$$\int_{C_S} dW e^{-Nf_0(W) - N^{1/3}f_2(W, 3)} h(W) \\ = e^{-Nf_0(3) - N^{1/3}f_2(3, 3)} O(e^{-\tilde{C}(S)N}) \text{ for some } \tilde{C}(S) > 0.$$

$$\Rightarrow \underbrace{\int_{I_0} dz \int_{C_0} dW f_{00}}_{(I_0, C_0)} = \underbrace{\int_{I_0} dz \int_{C_0} dW}_{(I_0, C_0)} (\dots) + \underbrace{\int_{I_S} dz \int_{C_S} dW}_{(I_S, C_S)} (\dots)$$

$$= e^{N(f_0(R) - f_0(3))} e^{N^{1/3}(f_2(R, 3) - f_2(3, 3))} \cdot \underbrace{(e^{-C(S)N} + e^{-\tilde{C}(S)N}) O(1)}_{(A)} \\ + \underbrace{\int_{I_S^c} dz \int_{C_S^c} dW (\dots)}_{(I_S^c, C_S^c)} (\dots) \quad (B)$$

Choose: $R = 1 + \mu N^{-1/3}$ and $3 = 1 - \mu N^{-1/3}$.

$$\text{Then, } N(f_0(R) - f_0(3)) + N^{1/3}(f_2(R, 3) - f_2(3, 3)) \\ = O(N^{2/3}) + N^{1/3}(f_2(1, 3) - f_2(1, 3)).$$

Plugging $(*)$ into (A) we get that

$$(A) = e^{N^{1/3}(f_2(1, 3) - f_2(1, 3))} (e^{-\frac{C(S)N}{2}} + e^{-\frac{\tilde{C}(S)N}{2}}) O(1).$$

Next, focus on (B) . The Taylor expansion and the change of variables $Z = 1 + zN^{-1/3}$, $W = 1 + wN^{-1/3}$ gives:

$$\textcircled{B}) = -\frac{1}{(2\pi i)^2} \int dz \int dw \frac{e^{z^3/3 - z^3}}{w^{3/3 - w^3}} \cdot \frac{1}{w-z} \cdot e^{O(N^{-1/3} z^4)} O(N^{-1/3})$$

$$e^{N^{1/3} \cdot (\chi_2(1,3) - \chi_2(1,5))} \rightarrow \text{cancellation.}$$

$$\text{cons} \equiv -\frac{1}{(2\pi i)^2} \int dz \int dw \frac{e^{z^3/3 - z^3}}{e^{w^3/3 - w^3}} \frac{1}{w-z} \cdot \left(1 + \left[e^{O(N^{-1/3} z^4; N^{-1/3} w^4)} - 1 \right] \right)$$

Finally, using: $|e^x - 1| \leq e^{|x|}$ we have

$$\textcircled{C}) = -\frac{1}{(2\pi i)^2} \int dz \int dw \frac{e^{z^3/3 - z^3}}{\frac{1}{w^3/3 - w^3} \frac{1}{w-z}} \quad \text{③}$$

$$+ -\frac{N^{-1/3}}{(2\pi i)^2} \int dz \int dw \frac{e^{z^3/3 - z^3}}{e^{w^3/3 - w^3}} \frac{1}{w-z} \cdot e^{O(N^{-1/3} z^4; N^{-1/4} w^4)} \cdot O(z^4, w^4)$$

④

The integrals in ③ are restricted to a $\delta N^{1/3}$ neighborhood of the origin. Extending the integrals to ∞ leads to an error term $O(e^{-\alpha(s)N} + e^{-\tilde{\alpha}(s)N})$ again.

By choosing δ small enough (but indep. of N) we have that the cubic term dominates the error terms and ④ remains $O(N^{-1/3}) \xrightarrow[N \rightarrow \infty]{} 0$. $\textcircled{*}$

Putting all together the statement to be proven holds true. $\textcircled{*}$

$\textcircled{A})$ The reason in this case is that the $O(w^4)$ term in $u(w)$ is $-\frac{1}{4}w^4$ as well as the $O(\varphi^4)$ term in $v(\varphi)$ is $-\frac{1}{4}\varphi^4$ (i.e., they help the avg.). Otherwise, one would need to take paths like: $\frac{1}{\sqrt{1}}$ instead of vertical ones.

Finally, we indicate the idea of the proof of Thm 32.

Sketch of the proof of Thm 32:

We have; for $E \in (-2, 2)$ fixed:

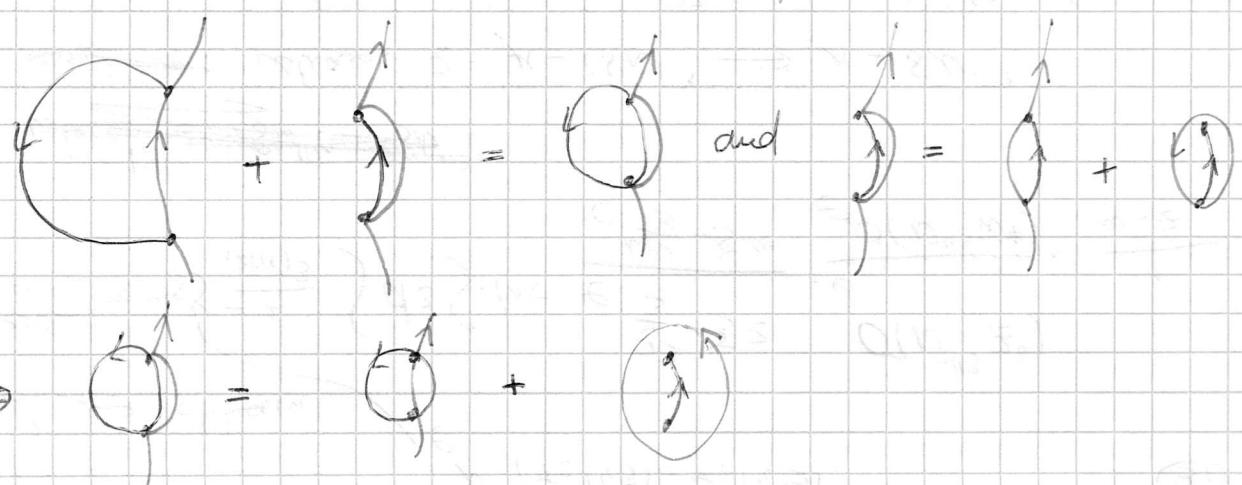
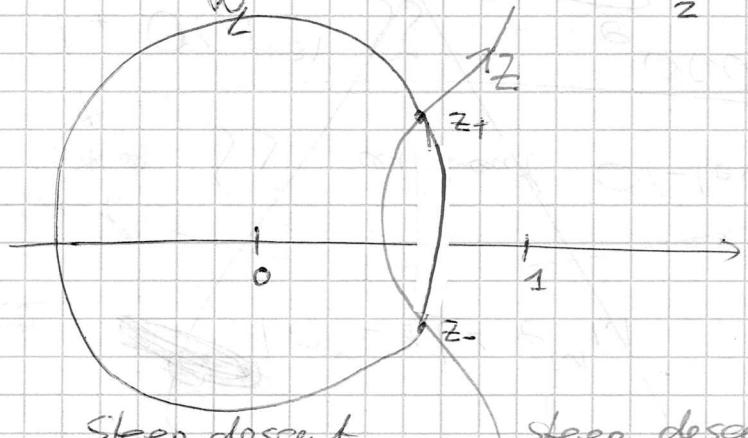
$$K_n^{\text{bulk}}(z, \bar{z}) = \frac{-1}{S_{\text{sc}}(E) (2\pi i)^2} \cdot \int d\omega d\bar{\omega} e^{\frac{N f_0(z) + f_1(z, \bar{z})}{e^{N f_0(\omega)} + f_1(\omega, \bar{z})}} \frac{1}{\omega - z}$$

$$\text{with } f_0(z) = \frac{z^2}{2} - E \cdot z + \ln z$$

$$\text{and } f_1(z, \bar{z}) = -\frac{z \bar{z}}{S_{\text{sc}}(E)}$$

Critical pts $\frac{df_0(z)}{dz} = z - E + \frac{1}{z} = 0$

$$\Leftrightarrow z = \frac{E \pm i\sqrt{4-E^2}}{2}$$



By steep descent analysis one sees that the leading contribution is



comes from a $N^{-1/2}$ -neighborhood of the critical points.

The contributions around the critical points can be easily estimated by Gaussian integrals and it is $O(N^{-1})$. The factor $\frac{1}{W-z}$ is not really a problem since this singularity is integrable (2 dimensions!).

It remains:

$$\begin{aligned}
 & -\frac{1}{(2\pi i)^2 S_{\text{sc}}(E)} \cdot \int_{\Gamma_W} dz \oint_{\gamma} dz \frac{e^{Nf_0(z) + f_1(z, \bar{z})}}{e^{Nf_0(W) + f_1(W, \bar{z})}} \cdot \frac{1}{W-z} \\
 & = \frac{+1}{2\pi i \cdot S_{\text{sc}}(E)} \int_{\gamma} dz e^{-W(\bar{z}-z)/S_{\text{sc}}(E)} \\
 & = -\frac{1}{2\pi i} \cdot \frac{1}{\bar{z}-z} \cdot \left(e^{-\frac{-(\bar{z}-z)}{S_{\text{sc}}(E)}} - e^{-\frac{-(\bar{z}-z)}{S_{\text{sc}}(E)}} \right) \\
 & = e^{-\frac{-(\bar{z}-z)}{2 S_{\text{sc}}(E)}} \cdot \underbrace{\frac{-1}{2\pi i(\bar{z}-z)} \cdot \left(e^{-\pi i(\bar{z}-z)} - e^{\pi i(\bar{z}-z)} \right)}_{\text{conjugation}} \\
 & z_{\pm} = \frac{E}{2} \pm \frac{i\sqrt{4-E^2}}{2}
 \end{aligned}$$

$$\begin{aligned}
 & \text{Ans.} \\
 & \equiv \frac{\sin(\pi(\bar{z}-z))}{\pi(\bar{z}-z)} \quad \text{#}
 \end{aligned}$$