

Zeros of Gaussian Analytic Functions and Determinantal Point Processes

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Preface

Random configurations of points in space, also known as point processes, have been studied in mathematics, statistics and physics for many decades. In mathematics and statistics, the emphasis has been on the Poisson process, which can be thought of as a limit of picking points independently and uniformly in a large region. Taking a different perspective, a finite collection of points in the plane can always be considered as the roots of a polynomial; in this coordinate system, taking the coefficients of the polynomial to be independent is natural. Limits of these random polynomials and their zeros are a core subject of this book; the other class consists of processes with joint intensities of determinantal form. The intersection of the two classes receives special attention, in Chapter 5 for instance. Zeros of random polynomials and determinantal processes have been studied primarily in mathematical physics. In this book we adopt a probabilistic perspective, exploiting independence whenever possible.

The book is designed for graduate students in probability, analysis, and mathematical physics, and exercises are included. No familiarity with physics is assumed, but we do assume that the reader is comfortable with complex analysis as in Ahlfors' text (1) and with graduate probability as in Durrett (19) or Billingsley (6). Possible ways to read the book are indicated graphically below, followed by an overview of the various chapters.

The book is organized as follows:

Chapter 1 starts off with a quick look at how zeros of a random polynomial differ from independently picked points, and the ubiquitous Vandermonde factor makes its first appearance in the book. Following that, we give definitions of basic notions such as point processes and their joint intensities.

Chapter 2 provides an introduction to the theory of Gaussian analytic functions (GAFs) and gives a formula for the first intensity of zeros. We introduce three important classes of geometric GAFs: planar, hyperbolic and spherical GAFs, whose zero sets are invariant in distribution under isometries preserving the underlying geometric space. Further we show that the intensity of zeros of a GAF determines the distribution of the GAF (Calabi's rigidity).

Chapter 3 We prove a formula due to Hammersley for computing the joint intensities of zeros for an arbitrary GAF.

Chapter 4 introduces determinantal processes which are used to model fermions in quantum mechanics and also arise naturally in many other settings. We show that general determinantal processes may be realized as mixtures of "determinantal projection processes", and use this result to give simple proofs of existence and central limit theorems. We also present similar results for permanental processes, which are used to model bosons in quantum mechanics.

Chapter 5 gives a deeper analysis of the hyperbolic GAF. Despite the many similarities between determinantal processes and zeros of GAFs, this function provides the only known link between the two fields. For a certain value of the parameter, the zero set of the hyperbolic GAF is indeed a determinantal process and this discovery allows one to say a great deal about its distribution. In particular, we give a simple description of the distribution of the moduli of zeros and obtain sharp asymptotics for the “hole probability” that a disk of radius r contains no zeros. We also obtain a law of large numbers and reconstruction result for the hyperbolic GAFs, the proofs of these do not rely on the determinantal property.

Chapter 6 studies a number of examples of determinantal point processes that arise naturally in combinatorics and probability. This includes the classical Ginibre and circular unitary ensembles from random matrix theory, as well as examples arising from non-intersecting random walks and random spanning trees. We give proofs that these point processes are determinantal.

Chapter 7 turns to the topic of large deviations. First we prove a very general result due to Offord which may be applied to an arbitrary GAF. Next we present more specialized techniques developed by Sodin and Tsirelson which can be used to determine very precisely, the asymptotic decay of the hole probability for the zero set of the planar GAF. The computation is more difficult in this setting, since this zero set is not a determinantal process.

Chapter 8 touches on two advanced topics, dynamical Gaussian analytic functions and allocation of area to zeros.

In the section on dynamics, we present a method by which the zero set of the hyperbolic GAF can be made into a time-homogeneous Markov process. This construction provides valuable insight into the nature of the repulsion between zeros, and we give an SDE description for the evolution of a single zero. This description can be generalized to simultaneously describe the evolution of all the zeros.

In the section on allocation, we introduce the reader to a beautiful scheme discovered by Sodin and Tsirelson for allocating Lebesgue measure to the zero set of the planar GAF. The allocation is obtained by constructing a random potential as a function of the planar GAF and then allowing points in the plane to flow along the gradient curves of the potential in the direction of decay. This procedure partitions the plane into basins of constant area, and we reproduce an argument due to Nazarov, Sodin and Volberg that the diameter of a typical basin has super-exponentially decaying tails.

The inter-dependence of the chapters is shown in Figure 1 schematically.

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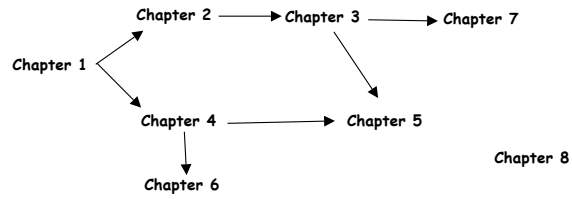


FIGURE 1. Dependence among chapters.

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Introduction

1.1. Random polynomials and their zeros

The primary objects of study in this book are *point processes*, which are random variables taking values in the space of discrete subsets of a metric space, where, by a discrete set we mean a countable set with no accumulation points. Precise definitions of relevant notions will be given later. Many physical phenomena can be modeled by random discrete sets. For example, the arrival times of people in a queue, the arrangement of stars in a galaxy, energy levels of heavy nuclei of atoms etc. This calls upon probabilists to find point processes that can be mathematically analysed in some detail, as well as capture various qualitative properties of naturally occurring random point sets.

The single most important such process, known as the **Poisson process** has been widely studied and applied. The Poisson process is characterized by independence of the process when restricted to disjoint subsets of the underlying space. More precisely, for any collection of mutually disjoint measurable subsets of the underlying space, the numbers of points of a Poisson process that fall in these subsets are stochastically independent. The number of points that fall in A has Poisson distribution with a certain mean $\mu(A)$ depending on A . Then, it is easy to see then that μ must be a measure, and it is called the intensity measure of the Poisson process. This assumption of independence is acceptable in some examples, but naturally, not all. For instance if one looks at outbreaks of a rare disease in a province, then knowing that there is a case in a particular location makes it more likely that there are more such cases in a neighbourhood of that location. On the other hand, if one looks at the distribution of like-charged particles confined by an external field (physicists call it a “one component plasma”), then the opposite is true. Knowing that a particular location holds a particle makes it unlikely for there to be any others close to it. These two examples indicate two ways of breaking the independence assumption, inducing more clumping (“positively correlated”) as in the first example or less clumping (“negatively correlated”) as in the second.

A natural question is, are there probabilistic mechanisms to generate such clumping or anti-clumping behaviour? A simple recipe that gives rise to positively correlated point processes is well-known to statisticians: First sample $X(\cdot)$, a continuous random function on the underlying space that takes values in \mathbb{R}_+ , and then, sample a Poisson process whose intensity measure has density $X(\cdot)$ with respect to a fixed reference measure ν on the underlying space. These kinds of processes are now called Cox processes, and it is clear why they exhibit clumping - more points fall where X is large, and if X is large at one location in space, it is large in a neighbourhood. We shall encounter a particular subclass of Cox processes, known as permanental processes, in Chapter 4, only to compare their properties with determinantal processes,

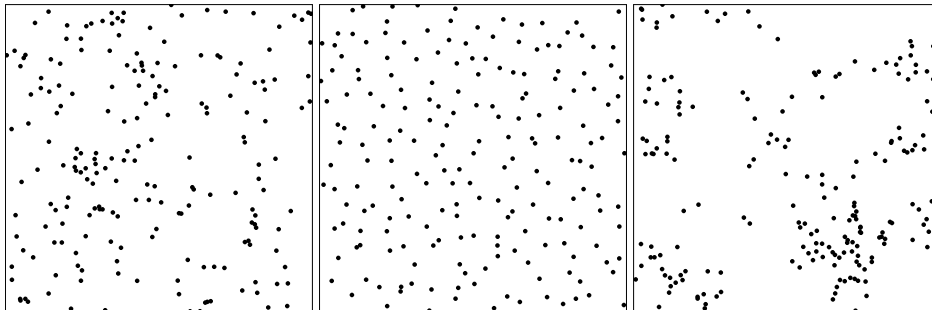


FIGURE 1. Samples of translation invariant point processes in the plane: Poisson (left), determinantal (center) and permanental for $\mathbb{K}(z, w) = \frac{1}{\pi} e^{z\bar{w} - \frac{1}{2}(|z|^2 + |w|^2)}$. Determinantal processes exhibit repulsion, while permanental processes exhibit clumping.

one of two important classes of point processes having negative correlations that we study in this book.

This brings us to the next natural question and that is of central importance to this book. Are there interesting point processes that have *less* clumping than Poisson processes? As we shall see, one natural way of getting such a process without putting in the anti-clumping property by hand, is to extract zero sets of random polynomials or analytic functions, for instance, zeros of random polynomials with stochastically independent coefficients. On the other hand it is also possible to build anti-clumping into the very definition. A particularly nice class of such processes, known as determinantal point processes, is another important object of study in this book.

We study these point processes only in the plane and give some examples on the line, that is, we restrict ourselves to random analytic functions in one variable. One can get point processes in \mathbb{R}^{2n} by considering the joint zeros of n random analytic functions on \mathbb{C}^n , but we do not consider them in this book. Determinantal processes have no dimensional barrier, but it should be admitted that most of the determinantal processes studied have been in one and two dimensions. In contrast to Cox processes that we described earlier, determinantal point processes seem mathematically more interesting to study because, for one, they are apparently not just built out of Poisson processes¹.

Next we turn to the reason why these processes (zeros of random polynomials and determinantal processes) have less clustering of points than Poisson processes. Determinantal processes have this anti-clustering or repulsion built into their definition (chapter 4, definition 4.2.1), and below we give an explanation as to why zeros of random polynomials tend to repel in general. Before going into this, we invite the reader to look at Figure 1. All the three samples shown are portions of certain translation invariant point processes in the plane, with the same average number of points per unit area. Nevertheless, they visibly differ from each other qualitatively, in terms of the clustering they exhibit.

¹“Do not listen to the prophets of doom who preach that every point process will eventually be found out to be a Poisson process in disguise!” - Gian-Carlo Rota.

Now we “explain” the repulsion of points in point processes arising from zeros of random analytic functions (Of course, any point process in the plane is the zero set of a random analytic function, and hence one may wonder if we are making an empty or false claim. However, when we use the term random analytic function, we tacitly mean that we have somehow specified the distribution of coefficients, and that there is a certain amount of independence therein). Consider a polynomial

$$(1.1.1) \quad p(z) = z^n + a_{n-1}z^{n-1} + \dots + a_1z + a_0.$$

We let the coefficients be random variables and see how the (now random) roots of the polynomial are distributed. This is just a matter of change of variables, from coefficients to the roots, and the Jacobian determinant of this transformation is given by the following well known fact (see the book (2) p. 411-412, for instance).

LEMMA 1.1.1. *Let $p(z) = \prod_{k=1}^n (z - z_k)$ have coefficients a_k , $0 \leq k \leq n-1$ as in (1.1.1). Then the transformation $T : \mathbb{C}^n \rightarrow \mathbb{C}^n$ defined by*

$$T(z_1, \dots, z_n) = (a_{n-1}, \dots, a_0),$$

has Jacobian determinant $\prod_{i < j} |z_i - z_j|^2$.

PROOF. Note that we are looking for the real Jacobian determinant, which is equal to

$$\left| \det \left(\frac{\partial T(z_1, \dots, z_n)}{\partial (z_1, \dots, z_n)} \right) \right|^2.$$

To see this in the simplest case of one complex variable, observe that if $f = u + iv : \mathbb{C} \rightarrow \mathbb{C}$, its Jacobian determinant is

$$\det \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix},$$

which is equal to $|f'|^2$, provided f is complex analytic. See Exercise 1.1.2 for the relationship between real and complex Jacobian determinants in general.

Let us write

$$T_n(k) = a_{n-k} = (-1)^k \sum_{1 \leq i_1 < \dots < i_k \leq n} z_{i_1} \dots z_{i_k}.$$

$T_n(k)$ and all its partial derivatives are polynomials in z_j s. Moreover, by the symmetry of $T_n(k)$ in the z_j s, it follows that if $z_i = z_j$ for some $i \neq j$, then the i^{th} and j^{th} columns of $\frac{\partial T(z_1, \dots, z_n)}{\partial (z_1, \dots, z_n)}$ are equal, and hence the determinant vanishes. Therefore, the polynomial $\det \left(\frac{\partial T_n(k)}{\partial z_j} \right)_{1 \leq j, k \leq n}$ is divisible by $\prod_{i < j} (z_i - z_j)$. As the degree of

$\det \left(\frac{\partial T_n(k)}{\partial z_j} \right)_{1 \leq j, k \leq n}$ is equal to $\sum_{k=1}^n (k-1) = \frac{1}{2}n(n-1)$, it must be that

$$\det \left(\frac{\partial T(z_1, \dots, z_n)}{\partial (z_1, \dots, z_n)} \right) = C_n \prod_{i < j} (z_i - z_j).$$

To find the constant C_n , we compute the coefficient of the monomial $\prod z_j^{j-1}$ on both sides. On the right hand side the coefficient is easily seen to be $D_n := (-1)^{n(n-1)/2} C_n$. On the left, we begin by observing that $T_n(k) = -z_n T_{n-1}(k-1) + T_{n-1}(k)$, whence

$$(1.1.2) \quad \frac{\partial T_n(k)}{\partial z_j} = -z_n \frac{\partial T_{n-1}(k-1)}{\partial z_j} + \frac{\partial T_{n-1}(k)}{\partial z_j} - \delta_{jn} T_{n-1}(k-1).$$

The first row in the Jacobian matrix of T has all entries equal to -1 . Further, the entries in the last column (when $j = n$) are just $-T_{n-1}(k-1)$, in particular, independent of z_n . Thus when we expand $\det\left(\frac{\partial T_n(k)}{\partial z_j}\right)$ by the first row, to get z_n^{n-1} we must take the $(1, n)$ entry in the first row and in every other row we must use the first summand in (1.1.2) to get a factor of z_n . Therefore

$$\begin{aligned} D_n &= \text{coefficient of } \prod_{j=1}^n z_j^{j-1} \text{ in } \det\left(\frac{\partial T_n(k)}{\partial z_j}\right)_{1 \leq k, j \leq n} \\ &= (-1)^n \text{coefficient of } \prod_{j=1}^{n-1} z_j^{j-1} \text{ in } \det\left(-\frac{\partial T_{n-1}(k-1)}{\partial z_j}\right)_{\substack{2 \leq k \leq n \\ 1 \leq j \leq n-1}} \\ &= -D_{n-1}. \end{aligned}$$

Thus $C_n = (-1)^n C_{n-1} = (-1)^{n(n+1)/2}$ because $C_1 = -1$. Therefore the real Jacobian determinant of T is $\prod_{i < j} |z_i - z_j|^2$. \square

The following relationship between complex and real Jacobians was used in the proof of the lemma.

EXERCISE 1.1.2. Let $(T_1, \dots, T_n) : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be complex analytic in each argument. Let $A_{ij} = \frac{\partial \operatorname{Re} T_i(\mathbf{z})}{\partial x_j}$ and $B_{ij} = \frac{\partial \operatorname{Re} T_i(\mathbf{z})}{\partial y_j}$ where $z_j = x_j + iy_j$. Then the real Jacobian determinant of $(\operatorname{Re} T_1, \dots, \operatorname{Re} T_n, \operatorname{Im} T_1, \dots, \operatorname{Im} T_n)$ at $(x_1, \dots, x_n, y_1, \dots, y_n)$, is

$$\det \begin{bmatrix} A & B \\ -B & A \end{bmatrix}$$

which is equal to $|\det(A - iB)|^2$, the absolute square of the complex Jacobian determinant.

We may state Lemma 1.1.1 in the reverse direction. But first a remark that will be relevant throughout the book.

REMARK 1.1.3. Let z_k , $1 \leq k \leq n$ be the zeros of a polynomial. Then z_i s do not come with any natural order, and usually we do not care to order them. In that case we identify the set $\{z_k\}$ with the measure $\sum \delta_{z_k}$. However sometimes we might also arrange the zeros as a vector $(z_{\pi_1}, \dots, z_{\pi_n})$ where π is any permutation. If we randomly pick π with equal probability to be one of the $n!$ permutations, we say that the zeros are in **exchangeable random order** or uniform random order. We do this when we want to present joint probability densities of zeros of a random polynomial. Needless to say, the same applies to eigenvalues of matrices or any other (finite) collection of unlabeled points.

Endow the coefficients of a monic polynomial with product Lebesgue measure. The induced measure on the vector of zeros of the polynomial (taken in exchangeable random order) is

$$\left(\prod_{i < j} |z_i - z_j|^2 \right) \prod_{k=1}^n dm(z_k).$$

Here dm denotes the Lebesgue measure on the complex plane.

One can get a probabilistic version of this by choosing the coefficients from Lebesgue measure on a domain in \mathbb{C}^n . Then the roots will be distributed with density proportional to $\prod_{i < j} |z_i - z_j|^2$ for (z_1, \dots, z_n) in a certain symmetric domain of \mathbb{C}^n .

A similar phenomenon occurs in random matrix theory. We just informally state the result here and refer the reader to (6.3.5) in chapter 6 for a precise statement and proof.

FACT 1.1.4. Let $(a_{i,j})_{i,j \leq n}$ be a matrix with complex entries and let z_1, \dots, z_n be the eigenvalues of the matrix. Then it is possible to choose a set of auxiliary variables which we just denote \mathbf{u} (so that \mathbf{u} has $2n(n-1)$ real parameters) so that the transformation $T(z, \mathbf{u}) = (a_{i,j})$ is essentially one-to-one and onto and has Jacobian determinant

$$f(\mathbf{u}) \prod_{i < j} |z_i - z_j|^2$$

for some function f .

REMARK 1.1.5. Unlike in Lemma 1.1.1, to make a change of variables from the entries of the matrix, we needed auxiliary variables in addition to eigenvalues. If we impose product Lebesgue measure on $a_{i,j}$ s, the measure induced on $(z_1, \dots, z_n, \mathbf{u})$ is a product of a measure on the eigenvalues and a measure on \mathbf{u} . However, the measures are infinite and hence it does not quite make sense to talk of "integrating out the auxiliary variables" to obtain

$$(1.1.3) \quad \prod_{i < j} |z_i - z_j|^2 \prod_{k=1}^n dm(z_k)$$

as the "induced measure on the eigenvalues". We can however make sense of similar statements as explained below.

Lemma 1.1.1 and Fact 1.1.4 give a technical intuition as to why zeros of random analytic functions as well as eigenvalues of random matrices often exhibit repulsion. To make genuine probability statements however, we would have to endow the coefficients (or entries) with a probability distribution and use the Jacobian determinant to compute the distribution of zeros (or eigenvalues). In very special cases, one can get an explicit and useful answer, often of the kind

$$(1.1.4) \quad \prod_{i < j} |z_i - z_j|^2 \prod_k e^{-V(z_k)} \prod_{k=1}^n dm(z_k) = \exp \left\{ - \left[\sum_{k=1}^n V(z_k) - \sum_{i \neq j} \log |z_i - z_j| \right] \right\} \prod_{k=1}^n dm(z_k).$$

This density may be regarded as a one component plasma with external potential V and at a particular temperature (see Remark 1.1.6 below). Alternately one may regard it as a "determinantal point process". However it should be pointed out that in most cases, the distribution of zeros (or eigenvalues) is not exactly of this form, and then it is not to be hoped that one can get any explicit and tractable expression of the density. Nevertheless the property of repulsion is generally valid at short distances. Figure 2 shows a determinantal process and a process of zeros of a random analytic function both having the same intensity (the average number of points per unit area).

REMARK 1.1.6. Let us make precise the notion of a one component plasma of n particles with unit charge in the plane with potential V and temperature β^{-1} . This is just the probability density (with respect to Lebesgue measure on \mathbb{C}^n) proportional to

$$\exp \left\{ - \frac{\beta}{2} \left[\sum_{k=1}^n V(z_k) - \sum_{j \neq k} \log |z_j - z_k| \right] \right\} \prod_{k=1}^n dm(z_k).$$

This expression fits the statistical mechanical paradigm, namely it is of the form $\exp\{-\beta H(x)\}$, where H has the interpretation of the energy of a configuration and $1/\beta$ has the physical interpretation of temperature. In our setting we have

$$(1.1.5) \quad H(z_1, \dots, z_n) = \sum_{k=1}^n V(z_k) - \sum_{j \neq k} \log |z_j - z_k|.$$

If we consider n unit negative charges placed in an external potential V at locations z_1, \dots, z_n , then the first term gives the total potential energy due the external field and the second term the energy due to repulsion between the charges. According to Coulomb's law, in three dimensional space the electrical potential due to a point charge is proportional to the inverse distance from the charge. Since we are in two dimensions, the appropriate potential is $\log |z - w|$, which is the Green's function for the Laplacian on \mathbb{R}^2 . However in the density (1.1.4) that (sometimes) comes from random matrices, the temperature parameter is set equal to the particular value $\beta = 2$, which correspond to determinantal processes. Surprisingly, this particular case is much easier to analyse as compared to other values of β !

We study here two kinds of processes (determinantal and zero sets), focusing particularly on specific examples that are invariant under a large group of transformations of the underlying space (translation-invariance in the plane, for instance). Moreover there are certain very special cases of random analytic functions, whose zero sets turn out to be determinantal and we study them in some detail. Finally, apart from these questions of exact distributional calculations, we also present results on large deviations, central limit theorems and also (in a specific case) the stochastic geometry of the zeros. In the rest of the chapter we define some basic notions needed throughout, and give a more detailed overview of the contents of the book.

1.2. Basic notions and definitions

Now we give precise definitions of the basic concepts that will be used throughout the book. Let Λ be a locally compact Polish space (i.e., a topological space that can be topologized by a complete and separable metric). Let μ be a Radon measure on Λ (recall that a Radon measure is a Borel measure which is finite on compact sets). For all examples of interest it suffices to keep the following two cases in mind.

- Λ is an open subset of R^d and μ is the d -dimensional Lebesgue measure restricted to Λ .
- Λ is a finite or countable set and μ assigns unit mass to each element of Λ (the **counting measure** on Λ).

Our point processes (to be defined) will have points in Λ and μ will be a reference measure with respect to which we shall express the probability densities and other similar quantities. So far we informally defined a point process to be a random discrete subset of Λ . However the standard setting in probability theory is to have a sample space that is a complete separable metric space and the set of all discrete subsets of Λ is not such a space, in general. However, a discrete subset of Λ may be identified with the counting measure on the subset (the Borel measure on Λ that assigns unit mass to each element of the subset), and therefore we may define a point process as a random variable taking values in the space $\mathcal{M}(\Lambda)$ of sigma-finite Borel measures on Λ . This latter space is well-known to be a complete separable metric space (see (67), for example).

A **point process** \mathcal{X} on Λ is a random integer-valued positive Radon measure on Λ . If \mathcal{X} almost surely assigns at most measure 1 to singletons, it is a **simple** point process; in this case \mathcal{X} can be identified with a random discrete subset of Λ , and $\mathcal{X}(D)$ represents the number of points of this set that fall in D .

How does one describe the distribution of a point process? Given any $m \geq 1$, any Borel sets D_1, \dots, D_m of Λ , and open intervals $I_1, \dots, I_m \subset [0, \infty)$, we define a subset of $\mathcal{M}(\Lambda)$ consisting of all measures θ such that $\theta(D_k) \in I_k$, for each $k \leq m$. These are called cylinder sets and they generate the sigma field on $\mathcal{M}(\Lambda)$. Therefore, the distribution of a point process \mathcal{X} is determined by the probabilities of cylinder sets, i.e., by the numbers $\mathbf{P}[\mathcal{X}(D_k) = n_k, 1 \leq k \leq m]$ for Borel subsets D_1, \dots, D_m of Λ .

Conversely, one may *define* a point process by consistently assigning probabilities to cylinder sets. Consistency means that

$$\sum_{0 \leq n_m \leq \infty} \mathbf{P}[\mathcal{X}(D_k) = n_k, 1 \leq k \leq m]$$

should be the same as $\mathbf{P}[\mathcal{X}(D_k) = n_k, 1 \leq k \leq m-1]$. (Of course, the usual properties of finite additivity should hold as should the fact that these numbers are between zero and one!). For example the Poisson process may be defined in this manner.

EXAMPLE 1.2.1. For $m \geq 1$ and mutually disjoint Borel subsets $D_k, 1 \leq k \leq m$, of Λ , let

$$p((D_1, n_1), \dots, (D_m, n_m)) = \prod_{k=1}^m e^{-\mu(D_k)} \frac{\mu(D_k)^{n_k}}{n_k!}.$$

The right hand side is to be interpreted as zero if at least one of the D_k s has infinite μ -measure. Then Kolmogorov's existence theorem asserts that there exists a point process \mathcal{X} such that

$$\mathbf{P}[\mathcal{X}(D_k) = n_k, 1 \leq k \leq m] = p((D_1, n_1), \dots, (D_m, n_m)).$$

This is exactly what we informally defined as the Poisson process with intensity measure μ .

Nevertheless, specifying the joint distributions of the counts $\mathcal{X}(D)$, $D \subset \Lambda$ may not be the simplest or the most useful way to define or to think about the distribution of a point process. Alternately, the distribution of a point process can be described by its **joint intensities** (also known as **correlation functions**). We give the definition for simple point processes only, but see remark 1.2.3 for trick to extend the same to general point processes.

DEFINITION 1.2.2. Let \mathcal{X} be a simple point process. The joint intensities of a point process \mathcal{X} w.r.t. μ are functions (if any exist) $\rho_k : \Lambda^k \rightarrow [0, \infty)$ for $k \geq 1$, such that for any family of mutually disjoint subsets D_1, \dots, D_k of Λ ,

$$(1.2.1) \quad \mathbf{E} \left[\prod_{i=1}^k \mathcal{X}(D_i) \right] = \int_{\prod_i D_i} \rho_k(x_1, \dots, x_k) d\mu(x_1) \dots d\mu(x_k).$$

In addition, we shall require that $\rho_k(x_1, \dots, x_k)$ vanish if $x_i = x_j$ for some $i \neq j$.

As joint intensities are used extensively throughout the book, we spend the rest of the section clarifying various points about their definition.

The first intensity is the easiest to understand - we just define the measure $\mu_1(D) := \mathbf{E}[\mathcal{X}(D)]$, we call it the **first intensity measure** of \mathcal{X} . If it happens to be absolutely continuous to the given measure μ , then the Radon Nikodym derivative

ρ_1 is called the first intensity function. From definition 1.2.2 it may appear that the k -point intensity measure μ_k is the first intensity measure of $\mathcal{X}^{\otimes k}$ (the k -fold product measure on Λ^k) and that the k -point intensity function is the Radon Nikodym derivative of μ_k with respect to $\mu^{\otimes k}$, in cases when μ_k is absolutely continuous to $\mu^{\otimes k}$. However, this is incorrect, because (1.2.1) is valid only for pairwise disjoint D_i s. For general subsets of Λ^k , for example, $D_1 \times \dots \times D_k$ with overlapping D_i s, the situation is more complicated as we explain now.

REMARK 1.2.3. Restricting attention to simple point processes, ρ_k is not the intensity measure of \mathcal{X}^k , but that of $\mathcal{X}^{\wedge k}$, the set of ordered k -tuples of distinct points of \mathcal{X} . First note that (1.2.1) by itself does not say anything about ρ_k on the diagonals, that is, for (x_1, \dots, x_k) with $x_i = x_j$ for some $i \neq j$. That is why we added to the definition, the requirement that ρ_k vanish on the diagonal. Then, as we shall explain, equation (1.2.1) implies that for any Borel set $B \subset \Lambda^k$ we have

$$(1.2.2) \quad \mathbf{E} \#(B \cap \mathcal{X}^{\wedge k}) = \int_B \rho_k(x_1, \dots, x_k) d\mu(x_1) \dots d\mu(x_k).$$

When $B = \prod_{i=1}^r D_i^{\otimes k_i}$ for a mutually disjoint family of subsets D_1, \dots, D_r of Λ , and $k = \sum_{i=1}^r k_i$, the left hand side becomes

$$(1.2.3) \quad \mathbf{E} \left[\prod_{i=1}^r \binom{\mathcal{X}(D_i)}{k_i} k_i! \right].$$

For a general point process \mathcal{X} , observe that it can be identified with a simple point process \mathcal{X}^* on $\Lambda \times \{1, 2, 3, \dots\}$ such that $\mathcal{X}^*(D \times \{1, 2, 3, \dots\}) = \mathcal{X}(D)$ for Borel $D \subset \Lambda$. This way, one can deduce many facts about non-simple point processes from those for simple ones.

But why are (1.2.2) and (1.2.3) valid for a simple point process? It suffices to prove the latter. To make the idea transparent, we shall assume that Λ is a countable set and that μ is the counting measure and leave the general case to the reader (consult (53; 54; 68) for details). For simplicity, we restrict to $r = 1$ and $k_1 = 2$ in (1.2.3) and again leave the general case to the reader. We begin by computing $\mathbf{E}[\mathcal{X}(D)^2]$.

$$\begin{aligned} \mathbf{E}[\mathcal{X}(D)^2] &= \mathbf{E} \left[\left(\sum_{x \in D} \mathcal{X}(\{x\}) \right)^2 \right] \\ &= \mathbf{E} \left[\sum_{x \in D} \mathcal{X}(\{x\}) + \sum_{x \neq y} \mathbf{E}[\mathcal{X}(\{x\})\mathcal{X}(\{y\})] \right] \\ &= \mathbf{E}[\mathcal{X}(D)] + \int_{D \times D} \rho_2(x, y) d\mu(x) d\mu(y). \end{aligned}$$

Here we used two facts. Firstly, $\mathcal{X}(\{x\})$ is 0 or 1 (and 0 for all but finitely many $x \in D$) and secondly, from (1.2.1), for $x \neq y$ we get $\mathbf{E}[\mathcal{X}(\{x\})\mathcal{X}(\{y\})] = \rho_2(x, y)$ while $\rho_2(x, x) = 0$ for all x . Thus

$$(1.2.4) \quad \mathbf{E}[\mathcal{X}(D)(\mathcal{X}(D) - 1)] = \int_{D \times D} \rho_2(x, y) d\mu(x) d\mu(y)$$

as claimed.

Do joint intensities determine the distribution of a point process? The following remark says yes, under certain restrictions.

REMARK 1.2.4. Suppose that $\mathcal{X}(D)$ has exponential tails for all compact $D \subset \Lambda$. In other words, for every compact D , there is a constant $c > 0$ such that $\mathbf{P}[\mathcal{X}(D) > k] \leq e^{-ck}$ for all $k \geq 1$. We claim that under this assumption, the joint intensities (provided they exist) determine the law of \mathcal{X} .

This is because exponential tails for $\mathcal{X}(D)$ for any compact D ensures that for any compact D_1, \dots, D_k , the random vector $(\mathcal{X}(D_1), \dots, \mathcal{X}(D_k))$ has a convergent Laplace transform in a neighbourhood of 0. That is, for some $\epsilon > 0$ and any $s_1, \dots, s_k \in (-\epsilon, \epsilon)$, we have

$$(1.2.5) \quad \mathbf{E}[\exp\{s_1 \mathcal{X}(D_1) + \dots + s_k \mathcal{X}(D_k)\}] < \infty.$$

The Laplace transform determines the law of a random variable and is in turn determined by the moments, whence the conclusion. For these basic facts about moments and Laplace transform consult Billingsley's book (6).

Joint intensities are akin to densities: Assume that \mathcal{X} is simple. Then, the joint intensity functions may be interpreted as follows.

- If Λ is finite and μ is the *counting measure* on Λ , i.e., the measure that assigns unit mass to each element of Λ , then for distinct x_1, \dots, x_k , the quantity $\rho_k(x_1, \dots, x_k)$ is just the probability that $x_1, \dots, x_k \in \mathcal{X}$.
- If Λ is open in \mathbb{R}^d and μ is the Lebesgue measure, then for distinct x_1, \dots, x_k , and $\epsilon > 0$ small enough so that the balls $B_\epsilon(x_j)$ are mutually disjoint, by definition 1.2.2, we get

$$(1.2.6) \quad \int_{\prod_{j=1}^k B_\epsilon(x_j)} \rho_k(y_1, \dots, y_k) \prod_{j=1}^k dm(y_j) = \mathbf{E} \left[\prod_{j=1}^k \mathcal{X}(B_\epsilon(x_j)) \right] \\ = \sum_{\substack{(n_j) \\ n_j \geq 1}} \mathbf{P}(\mathcal{X}(B_\epsilon(x_j)) = n_j, j \leq k) \prod_{j=1}^k n_j.$$

In many examples the last sum is dominated by the term $n_1 = \dots = n_k = 1$. For instance, if we assume that for any compact K , the power series

$$(1.2.7) \quad \sum_{(n_j): j \leq k} \max\{\rho_{n_1+\dots+n_k}(t_1, \dots, t_{n_1+\dots+n_k}) : t_i \in K\} \frac{z_1^{n_1} \dots z_k^{n_k}}{n_1! \dots n_k!}$$

converges for z_i in a neighbourhood of 0, then it follows that for $n_j \geq 1$, by (1.2.2) and (1.2.3) that if $B_\epsilon(x_j) \subset K$ for $j \leq k$, then

$$\mathbf{P}(\mathcal{X}(B_\epsilon(x_j)) = n_j, j \leq k) \leq \mathbf{E} \left[\prod_{j=1}^k \binom{\mathcal{X}(B_\epsilon(x_j))}{n_j} \right] \\ = \frac{1}{n_1! \dots n_k!} \int_{B_\epsilon(x_1)^{n_1} \times \dots \times B_\epsilon(x_k)^{n_k}} \rho_{n_1+\dots+n_k}(y_1, \dots, y_{n_1+\dots+n_k}) \prod_j dm(y_j) \\ \leq \frac{\max\{\rho_{n_1+\dots+n_k}(t_1, \dots, t_{n_1+\dots+n_k}) : t_i \in K\}}{n_1! \dots n_k!} \prod_{j=1}^k m(B_\epsilon)^{n_j}.$$

Under our assumption 1.2.7, it follows that the term $\mathbf{P}(\mathcal{X}(B_\epsilon(x_j)) = 1, j \leq k)$ dominates the sum in (1.2.6). Further, as ρ_k is locally integrable, a.e.

(x_1, \dots, x_k) is a Lebesgue point and for such points we get

$$(1.2.8) \quad \rho_k(x_1, \dots, x_k) = \lim_{\epsilon \rightarrow 0} \frac{\mathbf{P}(\mathcal{X} \text{ has a point in } B_\epsilon(x_j) \text{ for each } j \leq k)}{m(B_\epsilon)^k}.$$

If a continuous version of ρ_k exists, then (1.2.8) holds for every $x_1, \dots, x_k \in \Lambda$. The following exercise demonstrates that for simple point processes with a deterministic finite total number of points, the joint intensities are determined by the top correlation (meaning k -point intensity for the largest k for which it is not identically zero). This fails if the number of points is random or infinite.

EXERCISE 1.2.5. (1) Let X_1, \dots, X_n be exchangeable real valued random variables with joint density $p(x_1, \dots, x_n)$ with respect to Lebesgue measure on \mathbb{R}^n . Let $\mathcal{X} = \sum \delta_{X_i}$ be the point process on \mathbb{R} that assigns unit mass to each X_i . Then show that the joint intensities of \mathcal{X} are given by

$$(1.2.9) \quad \rho_k(x_1, \dots, x_k) = \frac{n!}{(n-k)!} \int_{\mathbb{R}^{n-k}} p(x_1, \dots, x_n) dx_{k+1} \dots dx_n.$$

(2) Construct two simple point process on $\Lambda = \{1, 2, 3\}$ that have the same two-point intensities but not the same one-point intensities.

Moments of linear statistics: Joint intensities will be used extensively throughout the book. Therefore we give yet another way to understand them, this time in terms of **linear statistics**. If \mathcal{X} is a point process on Λ , and $\varphi : \Lambda \rightarrow \mathbb{R}$ is a measurable function, then the random variable

$$(1.2.10) \quad \mathcal{X}(\varphi) := \int_{\Lambda} \varphi d\mathcal{X} = \sum_{\alpha \in \Lambda} \varphi(\alpha) \mathcal{X}(\{\alpha\})$$

is called a linear statistic. If $\varphi = \mathbf{1}_D$ for some $D \subset \Lambda$, then $\mathcal{X}(\varphi)$ is just $\mathcal{X}(D)$.

Knowing the joint distributions of $\mathcal{X}(\varphi)$ for a sufficiently rich class of test functions φ , one can recover the distribution of the point process. For instance, the class of all indicator functions of compact subsets of Λ is rich enough, as explained earlier. Another example is the class of compactly supported continuous functions on Λ . Joint intensities determine the moments of linear statistics corresponding to indicator functions, as made clear in definition 1.2.2 and remark 1.2.4. Now we show how moments of any linear statistics can be expressed in terms of joint intensities. This is done below, but we state it so as to make it into an alternative definition of joint intensities. This is really a more detailed explanation of remark 1.2.3.

Let \mathcal{X} be a point process on Λ and let $C_c(\Lambda)$ be the space of compactly supported continuous functions on Λ . As always, we have a Radon measure μ on Λ .

(1) Define $T_1(\varphi) = \mathbf{E}[\mathcal{X}(\varphi)]$. Then, T_1 is a positive linear functional on $C_c(\Lambda)$. By Riesz's representation theorem, there exists a unique positive regular Borel measure μ_1 such that

$$(1.2.11) \quad T_1(\varphi) = \int \varphi d\mu_1.$$

The measure μ_1 is called the **first intensity measure** of \mathcal{X} .

If it happens that μ_1 is absolutely continuous to μ , then we write $d\mu_1 = \rho_1 d\mu$ and call ρ_1 the first intensity function of \mathcal{X} (with respect to the measure μ). We leave it to the reader to check that this coincides with ρ_1 in definition 1.2.2.

- (2) Define a positive bilinear functional on $C_c(\Lambda) \times C_c(\Lambda)$ by $T_2(\varphi, \psi) = \mathbf{E} [\mathcal{X}(\varphi)\mathcal{X}(\psi)]$ which induces a positive linear functional on $C_c(\Lambda^2)$. Hence, there is a unique positive regular Borel measure $\tilde{\mu}_2$ on Λ^2 such that

$$T_2(\varphi, \psi) = \int_{\Lambda^2} \varphi(x)\psi(y)d\tilde{\mu}_2(x, y).$$

However, in general $\tilde{\mu}_2$ should not be expected to be absolutely continuous to $\mu \otimes \mu$. This is because the random measure $\mathcal{X} \otimes \mathcal{X}$ has atoms on the diagonal $\{(x, x) : x \in \Lambda\}$. In fact,

$$(1.2.12) \quad \mathbf{E} [\mathcal{X}(\varphi)\mathcal{X}(\psi)] = \mathbf{E} [\mathcal{X}(\varphi\psi)] + \mathbf{E} \left[\sum_{(x,y) \in \Lambda^2} \varphi(x)\psi(y)\mathbf{1}_{x \neq y} \mathcal{X}(\{x\})\mathcal{X}(\{y\}) \right].$$

Both terms define positive bilinear functionals on $C_c(\Lambda) \times C_c(\Lambda)$ and are represented by two measures $\hat{\mu}_2$ and μ_2 that are supported on the diagonal $D := \{(x, x) : x \in \Lambda\}$ and $\Lambda^2 \setminus D$, respectively. Naturally, $\tilde{\mu}_2 = \mu_2 + \hat{\mu}_2$.

The measure $\hat{\mu}_2$ is singular with respect to $\mu \otimes \mu$ and is in fact the same as the first intensity measure μ_1 , under the natural identification of D with Λ . The second measure μ_2 is called the **two point intensity measure** of \mathcal{X} and if it so happens that μ_2 is absolutely continuous to $\mu \otimes \mu$, then its Radon-Nikodym derivative $\rho_2(x, y)$ is called the two point intensity function. The reader may check that this coincides with the earlier definition. For an example where the second intensity measure is not absolutely continuous to $\mu \otimes \mu$, look at the point process $\mathcal{X} = \delta_a + \delta_{a+1}$ on \mathbb{R} , where a has $N(0, 1)$ distribution.

- (3) Continuing, for any $k \geq 1$ we define a positive multilinear functional on $C_c(\Lambda)^k$ by

$$(1.2.13) \quad T_k(\psi_1, \dots, \psi_k) = \mathbf{E} \left[\prod_{i=1}^k \mathcal{X}(\psi_i) \right]$$

which induces a linear functional on $C_c(\Lambda)^{\otimes k}$ and hence, is represented by a unique positive regular Borel measure $\tilde{\mu}_k$ on Λ^k . We write $\tilde{\mu}_k$ as $\hat{\mu}_k + \mu_k$, where $\hat{\mu}_k$ is supported on the diagonal $D_k = \{(x_1, \dots, x_k) : x_i = x_j \text{ for some } i \neq j\}$ and μ_k is supported on the complement of the diagonal in Λ^k . We call μ_k the **k point intensity measure** and if it happens to be absolutely continuous to $\mu^{\otimes k}$, then we refer to its Radon Nikodym derivative as the k -point intensity function. This agrees with our earlier definition.

1.3. Hints and solutions

Exercise 1.1.2 Consider $\begin{bmatrix} A & B \\ -B & A \end{bmatrix}$. Multiply the second row by i and add to the first to get $\begin{bmatrix} A-iB & B+iA \\ -B & A \end{bmatrix}$. Then multiply the first column by $-i$ and add to the second to get $\begin{bmatrix} A-iB & 0 \\ -B & A+iB \end{bmatrix}$. Since both these operations do not change the determinant, we see that the original matrix has determinant equal to $\det(A-iB)\det(A+iB) = |\det(A-iB)|^2$.

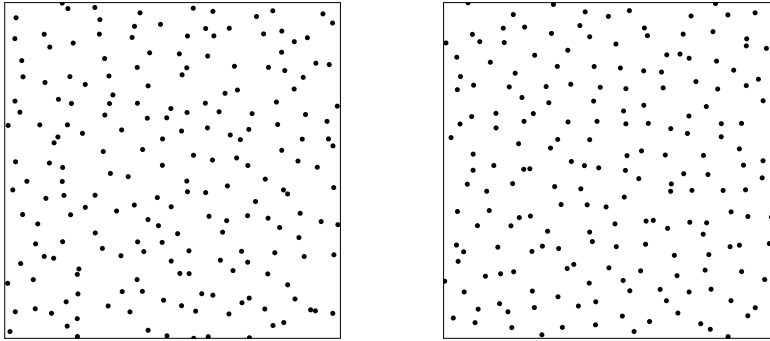


FIGURE 2. Samples of a translation invariant determinantal process (left) and zeros of a Gaussian analytic function. Determinantal processes exhibit repulsion at all distances, and the zeros repel at short distances only. However, the distinction is not evident in the pictures.

Gaussian Analytic Functions

2.1. Complex Gaussian distribution

Throughout this book, we shall encounter complex Gaussian random variables. As conventions vary, we begin by establishing our terminology. By $N(\mu, \sigma^2)$, we mean the distribution of the real-valued random variable with probability density $\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$. Here $\mu \in \mathbb{R}$ and $\sigma^2 > 0$ are the mean and variance respectively.

A **standard complex Gaussian** is a complex-valued random variable with probability density $\frac{1}{\pi} e^{-|z|^2}$ w.r.t the Lebesgue measure on the complex plane. Equivalently, one may define it as $X+iY$, where X and Y are i.i.d. $N(0, \frac{1}{2})$ random variables.

Let a_k , $1 \leq k \leq n$ be i.i.d. standard complex Gaussians. Then we say that $\mathbf{a} := (a_1, \dots, a_n)^t$ is a standard complex Gaussian vector. Then if B is a (complex) $m \times n$ matrix, $B\mathbf{a} + \mu$ is said to be an m -dimensional complex Gaussian vector with mean μ (an $m \times 1$ vector) and covariance $\Sigma = BB^*$ (an $m \times m$ matrix). We denote its distribution by $N_{\mathbb{C}}^m(\mu, \Sigma)$.

- EXERCISE 2.1.1. i. Let U be an $n \times n$ unitary matrix, i.e. $UU^* = I_n$, (here U^* is the conjugate transpose of U), and \mathbf{a} an n -dimensional standard complex Gaussian vector. Show that $U\mathbf{a}$ is also an n -dimensional standard complex Gaussian vector.
- ii. Show that the mean and covariance of a complex Gaussian random vector determines its distribution.

REMARK 2.1.2. Although a complex Gaussian can be defined as one having i.i.d. $N(0, \frac{1}{2})$ real and imaginary parts, we advocate thinking of it as a single entity, if not to think of a real Gaussian as merely the real part of a complex Gaussian! Indeed, one encounters the complex Gaussian variable in basic probability courses, for instance in computing the normalizing constant for the density $e^{-x^2/2}$ on the line (by computing the normalizing constant for a complex Gaussian and then taking square roots); and also in generating a random normal on the computer (by generating a complex Gaussian and taking its real part). The complex Gaussian is sometimes easier to work with because it can be represented as a pair of independent random variables in two co-ordinate systems, Cartesian as well as polar (as explained below in more detail). At a higher level, in the theory of random analytic functions and random matrix theory, it is again true that many more exact computations are possible when we use complex Gaussian coefficients (or entries) than when real Gaussians are used.

Here are some other basic properties of complex Gaussian random variables.

- If \mathbf{a} has $N_{\mathbb{C}}^m(\mu, \Sigma)$ distribution, then for every $j, k \leq n$ (not necessarily distinct), we have

$$\mathbf{E}[(a_k - \mu_k)(a_j - \mu_j)] = 0 \text{ and } \mathbf{E}[(a_j - \mu_j)\overline{(a_k - \mu_k)}] = \Sigma_{j,k}.$$

- If a is a standard complex Gaussian, then $|a|^2$ and $\frac{a}{|a|}$ are independent, and have exponential distribution with mean 1 and uniform distribution on the circle $\{z : |z| = 1\}$, respectively.
- Suppose \mathbf{a} and \mathbf{b} are m and n -dimensional random vectors such that

$$\begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} \sim N_{\mathbb{C}}^{m+n} \left(\begin{bmatrix} \mu \\ \nu \end{bmatrix}, \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \right),$$

where the mean vector and covariance matrices are partitioned in the obvious way. Then Σ_{11} and Σ_{22} are Hermitian, while $\Sigma_{12}^* = \Sigma_{21}$. Assume that Σ_{11} is non-singular. Then the distribution of \mathbf{a} is $N_{\mathbb{C}}^m(\mu, \Sigma_{11})$ and the conditional distribution of \mathbf{b} given \mathbf{a} is

$$N_{\mathbb{C}}^n(\nu + \Sigma_{21}\Sigma_{11}^{-1}(\mathbf{a} - \mu), \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}).$$

EXERCISE 2.1.3. Prove this.

- Weak limits of complex Gaussians are complex Gaussians. More precisely,

EXERCISE 2.1.4. If \mathbf{a}_n has $N_{\mathbb{C}}(\mu_n, \Sigma_n)$ distribution and $\mathbf{a}_n \xrightarrow{d} \mathbf{a}$, then $\{\mu_n\}$ and $\{\Sigma_n\}$ must converge, say to μ and Σ , and \mathbf{a} must have $N_{\mathbb{C}}(\mu, \Sigma)$ distribution.

Conversely, if $\{\mu_n\}$ and $\{\Sigma_n\}$ converge to μ and Σ , then \mathbf{a}_n converges weakly to $N_{\mathbb{C}}(\mu, \Sigma)$ distribution.

- The moments of products of complex Gaussians can be computed in terms of the covariance matrix by the Wick or the Feynman diagram formula. First we recall the notion of “permanent” of a matrix, well-known to combinatorists but less ubiquitous in mathematics than its more famous sibling, the determinant.

DEFINITION 2.1.5. For an $n \times n$ matrix M , its permanent, denoted $\text{per}(M)$ is defined by

$$\text{per}(M) = \sum_{\pi \in \mathbb{S}_n} \prod_{k=1}^n M_{k\pi_k}.$$

The sum is over all permutations of $\{1, 2, \dots, n\}$.

REMARK 2.1.6. The analogy with the determinant is clear - the signs of the permutations have been omitted in the definition. But note that this makes a huge difference in that $\text{per}(A^{-1}MA)$ is not in general equal to $\text{per}(M)$. This means that the permanent is a basis-dependent notion and thus has no geometric meaning unlike the determinant. As such, it can be expected to occur only in those contexts where the entries of the matrices themselves are important, as often happens in combinatorics and also in probability.

Now we return to computing moments of products of complex Gaussians. The books of Janson (38) or Simon (77) have such formulas, also in the real Gaussian case.

LEMMA 2.1.7 (Wick formula). *Let $(\mathbf{a}, \mathbf{b}) = (a_1, \dots, a_n, b_1, \dots, b_n)^t$ have $N_{\mathbb{C}}(\mathbf{0}, \Sigma)$ distribution, where*

$$(2.1.1) \quad \Sigma = \begin{bmatrix} \Sigma_{1,1} & \Sigma_{1,2} \\ \Sigma_{2,1} & \Sigma_{2,2} \end{bmatrix}.$$

Then,

$$(2.1.2) \quad \mathbf{E} \left[a_1 \cdots a_n \bar{b}_1 \cdots \bar{b}_n \right] = \text{per}(\Sigma_{1,2}).$$

In particular

$$\mathbf{E} \left[|a_1 \cdots a_n|^2 \right] = \text{per}(\Sigma_{1,1}).$$

PROOF. First we prove that

$$\mathbf{E}[a_1 \cdots a_n \bar{b}_1 \cdots \bar{b}_n] = \sum_{\pi} \prod_{j=1}^k \mathbf{E} a_j \bar{b}_{\pi(j)} = \text{per} \left(\mathbf{E} a_j \bar{b}_k \right)_{jk},$$

where the sum is over all permutations $\pi \in S_n$. Both sides are linear in each a_j and \bar{b}_j , and we may assume that the a_j, b_j are complex linear combinations of some finite i.i.d. standard complex Gaussian sequence $\{V_j\}$. The formula is proved by induction on the total number of nonzero coefficients that appear in the expression of the a_j and b_j in terms of the V_j . If the number of nonzero coefficients is more than one for one of a_j or b_j , then we may write that variable as a sum and use induction and linearity. If it is 1 or 0 for all a_j, b_j , then the formula is straightforward to verify; in fact, using independence it suffices to check that $V = V_j$ has $\mathbf{E} V^n \bar{V}^m = n! \mathbf{1}_{\{m=n\}}$. For $n \neq m$ this follows from the fact that V has a rotationally symmetric distribution. Otherwise, $|V|^{2n}$ has the distribution of the n th power of a rate 1 exponential random variable, so its expectation equals $n!$.

The second statement follows immediately from the first, applied to the vector (\mathbf{a}, \mathbf{a}) . \square

- If $a_n, n \geq 1$ are i.i.d. $N_{\mathbb{C}}(0, 1)$, then

$$(2.1.3) \quad \limsup_{n \rightarrow \infty} |a_n|^{\frac{1}{n}} = 1, \quad \text{almost surely.}$$

In fact, equation (2.1.3) is valid for any i.i.d. sequence of complex valued random variables a_n , such that

$$(2.1.4) \quad \mathbf{E}[\max\{\log|a_1|, 0\}] < \infty, \text{ provided } \mathbf{P}[a_1 = 0] < 1.$$

We leave the proof as a simple exercise for the reader not already familiar with it. We shall need this fact later, to compute the radii of convergence of random power series with independent coefficients.

2.2. Gaussian analytic functions

Endow the space of analytic functions on a region $\Lambda \subset \mathbb{C}$ with the topology of uniform convergence on compact sets. This makes it a complete separable metric space which is the standard setting for doing probability theory (To see completeness, if $\{f_n\}$ is a Cauchy sequence, then f_n converges uniformly on compact sets to some continuous function f . Then Morera's theorem assures that that f must be analytic because its contour integral vanishes on any closed contour in Λ , since $\int f = \lim_{n \rightarrow \infty} \int f_n$ and the latter vanishes for every n by analyticity of f_n).

DEFINITION 2.2.1. Let \mathbf{f} be a random variable on a probability space taking values in the space of analytic functions on a region $\Lambda \subset \mathbb{C}$. We say \mathbf{f} is a Gaussian analytic function (GAF) on Λ if $(\mathbf{f}(z_1), \dots, \mathbf{f}(z_n))$ has a mean zero complex Gaussian distribution for every $n \geq 1$ and every $z_1, \dots, z_n \in \Lambda$.

It is easy to see the following properties of GAFs

- $\{\mathbf{f}^{(k)}\}$ are jointly Gaussian, i.e., the joint distribution of \mathbf{f} and finitely many derivatives of \mathbf{f} at finitely many points,

$$\{\mathbf{f}^{(k)}(z_j) : 0 \leq k \leq n, 1 \leq j \leq m\},$$

has a (mean zero) complex Gaussian distribution. (Hint: Weak limits of Gaussians are Gaussians and derivatives are limits of difference coefficients).

- For any $n \geq 1$ and any $z_1, \dots, z_n \in \Lambda$, the \mathbb{C}^n -valued random vector $(\mathbf{f}(z_1), \dots, \mathbf{f}(z_n))$ has a complex Gaussian distribution with mean zero and covariance matrix $(K(z_i, z_j))_{i,j \leq n}$. By Exercise 2.1.1 it follows that the covariance kernel K determines all the finite dimensional marginals of \mathbf{f} . Since \mathbf{f} is almost surely continuous, it follows that the distribution of \mathbf{f} is determined by K .
- Analytic extensions of GAFs are GAFs.

EXERCISE 2.2.2. In other words, if \mathbf{f} is a random analytic function on Λ and is Gaussian when restricted to a domain $D \subset \Lambda$, then \mathbf{f} is a GAF on the whole of Λ .

The following lemma gives a general recipe to construct Gaussian analytic functions.

LEMMA 2.2.3. Let ψ_n be holomorphic functions on Λ . Assume that $\sum_n |\psi_n(z)|^2$ converges uniformly on compact sets in Λ . Let a_n be i.i.d. random variables with zero mean and unit variance. Then, almost surely, $\sum_n a_n \psi_n(z)$ converges uniformly on compact subsets of Λ and hence defines a random analytic function.

In particular, if a_n has standard complex Gaussian distribution, then $\mathbf{f}(z) := \sum_n a_n \psi_n(z)$ is a GAF with covariance kernel $K(z, w) = \sum_n \psi_n(z) \overline{\psi_n(w)}$.

If (c_n) is any square summable sequence of complex numbers, and a_n s are i.i.d. with zero mean and unit variance, then $\sum c_n a_n$ converges almost surely, because by Kolmogorov's inequality

$$\begin{aligned} \mathbf{P} \left[\sup_{k \geq N} \left| \sum_{j=N}^k c_j a_j \right| \geq t \right] &\leq \frac{1}{t^2} \sum_{j=N}^{\infty} |c_j|^2 \\ &\rightarrow 0 \quad \text{as } N \rightarrow \infty. \end{aligned}$$

Thus, for fixed z , the series of partial sums for $\mathbf{f}(z)$ converge almost surely. However, it is not clear that the series converges for all z simultaneously, even for a single sample point. The idea of the proof is to regard $\sum a_n \psi_n$ as a Hilbert space valued series and prove a version of Kolmogorov's inequality for such series. This part is taken from chapter 3 of Kahane's book (42). That gives convergence in the Hilbert space, and by Cauchy's formulas we may deduce uniform convergence on compacta.

PROOF. Let K be any compact subset of Λ . Regard the sequence $X_n = \sum_{k=1}^n a_k \psi_k$ as taking values in $L^2(K)$ (with respect to Lebesgue measure). Let $\|\cdot\|^2$ denote the

norm in $L^2(K)$. It is easy to check that for any $k < n$ we have

$$(2.2.1) \quad \mathbf{E}[\|X_n\|^2 | a_j, j \leq k] = \|X_k\|^2 + \sum_{j=k+1}^n \|\psi_j\|^2.$$

Define the stopping time $\tau = \inf\{n : \|X_n\| > \epsilon\}$. Then,

$$\begin{aligned} \mathbf{E}[\|X_n\|^2] &\geq \sum_{k=1}^n \mathbf{E}[\|X_n\|^2 \mathbf{1}_{\tau=k}] \\ &= \sum_{k=1}^n \mathbf{E}[\mathbf{1}_{\tau=k} \mathbf{E}[\|X_n\|^2 | a_j, j \leq k]] \\ &\geq \sum_{k=1}^n \mathbf{E}[\mathbf{1}_{\tau=k} \|X_k\|^2] \quad \text{by (2.2.1)} \\ &\geq \epsilon^2 \mathbf{P}[\tau \leq n]. \end{aligned}$$

Thus

$$(2.2.2) \quad \mathbf{P}\left[\sup_{j \leq n} \|X_j\| \geq \epsilon\right] \leq \frac{1}{\epsilon^2} \sum_{j=1}^n \|\psi_j\|^2.$$

We have just proved Kolmogorov's inequality for Hilbert space valued random variables. Apply this to the sequence $\{X_{N+n} - X_N\}_n$ to get

$$(2.2.3) \quad \mathbf{P}\left[\sup_{m, n \geq N} \|X_m - X_n\| \geq 2\epsilon\right] \leq \mathbf{P}\left[\sup_{n \geq 1} \|X_{N+n} - X_N\| \geq \epsilon\right] \leq \frac{1}{\epsilon^2} \sum_{j=N+1}^{\infty} \|\psi_j\|^2$$

which converges to zero as $N \rightarrow \infty$. Thus

$$\mathbf{P}[\exists N \text{ such that } \forall n, \|X_{N+n} - X_N\| \leq \epsilon] = 1.$$

In other words, almost surely X_n is a Cauchy sequence in $L^2(K)$.

To show uniform convergence on compact subsets, consider any disk $D(z_0, 4R)$ contained in Λ . Since X_n is an analytic function on Λ for each n , Cauchy's formula says

$$(2.2.4) \quad X_n(z) = \frac{1}{2\pi i} \int_{C_r} \frac{X_n(\zeta)}{\zeta - z} d\zeta$$

where $C_r(t) = z_0 + r e^{it}$, $0 \leq t \leq 2\pi$ and $|z - z_0| < r$. For any $z \in D(z_0, R)$, average equation (2.2.4) over $r \in (2R, 3R)$ to deduce that

$$\begin{aligned} X_n(z) &= \frac{1}{2\pi i R} \int_{2R}^{3R} \int_0^{2\pi} \frac{X_n(z_0 + r e^{i\theta})}{z_0 + r e^{i\theta} - z} i e^{i\theta} d\theta dr \\ &= \frac{1}{2\pi} \int_A X_n(\zeta) \varphi_z(\zeta) dm(\zeta) \end{aligned}$$

where A denotes the annulus around z_0 of radii $2R$ and $3R$ and $\varphi_z(\cdot)$ is defined by the equality. The observation that we shall need is that the collection $\{\varphi_z\}_{z \in D(z_0, R)}$ is uniformly bounded in $L^2(A)$.

We proved that almost surely $\{X_n\}$ is a Cauchy sequence in $L^2(K)$ where $K := D(z_0, 4R)$. Therefore there exists $X \in L^2(K)$ such that $X_n \rightarrow X$ in $L^2(K)$. Therefore the integral above converges to $\frac{1}{2\pi} \int_A X(\zeta) \varphi_z(\zeta) dm(\zeta)$ uniformly over $z \in D(z_0, R)$.

Thus we conclude that $X_n \rightarrow X$ uniformly on compact sets in Λ and that X is an analytic function on Λ .

If a_n s are complex Gaussian, it is clear that X_n is a GAF for each n . Since limits of Gaussians are Gaussians, we see that X is also a GAF. The formula for the covariance $\mathbf{E}[\mathbf{f}(z)\overline{\mathbf{f}(w)}]$ is obvious. \square

2.3. Isometry-invariant zero sets

As explained in Chapter 1, our interest is in the zero set of a random analytic function. Unless one's intention is to model a particular physical phenomenon by a point process, there is one criterion that makes some point processes more interesting than others, namely, *invariance* under a large group of transformations (invariance of a measure means that its distribution does not change under the action of a group, i.e., symmetry). There are three particular two dimensional domains (up to conformal equivalence) on which the group of conformal automorphisms act transitively (There are two others that we do not consider here, the cylinder or the punctured plane, and the two dimensional torus). We introduce these domains now.

- **The Complex Plane \mathbb{C} :** The group of transformations

$$(2.3.1) \quad \varphi_{\lambda,\beta}(z) = \lambda z + \beta, \quad z \in \mathbb{C}$$

where $|\lambda| = 1$ and $\beta \in \mathbb{C}$, is nothing but the Euclidean motion group. These transformations preserve the Euclidean metric $ds^2 = dx^2 + dy^2$ and the Lebesgue measure $dm(z) = dx dy$ on the plane.

- **The Sphere \mathbb{S}^2 :** The group of rotations act transitively on the two-dimensional sphere. Moreover the sphere inherits a complex structure from the complex plane by stereographic projection which identifies the sphere with the extended complex plane. In this book we shall always refer to $\mathbb{C} \cup \{\infty\}$ as the sphere. The rotations of the sphere become linear fractional transformations mapping $\mathbb{C} \cup \{\infty\}$ to itself bijectively. That is, they are given by

$$(2.3.2) \quad \varphi_{\alpha,\beta}(z) = \frac{\alpha z + \beta}{-\beta z + \bar{\alpha}}, \quad z \in \mathbb{C} \cup \{\infty\}$$

where $\alpha, \beta \in \mathbb{C}$ and $|\alpha|^2 + |\beta|^2 = 1$. These transformations preserve the spherical metric $ds^2 = \frac{dx^2 + dy^2}{(1+|z|^2)^2}$ and the spherical measure $\frac{dm(z)}{(1+|z|^2)^2}$. It is called the spherical metric because it is the push forward of the usual metric (inherited from \mathbb{R}^3) on the sphere onto $\mathbb{C} \cup \{\infty\}$ under the stereographic projection, and the measure is the push forward of the spherical area measure.

EXERCISE 2.3.1. (i) Show that the transformations $\varphi_{\alpha,\beta}$ in (2.3.2) preserve the spherical metric and the spherical measure.

- (ii) Show that the radius and area of the disk $D(0, r)$ in the spherical metric and spherical measure are $\arctan(r)$ and $\frac{\pi r^2}{1+r^2}$, respectively.

- **The Hyperbolic Plane \mathbb{D} :** The group of transformations

$$(2.3.3) \quad \varphi_{\alpha,\beta}(z) = \frac{\alpha z + \beta}{\beta z + \bar{\alpha}}, \quad z \in \mathbb{D}$$

where $\alpha, \beta \in \mathbb{C}$ and $|\alpha|^2 - |\beta|^2 = 1$, is the group of linear fractional transformations mapping the unit disk $\mathbb{D} = \{z : |z| < 1\}$ to itself bijectively. These

transformations preserve the hyperbolic metric $ds^2 = \frac{dx^2+dy^2}{(1-|z|^2)^2}$ and the hyperbolic area measure $\frac{dm(z)}{(1-|z|^2)^2}$ (this normalization differs from the usual one, with curvature -1 , by a factor of 4, but it makes the analogy with the other two cases more formally similar). This is one of the many models for the hyperbolic geometry of Bolyai, Gauss and Lobachevsky (see (13) or (34) for an introduction to hyperbolic geometry).

EXERCISE 2.3.2. (i) Show that the transformations $\varphi_{\alpha,\beta}$ in (2.3.3) preserve the hyperbolic metric and the hyperbolic measure.

(ii) Show that the radius and area of the disk $D(0,r)$, $r < 1$ in the hyperbolic metric and hyperbolic measure are $\operatorname{arctanh}(r)$ and $\frac{\pi r^2}{1-r^2}$, respectively.

Note that in each case, the group of transformations acts transitively on the corresponding space, i.e., for every z, w in the domain, there is a transformation φ such that $\varphi(z) = w$. This means that in these spaces every point is just like every other point. Now we introduce three families of GAFs whose relation to these symmetric spaces will be made clear in Proposition 2.3.4.

In each case, the domain of the random analytic function can be found using Lemma 2.2.3 or directly from equation (2.1.3).

- **The Complex Plane \mathbb{C} :** Define for $L > 0$,

$$(2.3.4) \quad \mathbf{f}(z) = \sum_{n=0}^{\infty} a_n \frac{\sqrt{L^n}}{\sqrt{n!}} z^n.$$

For every $L > 0$, this is a random analytic function in the entire plane with covariance kernel $\exp\{Lz\bar{w}\}$.

- **The Sphere \mathbb{S}^2 :** Define for $L \in \mathbb{N} = \{1, 2, 3, \dots\}$,

$$(2.3.5) \quad \mathbf{f}(z) = \sum_{n=0}^L a_n \frac{\sqrt{L(L-1)\dots(L-n+1)}}{\sqrt{n!}} z^n.$$

For every $L \in \mathbb{N}$, this is a random analytic function on the complex plane with covariance kernel $(1+z\bar{w})^L$. Since it is a polynomial, we may also think of it as an analytic function on $\mathbb{S}^2 = \mathbb{C} \cup \{\infty\}$ with a pole at ∞ .

- **The Hyperbolic Plane \mathbb{D} :** Define for $L > 0$,

$$(2.3.6) \quad \mathbf{f}(z) = \sum_{n=0}^{\infty} a_n \frac{\sqrt{L(L+1)\dots(L+n-1)}}{\sqrt{n!}} z^n.$$

For every $L > 0$, this is a random analytic function in the unit disk $\mathbb{D} = \{z : |z| < 1\}$ with covariance kernel $(1-z\bar{w})^{-L}$. When L is not an integer, the question of what branch of the fractional power to take, is resolved by the requirement that $K(z, z)$ be positive.

It is natural to ask whether the unit disk is the natural domain for the hyperbolic GAF or if it has an analytic continuation to a larger region. To see that almost surely it does not extend to any larger open set, consider an open disk D intersecting \mathbb{D} but not contained in \mathbb{D} , and let C_D be the event that there exists an analytic continuation of \mathbf{f} to $\mathbb{D} \cup D$. Note that C_D is a tail event, and therefore by Kolmogorov's zero-one law, if it has positive probability then it occurs almost surely. If $\mathbf{P}(C_D) = 1$ for some D , then by the rotational symmetry of complex Gaussian distribution, we see

that $\mathbf{P}(C_{e^{i\theta}D}) = 1$ for any $\theta \in [0, 2\pi]$. Choose finitely many rotations of D so that their union contains the unit circle. With probability 1, \mathbf{f} extends to all of these rotates of D , whence we get an extension of \mathbf{f} to a disk of radius strictly greater than 1. But the radius of convergence is 1 a.s. Therefore $\mathbf{P}(C_D) = 0$ for any D , which establishes our claim.

Another argument is pointed out in the notes. However, these arguments used the rotational invariance of complex Gaussian distribution very strongly. One may adapt an argument given in Billingsley (6), p. 292 to give a more robust proof that works for any symmetric distribution of the coefficients (that is, $-a \stackrel{d}{=} a$).

LEMMA 2.3.3. *Let a_n be i.i.d. random variables with a symmetric distribution in the complex plane. Assume that conditions (2.1.4) hold. Then $\sum_{n=0}^{\infty} a_n \frac{\sqrt{L(L+1)\dots(L+n-1)}}{\sqrt{n!}} z^n$ does not extend analytically to any domain larger than the unit disk.*

PROOF. Assuming (2.1.4), Borel-Cantelli lemmas show that the radius of convergence is at most 1. We need to consider only the case when it is equal to 1. As before, suppose that $\mathbf{P}(C_D) = 1$ for some disk D intersecting the unit disk but not contained in it. Fix k large enough so that an arc of the unit circle of length $\frac{2\pi}{k}$ is contained in D and set

$$(2.3.7) \quad \tilde{a}_n = \begin{cases} a_n & \text{if } n \not\equiv 0 \pmod{k} \\ -a_n & \text{if } n \equiv 0 \pmod{k} \end{cases}.$$

Let

$$(2.3.8) \quad \tilde{\mathbf{f}}(z) = \sum_{n=0}^{\infty} \tilde{a}_n \frac{\sqrt{L(L+1)\dots(L+n-1)}}{\sqrt{n!}} z^n$$

and define \tilde{C}_D in the obvious way. Since $\tilde{\mathbf{f}} \stackrel{d}{=} \mathbf{f}$ it follows that $\mathbf{P}(C_D) = \mathbf{P}(\tilde{C}_D)$. Now suppose both these events have probability one so that the function

$$(2.3.9) \quad \mathbf{g}(z) \stackrel{def}{=} \mathbf{f}(z) - \tilde{\mathbf{f}}(z) = 2 \sum_{n=0}^{\infty} a_{kn} \frac{\sqrt{L(L+1)\dots(L+kn-1)}}{\sqrt{(kn)!}} z^{kn}$$

may be analytically extended to $\mathbb{D} \cup D$ almost surely. Replacing z by $ze^{2\pi i/k}$ leaves $\mathbf{g}(z)$ unchanged, hence \mathbf{g} can be extended to $\mathbb{D} \cup (\cup_{\ell} D_{\ell})$ where $D_{\ell} = e^{2\pi i \ell/k} D$. In particular, \mathbf{g} can be analytically extended to $(1+\epsilon)\mathbb{D}$ for some $\epsilon > 0$ which is impossible since \mathbf{g} has radius of convergence equal to one. We conclude that C_D has probability zero. \square

Next we prove that the zero sets of the above analytic functions are isometry-invariant.

PROPOSITION 2.3.4. *The zero sets of the GAF \mathbf{f} in equations (2.3.4), (2.3.5) and (2.3.6) are invariant (in distribution) under the transformations defined in equations (2.3.1), (2.3.2) and (2.3.3) respectively. This holds for every allowed value of the parameter L , namely $L > 0$ for the plane and the disk and $L \in \mathbb{N}$ for the sphere.*

PROOF. For definiteness, let us consider the case of the plane. Fix $L > 0$. Then

$$\mathbf{f}(z) = \sum_{n=0}^{\infty} a_n \frac{\sqrt{L^n}}{\sqrt{n!}} z^n,$$

is a centered (mean zero) complex Gaussian process, and as such, its distribution is characterized by its covariance kernel $\exp\{Lz\bar{w}\}$. Now consider the function obtained by translating \mathbf{f} by an isometry in (2.3.1), i.e., fix $|\lambda| = 1$ and $\beta \in \mathbb{C}$, and set

$$\mathbf{g}(z) = \mathbf{f}(\lambda z + \beta).$$

\mathbf{g} is also a centered complex Gaussian process with covariance kernel

$$\begin{aligned} K_{\mathbf{g}}(z, w) &= K_{\mathbf{f}}(\lambda z + \beta, \lambda w + \beta) \\ &= e^{Lz\bar{w} + Lz\lambda\bar{\beta} + L\bar{w}\beta\bar{\lambda} + L|\beta|^2}. \end{aligned}$$

If we set

$$\mathbf{h}(z) = \mathbf{f}(z)e^{Lz\lambda\bar{\beta} + \frac{1}{2}L|\beta|^2},$$

then it is again a centered complex Gaussian process. Its covariance kernel $K_{\mathbf{h}}(z, w)$ is easily checked to be equal to $K_{\mathbf{g}}(z, w)$. This implies that

$$(2.3.10) \quad \mathbf{f}(\lambda z + \beta) \stackrel{d}{=} \mathbf{f}(z)e^{Lz\lambda\bar{\beta} + \frac{1}{2}L|\beta|^2},$$

where the equality in distribution is for the whole processes (functions), not just for a fixed z . Since the exponential function on the right hand side has no zeros, it follows that the zeros of $\mathbf{f}(\lambda z + \beta)$ and the zeros of $\mathbf{f}(z)$ have the same distribution. This proves that the zero set is translationally invariant in distribution.

The proof in the other two cases is exactly the same. If \mathbf{f} is one of the GAFs under consideration, and φ is an isometry of the corresponding domain, then by computing the covariance kernels one can easily prove that

$$(2.3.11) \quad \mathbf{f}(\varphi(\cdot)) \stackrel{d}{=} \mathbf{f}(\cdot)\Delta(\varphi, \cdot),$$

where, $\Delta(\varphi, z)$ is a deterministic nowhere vanishing analytic function of z . That immediately implies the desired invariance of the zero set of \mathbf{f} .

The function $\Delta(\varphi, z)$ is given explicitly by (we are using the expression for φ from equations (2.3.1), (2.3.2) and (2.3.3) respectively).

$$\Delta(\varphi, z) = \begin{cases} e^{Lz\lambda\bar{\beta} + \frac{1}{2}L|\beta|^2} & \text{domain} = \mathbb{C}. \\ \varphi'(z)^{\frac{L}{2}} & \text{domain} = \mathbb{S}^2. \\ \varphi'(z)^{-\frac{L}{2}} & \text{domain} = \mathbb{D}. \end{cases}$$

It is important to notice the following two facts or else the above statements do not make sense.

- (1) In the case of the sphere, by explicit computation we can see that $\varphi'(z)$ is $(-\bar{\beta}z + \bar{\alpha})^{-2}$. Therefore one may raise φ' to half-integer powers and get (single-valued) analytic functions.
- (2) In the case of the disk, again by explicit computation we can see that $\varphi'(z)$ is $(\bar{\beta}z + \bar{\alpha})^{-2}$, but since L is any positive number, to raise φ' to the power $L/2$ we should notice that $\varphi'(z)$ does not vanish for z in the unit disk (because $|\alpha|^2 - |\beta|^2 = 1$). And hence, a holomorphic branch of $\log \varphi'$ may be chosen and thus we may define φ' to the power $L/2$.

□

We shall see later (remark 2.4.5) that the first intensity of zero sets for these canonical GAFs is not zero. Translation invariance implies that the expected number of zeros of the planar and hyperbolic GAFs is almost surely infinite. However, mere translation invariance leaves open the possibility that with positive probability there

are no zeros at all! We rule out this ridiculous possibility by showing that the zero set is in fact ergodic. We briefly recall the definition of ergodicity.

DEFINITION 2.3.5. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space and let G be a group of measure preserving transformations of Ω to itself, that is, $\mathbf{P} \circ \tau^{-1} = \mathbf{P}$ for every $\tau \in G$. An invariant event is a set $A \in \mathcal{F}$ such that $\tau(A) = A$ for every $\tau \in G$. The action of G is said to be **ergodic** if every invariant set has probability equal to zero or one. In this case we may also say that \mathbf{P} is ergodic under the transformations G .

EXAMPLE 2.3.6. Let \mathbf{P} be the distribution of the zero set of the planar GAF \mathbf{f} . Then by Proposition 2.3.4 we know that the Euclidean motion group acts in a measure preserving manner. The event that \mathbf{f} has infinitely many zeros is an invariant set. Another example is the event that

$$(2.3.12) \quad \lim_{a \rightarrow \infty} \frac{1}{4a^2} \{\text{Number of zeros of } \mathbf{f} \text{ in } [-a, a]^2\} = c$$

where c is a fixed constant. In Proposition 2.3.7 below, we shall see that the action of the translation group (and hence the whole motion group) is ergodic and hence all these invariant events have probability zero or one. We shall see later that the expected number of zeros is positive, which shows that the number of zeros is almost surely infinite. Similarly, the event in (2.3.12) has probability 1 for $c = 1/\pi$ and zero for any other c .

PROPOSITION 2.3.7. *The zero sets of the GAF \mathbf{f} in equations (2.3.4), and (2.3.6) are ergodic under the action of the corresponding isometry groups.*

PROOF. We show the details in the planar case ($\Lambda = \mathbb{C}$) with $L = 1$. The proof is virtually identical in the hyperbolic case. For $\beta \in \mathbb{C}$, let $\mathbf{f}_\beta(z) = \mathbf{f}(z + \beta)e^{-z\bar{\beta} - \frac{1}{2}|\beta|^2}$. We saw in the proof of Proposition 2.3.4 that $\mathbf{f}_\beta \stackrel{d}{=} \mathbf{f}$. We compute

$$\mathbf{E} \left[\mathbf{f}_\beta(z) \bar{\mathbf{f}}(w) \right] = e^{-z\bar{\beta} - \frac{1}{2}|\beta|^2 + z\bar{w} + \beta\bar{w}}.$$

As $\beta \rightarrow \infty$ this goes to 0 uniformly for z, w in any compact set. By Cauchy's formula, the coefficients of the power series expansion of \mathbf{f}_β around 0 are given by

$$\frac{1}{2\pi i} \int_C \frac{\mathbf{f}_\beta(\zeta)}{\zeta^{n+1}} d\zeta,$$

where $C(t) = e^{it}$, $0 \leq t \leq 2\pi$. Therefore, for any n , the first n coefficients in the power series of \mathbf{f} and the first n coefficients in the power series of \mathbf{f}_β become uncorrelated and hence (by joint Gaussianity) independent, as $\beta \rightarrow \infty$.

Now let A be any invariant event. Then we can find an event A_n that depends only on the first n power series coefficients and satisfies $\mathbf{P}[A \Delta A_n] \leq c$. Then,

$$\left| \mathbf{E} [\mathbf{1}_A(\mathbf{f}) \mathbf{1}_A(\mathbf{f}_\beta)] - \mathbf{E} [\mathbf{1}_{A_n}(\mathbf{f}) \mathbf{1}_{A_n}(\mathbf{f}_\beta)] \right| \leq 2c.$$

Further, by the asymptotic independence of the coefficients of \mathbf{f} and \mathbf{f}_β , as $\beta \rightarrow \infty$,

$$\mathbf{E} [\mathbf{1}_{A_n}(\mathbf{f}) \mathbf{1}_{A_n}(\mathbf{f}_\beta)] \rightarrow \mathbf{E} [\mathbf{1}_{A_n}(\mathbf{f})] \mathbf{E} [\mathbf{1}_{A_n}(\mathbf{f}_\beta)] = (\mathbf{E} [\mathbf{1}_{A_n}(\mathbf{f})])^2.$$

Thus we get

$$(2.3.13) \quad \limsup_{\beta \rightarrow \infty} \left| \mathbf{E} [\mathbf{1}_A(\mathbf{f}) \mathbf{1}_A(\mathbf{f}_\beta)] - (\mathbf{E} [\mathbf{1}_A(\mathbf{f})])^2 \right| \leq 4c.$$

This is true for any $\epsilon > 0$ and further, by the invariance of A , we have $\mathbf{1}_A(\mathbf{f})\mathbf{1}_A(\mathbf{f}_\beta) = \mathbf{1}_A(\mathbf{f})$. Therefore

$$(2.3.14) \quad \mathbf{E}[\mathbf{1}_A(\mathbf{f})] = (\mathbf{E}[\mathbf{1}_A(\mathbf{f})])^2$$

showing that the probability of A is zero or one. Since the zeros of \mathbf{f}_β are just translates of the zeros \mathbf{f} , any invariant event that is a function of the zero set must have probability zero or one. In other words, the zero set is ergodic under translations. \square

REMARK 2.3.8. It is natural to ask whether these are the only GAFs with isometry-invariant zero sets on these domains. The answer is essentially yes, but we need to know a little more in general about zeros of GAFs before we can justify that claim.

2.4. Distribution of zeros - The first intensity

In this section, we show how to compute the first intensity or the one-point correlation function (see definition 1.2.2). The setting is that we have a GAF \mathbf{f} and the point process under consideration is the counting measure on $\mathbf{f}^{-1}\{0\}$ with multiplicities where \mathbf{f} is a GAF. The following lemma from (68) shows that in great generality almost surely each zero has multiplicity equal to 1.

LEMMA 2.4.1. *Let \mathbf{f} be a nonzero GAF in a domain Λ . Then \mathbf{f} has no nondeterministic zeros of multiplicity greater than 1. Furthermore, for any fixed complex number $w \neq 0$, $\mathbf{f} - w$ has no zeros of multiplicity greater than 1 (there can be no deterministic zeros for $w \neq 0$ since \mathbf{f} has zero mean).*

PROOF. To prove the first statement in the theorem, we must show that almost surely, there is no z such that $\mathbf{f}(z) = \mathbf{f}'(z) = 0$. Fix $z_0 \in \Lambda$ such that $K(z_0, z_0) \neq 0$. Then $\mathbf{h}(z) := \mathbf{f}(z) - \frac{K(z, z_0)}{K(z_0, z_0)}\mathbf{f}(z_0)$ is a GAF that is independent of $\mathbf{f}(z_0)$. For z such that $K(z, z_0) \neq 0$, we can also write

$$(2.4.1) \quad \frac{\mathbf{f}(z)}{K(z, z_0)} = \frac{\mathbf{h}(z)}{K(z, z_0)} + \frac{\mathbf{f}(z_0)}{K(z_0, z_0)}.$$

Thus if z is a multiple zero of \mathbf{f} , then either $K(z, z_0) = 0$ or z is also a multiple zero of the right hand side of (2.4.1). Since $K(\cdot, z_0)$ is an analytic function, its zeros constitute a deterministic countable set. Therefore, \mathbf{f} has no multiple zeros in that set unless it has a deterministic one. Thus we only need to consider the complement of this set.

Now restrict to the reduced domain Λ' got by removing from Λ all z for which $K(z, z_0) = 0$. Condition on \mathbf{h} . The double zeros of \mathbf{f} in Λ' are those z for which the right hand side of (2.4.1) as well as its derivative vanish. In other words, we must have

$$(2.4.2) \quad \left(\frac{\mathbf{h}(z)}{K(z, z_0)} \right)' = 0 \text{ and } \frac{\mathbf{f}(z_0)}{K(z_0, z_0)} = - \frac{\mathbf{h}(z)}{K(z, z_0)}.$$

Let S be the set of z such that $\left(\frac{\mathbf{h}(z)}{K(z, z_0)} \right)' = 0$. Almost surely, S is a countable set. Then the second event in (2.4.2) occurs if and only if

$$\frac{\mathbf{f}(z_0)}{K(z_0, z_0)} \in \left\{ - \frac{\mathbf{h}(z)}{K(z, z_0)} : z \in S \right\}.$$

The probability of this event is zero because the set on the right is countable and the conditional distribution of $\mathbf{f}(z_0)$ given $\mathbf{h}(\cdot)$ is not degenerate.

The same proof works with \mathbf{f} replaced by $\mathbf{f} - w$ because the mean 0 nature of \mathbf{f} did not really play a role. \square

We give three different ways to find a formula for the first intensity of $n_{\mathbf{f}}$, the counting measure (with multiplicities) on $\mathbf{f}^{-1}\{0\}$, when \mathbf{f} is a Gaussian analytic function. Part of the outcome will be that the first intensity does exist, except at the deterministic zeros (if any) of \mathbf{f} . The expressions that we obtain in the end can be easily seen to be equivalent.

2.4.1. First intensity by Green's formula. The first step is to note that for any analytic function f (not random), we have

$$(2.4.3) \quad dn_f(z) = \frac{1}{2\pi} \Delta \log |f(z)|.$$

Here the Laplacian Δ on the right hand side should be interpreted in the distributional sense. In other words, the meaning of (2.4.3) is just that for any smooth function φ compactly supported in Λ ,

$$(2.4.4) \quad \int_{\Lambda} \varphi(z) dn_f(z) = \int_{\Lambda} \Delta \varphi(z) \frac{1}{2\pi} \log |f(z)| dm(z).$$

To see this, write $f(z) = g(z) \prod_k (z - \alpha_k)^{m_k}$, where α_k are zeros of f (with multiplicities m_k) that are in the support of φ and g is an analytic function with no zeros in the support of φ . Since φ is compactly supported, there are only finitely many α_k . Thus

$$\log |f(z)| = \log |g(z)| + \sum_k m_k \log |z - \alpha_k|.$$

Now, $\Delta \log |g|$ is identically zero on the support of φ because $\log |g|$ is, locally, the real part of an analytic function (of any continuous branch of $\log(g)$). Moreover, $\frac{1}{2\pi} \log |z - \alpha_k| = G(\alpha_k, z)$, the Green's function for the Laplacian in the plane implying that

$$\int_{\Lambda} \Delta \varphi(z) \frac{1}{2\pi} \log |z - \alpha_k| = \varphi(\alpha_k).$$

Therefore (2.4.4) follows.

Now for a random analytic function \mathbf{f} , we get

$$(2.4.5) \quad \mathbf{E} \left[\int_{\Lambda} \varphi(z) dn_{\mathbf{f}}(z) \right] = \mathbf{E} \left[\int_{\Lambda} \Delta \varphi(z) \frac{1}{2\pi} \log |\mathbf{f}(z)| dm(z) \right]$$

$$(2.4.6) \quad = \int_{\Lambda} \Delta \varphi(z) \frac{1}{2\pi} \mathbf{E} [\log |\mathbf{f}(z)|] dm(z)$$

by Fubini's theorem. To justify applying Fubini's theorem, note that

$$\mathbf{E} \left[\int_{\Lambda} |\Delta \varphi(z)| \frac{1}{2\pi} |\log |\mathbf{f}(z)|| dm(z) \right] = \int_{\Lambda} |\Delta \varphi(z)| \frac{1}{2\pi} \mathbf{E} [|\log |\mathbf{f}(z)||] dm(z).$$

Now for a fixed $z \in \Lambda$, $\mathbf{f}(z)$ is a complex Gaussian with mean zero and variance $K(z, z)$. Therefore, if a denotes a standard complex Gaussian, then

$$\begin{aligned} \mathbf{E} [|\log |\mathbf{f}(z)||] &\leq \mathbf{E} [|\log |a||] + |\log \sqrt{K(z, z)}| \\ &= \frac{1}{2} \int_0^{\infty} |\log(r)| e^{-r} dr + \frac{1}{2} |\log K(z, z)| \\ &= C + \frac{1}{2} |\log K(z, z)| \end{aligned}$$

for a finite constant C . Observe that $|\log K(z, z)|$ is locally integrable everywhere in z . The only potential problem is at points z_0 for which $K(z_0, z_0) = 0$. But then, in a neighbourhood of z_0 we may write $K(z, z) = |z - z_0|^{2p} L(z, z)$ where $L(z_0, z_0)$ is not zero. Thus $\log K(z, z)$ grows as $\log|z - z_0|$ as $z \rightarrow z_0$, whence it is integrable in a neighbourhood of z_0 . Thus

$$\mathbf{E} \left[\int_{\Lambda} |\Delta\varphi(z)| |\log|\mathbf{f}(z)|| \frac{dm(z)}{2\pi} \right] < \infty.$$

This justifies the use of Fubini's theorem in (2.4.6) and we get

$$(2.4.7) \quad \mathbf{E} \left[\int_{\Lambda} \varphi(z) dn_{\mathbf{f}}(z) \right] = \int_{\Lambda} \varphi(z) \frac{1}{2\pi} \Delta \mathbf{E}[\log|\mathbf{f}(z)|] dm(z).$$

Again using the fact that $\frac{\mathbf{f}(z)}{\sqrt{K(z, z)}}$ is a standard complex Gaussian, we deduce that

$$\begin{aligned} \mathbf{E}[\log|\mathbf{f}(z)|] &= \mathbf{E}[\log|a|] + \frac{1}{2} \log K(z, z) \\ &= -\frac{\gamma}{2} + \log \sqrt{K(z, z)} \end{aligned}$$

where

$$\gamma = -\int_0^{\infty} \log(r) e^{-r} dr.$$

is in fact the negative of Euler's constant, but for our purpose we need only observe that it does not depend on z . Thus by comparing (2.4.7) which is valid for all C_c^2 functions, with (1.2.11) we deduce that the first intensity of $\mathbf{f}^{-1}\{0\}$ with respect to Lebesgue measure is given by

$$(2.4.8) \quad \rho_1(z) = \frac{1}{4\pi} \Delta \log K(z, z).$$

This is sometimes known as the **Edelman-Kostlan** formula. There is no problem with differentiating $\log K(z, z)$ which is real analytic. Exceptions are points where $K(z, z)$ vanish, and at such points the first intensity function does not exist and the first intensity measure has an atom (\mathbf{f} has a deterministic zero).

2.4.2. First intensity by linearization. This is a more probabilistic approach. Let $z \in \Lambda$. We want to estimate the probability that $\mathbf{f}(w) = 0$ for some $w \in D(z, \epsilon)$, up to order ϵ^2 . Expand \mathbf{f} as a power series around z :

$$\mathbf{f}(w) = \mathbf{f}(z) + \mathbf{f}'(z)(w - z) + \mathbf{f}''(z) \frac{(w - z)^2}{2!} + \dots$$

The idea is that up to an event of probability $o(\epsilon^2)$, \mathbf{f} and its linear approximant,

$$\mathbf{g}(w) := \mathbf{f}(z) + (w - z)\mathbf{f}'(z),$$

have the same number of zeros in $D(z, \epsilon)$. Assuming this, it follows from (1.2.8) that

$$\begin{aligned} \rho_1(z) &= \lim_{\epsilon \rightarrow 0} \frac{\mathbf{P}[\mathbf{f} \text{ has a zero in } D(z, \epsilon)]}{\pi \epsilon^2} \\ &= \lim_{\epsilon \rightarrow 0} \frac{\mathbf{P}[\mathbf{g} \text{ has a zero in } D(z, \epsilon)]}{\pi \epsilon^2} \\ &= \lim_{\epsilon \rightarrow 0} \frac{\mathbf{P}\left[\frac{-\mathbf{f}(z)}{\mathbf{f}'(z)} \in D(0, \epsilon)\right]}{\pi \epsilon^2} \\ &= \text{Probability density of } \frac{-\mathbf{f}(z)}{\mathbf{f}'(z)} \text{ at } 0. \end{aligned}$$

If a, b are complex-valued random variables then, by an elementary change of variables, we see that the density of a/b at 0 is equal to $\chi_a(0)\mathbf{E}[|b|^2 | a = 0]$, where χ_a is the density of a at 0 (assuming the density a and the second moment of b given $a = 0$ do exist).

When \mathbf{f} is Gaussian, $(\mathbf{f}(z), \mathbf{f}'(z))$ is jointly complex Gaussian with mean zero and covariance

$$\begin{bmatrix} K(z, z) & \frac{\partial}{\partial \bar{z}} K(z, z) \\ \frac{\partial}{\partial z} K(z, z) & \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} K(z, z) \end{bmatrix}.$$

Here we use the standard notation

$$\frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \text{ and } \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

The density of $\mathbf{f}(z)$ at 0 is $\frac{1}{\pi K(z, z)}$. Moreover, $\mathbf{f}'(z) |_{\mathbf{f}(z)=0}$ has

$$N_{\mathbb{C}} \left(0, \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} K(z, z) - \frac{1}{K(z, z)} \left(\frac{\partial}{\partial \bar{z}} K(z, z) \right) \left(\frac{\partial}{\partial z} K(z, z) \right) \right)$$

distribution. Thus we can write the first intensity as

$$\rho_1(z) = \frac{\frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} K(z, z) - \frac{1}{K(z, z)} \frac{\partial}{\partial \bar{z}} K(z, z) \frac{\partial}{\partial z} K(z, z)}{\pi K(z, z)}.$$

This is equivalent to the Edelman-Kostlan formula (2.4.8) as can be seen by differentiating $\log K(z, z)$ (since $\Delta = 4 \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}}$).

Now we justify replacing \mathbf{f} by its linearization \mathbf{g} . Without loss of generality, we can assume that $z = 0$ and expand \mathbf{f} as a power series. The following lemma is from Peres and Virág (68).

LEMMA 2.4.2. *Let $\mathbf{f}(z) = a_0 + a_1 z + \dots$ be a GAF. Assume that a_0 is not constant. Let A_ϵ denote the event that the number of zeros of \mathbf{f} in the disk $D(0, \epsilon)$ differs from the number of zeros of $\mathbf{g}(z) := a_0 + a_1 z$ in the same disk. Then for any $\delta > 0$, there exists $c > 0$ so that for all $\epsilon > 0$ we have*

$$\mathbf{P}[A_\epsilon] \leq c \epsilon^{3-2\delta}.$$

PROOF. By Rouché's theorem, if $|\mathbf{g}| > |\mathbf{f} - \mathbf{g}|$ on $\partial D(0, \epsilon)$, then \mathbf{f} and \mathbf{g} have the same number of zeros in $D(0, \epsilon)$.

We bound the maximum of $|\mathbf{f} - \mathbf{g}|$ by Lemma 2.4.4. For this we observe that for small enough ϵ ,

$$(2.4.9) \quad \max_{|z| < 2\epsilon} \mathbf{E}[|\mathbf{f}(z) - \mathbf{g}(z)|^2] \leq C \epsilon^4$$

since $\mathbf{f} - \mathbf{g}$ has a double root at 0. Thus Lemma 2.4.4 gives a constant γ such that

$$(2.4.10) \quad \mathbf{P} \left[\max\{|\mathbf{f}(z) - \mathbf{g}(z)| : z \in D(0, \epsilon)\} > \epsilon^{2-\delta} \right] < c_0 e^{-\gamma \epsilon^{-2\delta}}.$$

Now let Θ be the annulus $\partial D(0, |\alpha_1| \epsilon) + D(0, \epsilon^{2-\delta})$ (the Minkowski sum of the two sets), and consider the following events:

$$\begin{aligned} D_0 &= \{|\alpha_0| < 2\epsilon^{1-\delta}\}, \\ E &= \{|\alpha_1| < \epsilon^{-\delta}\}, \\ F &= \{\min\{|\mathbf{g}(z)| : z \in \partial D(0, \epsilon)\} < \epsilon^{2-\delta}\} = \{-\alpha_0 \in \Theta\}. \end{aligned}$$

Note that $\mathbf{P}[E^c] \leq c_2 \epsilon^3$ and that $E \cap F \subset D_0$. Given D_0 , the distribution of α_0 (recall our assumption that α_0 is not a constant) is approximately uniform on $D(0, 2\epsilon^{1-\delta})$ (in particular, its conditional density is $O(\epsilon^{2\delta-2})$). Since $\mathbf{P}[E]$ tends to one as $\epsilon \rightarrow 0$, this implies that

$$\mathbf{P}[F] \leq \mathbf{P}[F \cap E | D_0] \mathbf{P}[D_0] + \mathbf{P}[E^c] \leq c_4 \epsilon c_5 \epsilon^{2-2\delta} + c_2 \epsilon^3 \leq c_6 \epsilon^{3-2\delta}.$$

In the first term, the factor of ϵ comes from the area of Θ (as a fraction of the area of D_0) and the factor of $\epsilon^{2-2\delta}$ from the probability of D_0 . Together with (2.4.10), this gives the desired result. \square

REMARK 2.4.3. In the proof we used Lemma 2.4.4 to bound the maximum modulus of a Gaussian analytic function on a disk. In the literature there are deep and powerful theorems about the maximum of a general Gaussian process which we could have used instead. For instance, Borell's isoperimetric inequality (see Pollard (69); the inequality was also shown independently by Tsirelson-Ibragimov-Sudakov (86)) implies that for any collection of mean-zero (real) Gaussian variables with maximal standard deviation σ , the maximum M of the collection satisfies

$$(2.4.11) \quad \mathbf{P}[M > \text{median}(M) + b\sigma] \leq \mathbf{P}[\chi > b],$$

where χ is standard normal. We could have arrived at (2.4.10) by an application of (2.4.11) separately to the real and imaginary parts of $\frac{\mathbf{f}(z) - \mathbf{g}(z)}{z^2}$ (note that the median is just a finite quantity). However we preferred to use Lemma 2.4.4 as it is elementary and also exhibits some new tools for working with Gaussian analytic functions. One idea in the proof below comes from the paper of Nazarov, Sodin and Volberg (59), see Lemma 2.1 therein.

LEMMA 2.4.4. *Let \mathbf{f} be a Gaussian analytic function in a neighbourhood of the unit disk with covariance kernel K . Then for $r < \frac{1}{2}$, we have*

$$(2.4.12) \quad \mathbf{P} \left[\max_{|z| < r} |\mathbf{f}(z)| > t \right] \leq 2e^{-t^2/8\sigma_{2r}^2}$$

where $\sigma_{2r}^2 = \max\{K(z, z) : |z| \leq 2r\}$.

PROOF. Let $\gamma(t) = 2re^{it}$, $0 \leq t \leq 2\pi$. By Cauchy's integral formula, for $|z| < r$,

$$\begin{aligned} |\mathbf{f}(z)| &\leq \int_0^{2\pi} \frac{|\mathbf{f}(\gamma(t))|}{|z - \gamma(t)|} |\gamma'(t)| \frac{dt}{2\pi} \\ &\leq 2\sigma \int_0^{2\pi} |\hat{\mathbf{f}}(2re^{it})| \frac{dt}{2\pi} \end{aligned}$$

where $\hat{\mathbf{f}}(z) = \mathbf{f}(z)/\sqrt{K(z,z)}$ and we have written just σ for σ_{2r} .

$$\begin{aligned} \mathbf{P} \left[\max_{|z|<r} |\mathbf{f}(z)| > t \right] &\leq \mathbf{P} \left[\int_0^{2\pi} |\hat{\mathbf{f}}(2re^{it})| \frac{dt}{2\pi} > \frac{t}{2\sigma} \right] \\ &\leq e^{-t^2/8\sigma^2} \mathbf{E} \left[\exp \left\{ \frac{1}{2} \left(\int_0^{2\pi} |\hat{\mathbf{f}}(2re^{it})| \frac{dt}{2\pi} \right)^2 \right\} \right] \\ &\leq e^{-t^2/8\sigma^2} \mathbf{E} \left[\exp \left\{ \frac{1}{2} \int_0^{2\pi} |\hat{\mathbf{f}}(2re^{it})|^2 \frac{dt}{2\pi} \right\} \right] \end{aligned}$$

by Cauchy-Schwarz inequality. Now use the convexity of the exponential function to get

$$\mathbf{P} \left[\max_{|z|<r} |\mathbf{f}(z)| > t \right] \leq e^{-t^2/8\sigma^2} \mathbf{E} \left[\int_0^{2\pi} \exp \left\{ \frac{1}{2} |\hat{\mathbf{f}}(2re^{it})|^2 \right\} \frac{dt}{2\pi} \right].$$

Since $|\hat{\mathbf{f}}(w)|^2$ has exponential distribution with mean 1 for any w , the expectation of $\exp\{\frac{1}{2}|\hat{\mathbf{f}}(2re^{it})|^2\}$ is 2. Thus we arrive at

$$\mathbf{P} \left[\max_{|z|<r} |\mathbf{f}(z)| > t \right] \leq 2e^{-t^2/8\sigma^2}.$$

□

2.4.3. First intensity by integral geometry. This is a geometric approach to get the first intensity. We shall sketch the idea briefly. Interested readers are recommended to read the beautiful paper (22) for more along these lines.

Let \mathbf{f} be a GAF with covariance kernel K . Since K is Hermitian and positive definite, we can write $K(z,w) = \sum_n \psi_n(z)\bar{\psi}_n(w)$, where ψ_n are analytic functions on some domain in the plane. Then we see that $\mathbf{f}(z) = \sum_n a_n \psi_n(z)$, where a_n are i.i.d. standard complex Gaussians. (What we just said may be seen as a converse to Lemma 2.2.3).

First suppose that $\mathbf{f}(z) = \sum_{n=1}^N a_n \psi_n(z)$, where $N < \infty$. In the end let $N \rightarrow \infty$ to get the general case. This is possible by Rouché's theorem, for if the series $\mathbf{f}_N(z) = \sum_{n=1}^N a_n \psi_n(z)$ converges uniformly on compact sets to $\mathbf{f}(z) = \sum_{n=1}^{\infty} a_n \psi_n(z)$, then for any compact set, the number of zeros of \mathbf{f} and \mathbf{f}_N are equal, with high probability, for large N .

When N is finite, we may think of $\mathbf{f}(z)$ as the inner product of $\psi(z) = (\psi_1(z), \dots, \psi_N(z))$ with $(\bar{a}_1, \dots, \bar{a}_N)$. As z varies over Λ , $\psi(z)$ defines a complex curve in \mathbb{C}^N . Also $(\bar{a}_1, \dots, \bar{a}_N)$ has a spherically invariant distribution. Thus asking for the number of zeros of \mathbf{f} is equivalent to:

Choose a point uniformly at random on the unit sphere $\{(z_1, \dots, z_N) : \sum_{k=1}^N |z_k|^2 = 1\}$ in \mathbb{C}^N and ask for the number of times (counted with multiplicities) the hyper plane orthogonal to the chosen point intersects the fixed curve ψ .

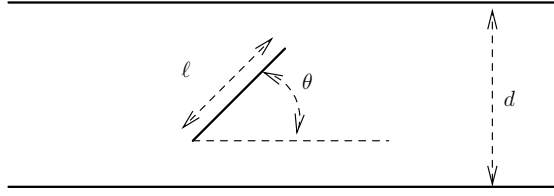


FIGURE 1. The Buffon needle problem.

Turning the problem around, fix z and let w vary over $D(z, \epsilon)$. Then the hyper-plane orthogonal to $\psi(w)$ sweeps out a certain portion of the unit sphere. The expected number of zeroes of \mathbf{f} in $D(z, \epsilon)$ is precisely the area of the region swept out (again counting multiplicities).

Now as w varies over $D(z, \epsilon)$, $\psi(w)$ varies over a disk of radius approximately $\|\psi'(z)\|\epsilon$ on the image of the curve ψ . However what matters to us is the projection of this disk orthogonal to the radial vector $\psi(z)$, and this has area $\left(\|\psi'(z)\|^2 - \frac{|\psi'(z) \cdot \psi(z)|^2}{\|\psi(z)\|^2}\right) \pi \epsilon^2$. However this disk is located at a distance $\|\psi(z)\|$ from the origin.

When a particle P moves a distance δ on a geodesic of the sphere of radius r , the hyper-plane P^\perp orthogonal to P , rotates by an angle of $\frac{\delta}{r}$. When $\delta = \pi$, the entire sphere is swept out by P^\perp exactly once. Putting these together, we find that the probability of having a zero in $D(z, \epsilon)$ is

$$\frac{\left(\|\psi'(z)\|^2 - \frac{|\psi'(z) \cdot \psi(z)|^2}{\|\psi(z)\|^2}\right)}{\pi \|\psi(z)\|^2} \epsilon^2,$$

and this gives $p(z)$. Since $K(z, w) = \psi(z) \cdot \psi(w)$, this is the same as what we got earlier.

REMARK 2.4.5. As a simple application of (2.4.8), one can check that the zero sets of the GAFs described in equations (2.3.4), (2.3.5) and (2.3.6) have first intensities equal to $\frac{L}{\pi}$, w.r.t, the Lebesgue measure $dm(z)$ on the plane, the Spherical measure $\frac{dm(z)}{(1+|z|^2)^2}$ on $\mathbb{S}^2 = \mathbb{C} \cup \{\infty\}$ and the Hyperbolic measure $\frac{dm(z)}{(1-|z|^2)^2}$ on the unit disk \mathbb{D} , respectively.

EXERCISE 2.4.6. Follow the steps outlined below to give a geometric solution to the classical Buffon needle problem: Consider a family of parallel lines in the plane with adjacent lines separated by a distance d . Drop a needle of length ℓ “at random” on the plane. What is the probability that the needle crosses one of the lines? See figure 1.

- i. Show that the probability of a crossing is $c\ell$ for some constant c , provided that $\ell < d$.
- ii. If a circle of circumference ℓ is dropped on the plane, deduce that the expected number of intersections of the circle with the family of parallel lines is again $c\ell$. Use this to compute c .

2.5. Intensity of zeros determines the GAF

In this section we present the result of Sodin (78) that two GAFs on Λ having the same intensity $\rho_1(z)dm(z)$ are essentially equal. In particular we get the remarkable conclusion that the distribution of the zero set $\mathbf{f}^{-1}\{0\}$ is completely determined by its

first intensity! We first prove a standard fact from complex analysis that will be used in the proof of Theorem 2.5.2.

LEMMA 2.5.1. *Let $K(z, w)$ be analytic in z and anti-analytic in w (i.e., analytic in \bar{w}) for $(z, w) \in \Lambda \times \Lambda$. If $K(z, z) = 0 \forall z \in \Lambda$, then $K(z, w) = 0 \forall z, w \in \Lambda$.*

PROOF. It is enough to prove that K vanishes in a neighbourhood of (z, z) for every $z \in \Lambda$. Without loss of generality take $z = 0$. Then around $(0, 0)$ we can expand K as $K(z, w) = \sum_{m, n \geq 1} a_{m, n} z^m \bar{w}^n$. Then $K(z, z) = \sum_{m, n \geq 1} a_{m, n} z^m \bar{z}^n$. Let $z = x + iy$. Note that

$$\frac{\partial^{m+n}}{\partial z^m \partial \bar{z}^n} z^k \bar{z}^\ell \Big|_{z=0} = \delta_{(m, n), (k, \ell)} m! n!.$$

Returning to $K(z, z) = \sum_{k, \ell \geq 1} a_{k, \ell} z^k \bar{z}^\ell$, this gives (since we have assumed that $K(z, z)$ is identically zero)

$$\begin{aligned} 0 &= \frac{\partial^{m+n}}{\partial z^m \partial \bar{z}^n} K(z, z) \Big|_{z=0} \\ &= m! n! a_{m, n}. \end{aligned}$$

Thus $K(z, w)$ vanishes identically in $\Lambda \times \Lambda$. \square

Sodin (78) discovered the following result and related it to Calabi's rigidity theorem in complex geometry.

THEOREM 2.5.2 (Calabi's rigidity). *Suppose \mathbf{f} and \mathbf{g} are two GAFs in a region Λ such that the first intensity measures of $\mathbf{f}^{-1}\{0\}$ and $\mathbf{g}^{-1}\{0\}$ are equal. Then there exists a nonrandom analytic function φ on Λ that does not vanish anywhere, such that $\mathbf{f} \stackrel{d}{=} \varphi \mathbf{g}$. In particular $\mathbf{f}^{-1}\{0\} \stackrel{d}{=} \mathbf{g}^{-1}\{0\}$.*

PROOF. For a $z \in \Omega$, we have $K_{\mathbf{f}}(z, z) = 0$ if and only if z is almost surely a zero of \mathbf{f} (and the corresponding orders of vanishing of $K_{\mathbf{f}}$ and \mathbf{f} at z match). Since \mathbf{f} and \mathbf{g} are assumed to have the same first intensity of zeros, the set of deterministic zeros of \mathbf{f} must coincide and have the same order of vanishing for \mathbf{f} and \mathbf{g} . By omitting all such zeros from Λ , we assume that $K_{\mathbf{f}}(z, z)$ and $K_{\mathbf{g}}(z, z)$ do not vanish anywhere in Λ . It suffices to prove the theorem for this reduced domain, for suppose that $\mathbf{f} = \varphi \mathbf{g}$ on $\Lambda - D$ where D is the discrete set that we have omitted, where φ is a non-vanishing analytic function on $\Lambda - D$. Since at each point z of D , the functions \mathbf{f} and \mathbf{g} vanish to the same order, we see that φ is bounded in a neighbourhood of z and thus φ extends as an analytic function to all of Λ . Again because \mathbf{f} and \mathbf{g} have the same order of vanishing at points of D , it is clear that φ cannot vanish anywhere.

Hence we assume that $K_{\mathbf{f}}(z, z)$ and $K_{\mathbf{g}}(z, z)$ are non-vanishing on Λ . By (2.4.8), the hypotheses imply that $\log K_{\mathbf{f}}(z, z) - \log K_{\mathbf{g}}(z, z)$ is harmonic in Λ . Therefore we can write

$$(2.5.1) \quad K_{\mathbf{f}}(z, z) = e^{u(z)} K_{\mathbf{g}}(z, z)$$

where u is a harmonic function in Λ .

If Λ is simply connected, we can find an analytic function ψ on Λ with $2\text{Re}(\psi) = u$. Set $\varphi = e^\psi$. Then the above equation says that the two functions $K_{\mathbf{f}}(z, w)$ and $\varphi(z) \overline{\varphi(w)} K_{\mathbf{g}}(z, w)$ are equal on the diagonal. As both of these are analytic in z and anti-analytic in w , Lemma 2.5.1 shows that they are identically equal. Hence $\mathbf{f} \stackrel{d}{=} \varphi \mathbf{g}$. As φ does not vanish this shows that $\mathbf{f}^{-1}\{0\}$ and $\mathbf{g}^{-1}\{0\}$ have the same distribution.

If Λ is not simply connected, fix a $z_0 \in \Lambda$ and an $r > 0$ such that $D(z_0, r) \subset \Lambda$. Then there exists a non-vanishing analytic function φ on $D(z_0, r)$ such that

$$(2.5.2) \quad K_{\mathbf{f}}(z, w) = \varphi(z)\overline{\varphi(w)}K_{\mathbf{g}}(z, w)$$

for every $z, w \in D(z_0, r)$. Then fix $w \in D(z_0, r)$ such that $\varphi(w) \neq 0$, and note that $\frac{K_{\mathbf{f}}(z, w)}{\varphi(w)K_{\mathbf{g}}(z, w)}$ is an analytic function on $\Lambda - \{z : K_{\mathbf{g}}(z, w) = 0\}$ and is equal to φ on $D(z_0, r)$. Taking the union over $w \in D(z_0, r)$ of all these analytic functions we get an analytic extension of φ to the whole of

$$(2.5.3) \quad \Lambda \setminus \{z : K_{\mathbf{g}}(z, w) = 0 \ \forall w \in D(z_0, r) \text{ s.t. } \varphi(w) \neq 0\}.$$

But if $K_{\mathbf{g}}(z, w) = 0$ for all w in an open set, then $K_{\mathbf{g}}(z, z) = 0$. By assumption this does not happen. Thus φ extends to the whole of Λ and the relationship

$$K_{\mathbf{f}}(z, w) = \varphi(z)\overline{\varphi(w)}K_{\mathbf{g}}(z, w)$$

persists. Thus $\varphi\mathbf{g}$ and \mathbf{f} have the same covariance kernel and by Gaussianity we get $\mathbf{f} \stackrel{d}{=} \varphi\mathbf{g}$ and φ is analytic on Λ . By inverting the roles of \mathbf{f} and \mathbf{g} , we see that $1/\varphi$ must also be analytic on Λ , which means that φ cannot vanish anywhere. \square

REMARK 2.5.3. Alternately, for the non-simply connected case, one could use the uniformization theorem to argue as follows. If Λ is a region of \mathbb{C} , let π be the covering map from \mathbb{D} or \mathbb{C} to Λ . Recall the definition of u from (2.5.1). Let $K_{\mathbf{f}}^*, K_{\mathbf{g}}^*, u^*$ be pull backs of $K_{\mathbf{f}}$ and $K_{\mathbf{g}}$ and u to \mathbb{D} . Then as before we can write $K_{\mathbf{f}}(z, w) = \varphi^*(z)\overline{\varphi^*(w)}K_{\mathbf{g}}^*(z, w)$ for a non-vanishing analytic function φ^* on \mathbb{D} . If $\pi(z_1) = \pi(z_2)$, then $\varphi^*(z_1) = \varphi^*(z_2)$ (Fix w and note that $K_{\mathbf{f}}^*(z_1, w) = K_{\mathbf{f}}^*(z_2, w)$ and $K_{\mathbf{g}}^*(z_1, w) = K_{\mathbf{g}}^*(z_2, w)$). Thus $\varphi = \varphi^* \pi^{-1}$ is well defined, does not vanish on Λ and satisfies, $K_{\mathbf{f}}(z, w) = \varphi(z)\overline{\varphi(w)}K_{\mathbf{g}}(z, w)$.

An immediate consequence is

COROLLARY 2.5.4. *The random power series described in equations (2.3.4), (2.3.5) and (2.3.6) are the only GAFs, up to multiplication by deterministic nowhere vanishing analytic functions, whose zeros are isometry-invariant under the respective group of isometries.*

Unfortunately, Theorem 2.5.2 is not constructive in that it does not tell us how to determine the k -point intensities of the zero set of a GAF if we know the first intensity. However, in the next section we shall see that it is possible to write general, although often intractable, formulas for the joint intensities of a GAF.

2.6. Notes

- The study of zeros of random polynomials goes back to Mark Kac (41) (but see also Paley and Wiener (66) which preceded Kac). He obtained the density of real zeros of various models of random polynomials, for example $a_0 + a_1x + \dots + a_nx^n$, where a_k are i.i.d. standard (real) Gaussians. These results can be obtained by the geometric proof presented here due to Edelman and Kostlan. See (22) for details. Following his papers, there was a significant amount of work done on this subject. Apart from zeros, there are many other interesting questions about random powers series as can be seen in the book of Kahane (42).
- The recent resurgence of interest in complex zeros is at least partly due to the work of many physicists such as Bogomolny, Bohigas and Leboeuf (7),(8), Hanay (30) and others. Apart from the Probabilistic perspective of these notes, there

are other frameworks in which these objects are studied. For instance see Shiffman, Zelditch (74) (and references therein) who study random sections of line bundles.

- The planar GAF models were introduced (in parts) by Bogomolny, Bohigas and Leboeuf (7) and (8), Kostlan (49), Shub and Smale (76). Some of them are natural generalizations to complex coefficients of random polynomials studied by Mark Kac. A special case in the unit disk ($L = 2$) was found by Diaconis and Evans (18) as the limit of the logarithmic derivative of characteristic polynomials of random unitary matrices.
- Subhroshekhar Ghosh pointed to us another proof that the hyperbolic GAF does not extend to any domain larger than the unit disk. If it did, the covariance kernel would be an extension of $(1 - z\bar{w})^{-L}$ and would remain analytic in z and anti-analytic in w . This is clearly impossible, as $(1 - |z|^2)^{-L}$ does not extend continuously to any point on the boundary of the unit disk.
- Theorem 2.5.2 is from Sodin (78), who found the result and related it to Calabi's rigidity theorem from differential geometry. A constructive way of recovering higher intensities from the first one is not known, and would be very desirable to have.

2.7. Hints and solutions

Exercise 2.1.1

- i. Note that X has density

$$(2.7.1) \quad f(z_1, \dots, z_d) = \frac{1}{(2\pi)^{d/2}} e^{-|z|^2/2}$$

where $|\cdot|$ denotes the Euclidean norm. By the transformation formula, the density of AX is $f(A^{-1}x)|\det(A^{-1})|^2$ (note that we use the real Jacobian here). The determinant is 1 and unitary matrices preserve the Euclidean norm, hence the density of X is invariant under A .

- ii. It suffices to consider the case $\mathbf{E}X = \mathbf{E}Y = 0$. By definition there are standard Gaussian random vectors \tilde{X} and \tilde{Y} and matrices A and B with $X = A\tilde{X}$ and $Y = B\tilde{Y}$. By adding columns of zeros to A or B , if necessary, we can assume that \tilde{X} and \tilde{Y} are both k -vectors, for some k , and A, B are both $d \times k$ matrices. Let \mathcal{A} and \mathcal{B} be the vector subspaces of \mathbb{C}^k generated by the row vectors of A and B , respectively. Suppose, WLOG, that the first $\ell \leq d$ row vectors of A form a basis of \mathcal{A} . Define the linear map $L: \mathcal{A} \rightarrow \mathcal{B}$ by

$$(2.7.2) \quad L(A_i) = B_i \text{ for } i = 1, \dots, \ell.$$

Here A_i is the i^{th} row vector of A , and B_i is the i^{th} row vector of B . Our aim is to show that L is an orthogonal isomorphism and then use the previous proposition. Let us first show that L is an isomorphism. The covariance assumption implies $AA^* = BB^*$. Suppose there is a vector $v_1A_1 + \dots + v_\ell A_\ell$ which maps to 0 under L . Then the vector

$$(2.7.3) \quad v = (v_1, \dots, v_\ell, 0, \dots, 0)$$

satisfies $vB = 0$. Hence

$$(2.7.4) \quad |vA|^2 = vAA^*v^* = vBB^*v^* = 0,$$

so $vA = 0$. Thus L is injective and $\dim \mathcal{A} \leq \dim \mathcal{B}$. Interchanging the roles of A and B shows that L is an isomorphism. The entry (i, j) of $AA^* = BB^*$ is the inner product of A_i and A_j as well as B_i and B_j , so the mapping L preserves inner products. Thus it can be extended on the orthocomplement of \mathcal{A} to give a unitary map $L: \mathbb{C}^k \rightarrow \mathbb{C}^k$ (or a unitary $k \times k$ matrix). Then $X = A\tilde{X}$ and $Y = B\tilde{Y} = AL^*\tilde{Y}$. From part i. we know that $L^*\tilde{Y}$ is standard complex normal, hence X and Y have the same distribution.

Exercise 2.2.2 If $z \in D$ and $D(z, r) \subset \Lambda$, then \mathbf{f} has a power series expansion in $D(z, r)$. By virtue of it being Gaussian in D , the coefficients of the power series have a jointly complex Gaussian distribution and hence \mathbf{f} is Gaussian on the whole of $D(z, r)$. In general, for any $w \in \Lambda \setminus D$, we can find a sequence of disks $D(z_1, r_1), \dots, D(z_n, r_n)$ contained in Λ such that $D(z_1, r_1) \subset D$, $D(z_n, r_n) \ni w$ and such that $z_j \in D(z_{j-1}, r_{j-1})$. Inductively, we apply our earlier observation about concentric disks to conclude that \mathbf{f} is Gaussian near z_2, \dots, z_n and hence near w .

Exercise 2.4.6

- i. Consider the needle as a union of shorter needles, use linearity of expectations. Each of the shorter needles will give the same expected number of intersections, provided that it is not too far from the center of mass of the original needle. And lastly, for $\ell < d$, the number of intersections is at most one, hence we get the probability of intersection.
- ii. The same argument used in i. shows that if a polygonal path of length ℓ is dropped uniformly between the two lines, the expected number of intersections is $c\ell$. Circles can be approximated arbitrarily well by polygonal paths, so the same is true for circles. A circle of diameter d has exactly two intersection, which yields $c = \frac{2}{\pi d}$.

Exercise 2.3.1

- i. Direct calculation shows that for φ as in (2.3.2), we have

$$\frac{|\varphi'(z)|}{1 + |\varphi(z)|^2} = \frac{1}{1 + |z|^2}.$$

This shows that the metric and area are preserved by φ .

- ii. The radius of $D(0, r)$ is given by $\int_0^r \frac{1}{1+t^2} dt$ and the area is given by $\int_{D(0,r)} \frac{1}{(1+|z|^2)^2} dm(z)$.

Straightforward calculations show that these integrals are equal to $\arctan(r)$ and $\frac{\pi r^2}{1+r^2}$, respectively.

Exercise 2.3.2

- i. This time we check easily that for φ as in (2.3.3), we have

$$\frac{|\varphi'(z)|}{1 - |\varphi(z)|^2} = \frac{1}{1 - |z|^2}.$$

This shows that the hyperbolic metric and area are preserved by φ .

- ii. The radius of $D(0, r)$ is given by $\int_0^r \frac{1}{1-t^2} dt$ and the area is given by $\int_{D(0,r)} \frac{1}{(1-|z|^2)^2} dm(z)$.

Straightforward calculations show that these integrals are equal to $\operatorname{arctanh}(r)$ and $\frac{\pi r^2}{1-r^2}$, respectively.

Joint Intensities

In chapter 2 we derived expressions for the first intensity of the zero set of a general Gaussian analytic function. In this chapter, in section 3.1 we find the joint intensities of zero sets. These are special cases of what are known as Kac-Rice formulas, and can be used to obtain explicit answers for low order intensities. In section 3.5, we find a different expressions for the two-point intensity of zeros and specialize them to the canonical Gaussian analytic functions of section 2.3. However our aim will not be to find the two-point intensities exactly, but to evaluate them asymptotically. This will lead us to surprising facts about the fluctuations of smooth linear statistics of zero sets. We shall also state asymptotic normality results on linear statistics due to Sodin and Tsirelson.

3.1. Introduction – Random polynomials

The goal of this section is to prove the k -point intensity formula for random analytic zeros which can be heuristically written as

$$(3.1.1) \quad p(x_1, \dots, x_k) = \mathbf{E} [|\mathbf{f}'(x_1) \cdots \mathbf{f}'(x_k)|^2; \mathbf{f}(x_1), \dots, \mathbf{f}(x_k) = 0]$$

in the strong and precise sense given in (3.1.2) below. For a random analytic function \mathbf{f} , let $\mu_k = \mu_{\mathbf{f},k}$ denote the expectation of the k -fold product of counting measure of the zero set. It is the measure satisfying

$$\int \varphi d\mu_k = \mathbf{E} \sum_{\mathcal{Z}_{\mathbf{f}}^k} \varphi(z_1, \dots, z_k)$$

for test functions φ . As discussed in text following definition 1.2.2, off the diagonals ($z_i = z_j$ for some $i \neq j$) μ_k agrees with the k -point intensity measure.

THEOREM 3.1.1. *Let \mathbf{f} be a random polynomial with a.s. bounded degree. Then we have the following weak* limit*

$$(3.1.2) \quad \mu_k = \lim_{\epsilon \rightarrow 0} \frac{\mathbf{E} [|\mathbf{f}'(x_1) \cdots \mathbf{f}'(x_k)|^2 \mathbf{1}(\mathbf{f}(x_1), \dots, \mathbf{f}(x_k) \in B_\epsilon)]}{\text{Vol}(B_\epsilon)^k} dx_1 \cdots dx_k.$$

In particular, the limit exists. Note that if the density in (3.1.2) converges uniformly on compact subsets of \mathbb{C}^k to some function ρ , then μ_k has density ρ .

In Section 3.3 we will extend this theorem to random analytic functions satisfying a simple moment condition 3.3.1. We shall show that this moment condition implies exponential tails for the number of zeros in any bounded set; this is proved in Section 3.2. The Gaussian analytic function case is proved in Section 3.4.

Our strategy is as follows. First of all, we can always think of the joint intensity of zeros of \mathbf{f} as the off-diagonal single intensity of the zeros of $(\mathbf{f}, \dots, \mathbf{f})$.

It would be nice to first prove a deterministic version of formula (3.1.1), and take expectations. But such version would involve point mass measures and would

be difficult to analyze. So we introduce two extra sources of averaging: first, instead of zeros, we consider near-zero values (or, more generally, values in a bounded set B). Second, we integrate a smooth test function over the locations where F takes these values.

After this averaging, we now have tractable deterministic version of (3.1.1), namely Lemma 3.1.2 below. It is a non-bijective change-of-variables formula (also called co-area formula) which we have not been able to find in the literature.

LEMMA 3.1.2 (Change of variables). *Let $F : \mathbb{C}^k \rightarrow \mathbb{C}^k$ be a function with continuous derivative. For any continuous function φ with compact support and any bounded $B \subset \mathbb{C}^k$ we have*

$$\int_B \sum_{x \in F^{-1}(y)} \varphi(x) dy = \int_{\mathbb{C}^k} \varphi(x) |F'(x)|^2 \mathbf{1}_B(F(x)) dx.$$

here $|F'(x)|$ is the absolute value of the Jacobian determinant of F .

PROOF. We may assume that $\varphi \geq 0$. Let S be the set of critical points of F , let K be the compact support of φ , and let S' be an open neighborhood of $S \cap K$. For each point $x \in K \setminus S$ by the inverse mapping theorem there is a neighborhood of x so that F is one-to-one. Using a finite subcover by such neighborhoods of $K \setminus S'$ and the usual change-of-variable formula for bijections, we get

$$(3.1.3) \quad \int_B \sum_{x \in F^{-1}(y) \setminus S'} \varphi(x) dy = \int_{\mathbb{C}^k \setminus S'} \varphi(x) |F'(x)|^2 \mathbf{1}_B(F(x)) dx,$$

Now we can let $S' \searrow S \cap K$, then by the monotone convergence theorem applied to both sides we can replace S' by S in (3.1.3). We can then drop “ $\setminus S$ ” on the right hand side because the integrand vanishes on S . We can drop it on the left hand side as well because of Sard’s Theorem ((35) p. 682), which states that for a differentiable function F the image $F(S)$ of the set of critical points S has measure 0. \square

The following deterministic lemma shows that in the case when $F = (\mathbf{f}, \dots, \mathbf{f})$ and \mathbf{f} is an analytic function, then some of the averaging from the formula of Lemma 3.1.2 can be removed. For such F and a test function $\varphi : \mathbb{C}^k \rightarrow \mathbb{R}$ use the shorthand

$$\varphi[F^{-1}(y)] = \sum_{x \in F^{-1}(y)} \varphi(x) = \sum_{\text{zeros } x \text{ of } F-y} \varphi(x).$$

Let B_ϵ denote the open disk of radius ϵ about 0.

LEMMA 3.1.3. *Let \mathbf{f} be an analytic function ($\mathbf{f} \neq 0$) on a domain D , and let \mathcal{Z} be the multi-set of its zeros. For any continuous function φ with compact support in D^k we have*

$$\frac{1}{\text{Vol}(B_\epsilon^k)} \int_{B_\epsilon^k} \varphi[F^{-1}(y)] dy \rightarrow \sum_{z \in \mathcal{Z}^k} \varphi(z_1, \dots, z_k).$$

PROOF. Let K be the union of the projection of $\text{supp}(\varphi)$ in all coordinate directions. Let \mathcal{W} denote the finite set (this time without multiplicity) of zeros of \mathbf{f} in K .

Consider a zero $w \in \mathcal{W}$ with multiplicity $m(w)$. Then there exists $\epsilon(w) > 0$ so that f restricted to a neighborhood of w has an inverse \mathbf{f}_w^{-1} that has exactly m -values on $B_{\epsilon(w)} \setminus \{0\}$, see Ahlfors (1) p. 133. Moreover, each value of $\mathbf{f}_w^{-1}(z)$ converges to w as $z \rightarrow 0$.

When $\epsilon < \min_{\mathcal{W}} \epsilon(w)$, then for $z \in B_\epsilon \setminus \{0\}$ the function $\mathbf{f}^{-1}(z)$ has exactly $m(w)$ values for every $w \in \mathcal{W}$ and they are close to the corresponding zero w . More precisely, adding φ to the picture we have

$$\sup_{y \in (B_\epsilon \setminus \{0\})^k} \left| \varphi[F^{-1}(y)] - \sum_{\mathcal{Z}^k} \varphi(z_1, \dots, z_k) \right| \rightarrow 0$$

as $\epsilon \rightarrow 0$. The claim follows. \square

Our proof of Theorem 3.1.1 works in the following, more general setting. It includes the case of polynomials with a.s. bounded degree.

LEMMA 3.1.4. *Theorem 3.1.1 holds for random analytic functions satisfying the following condition. For every compact set K , the random variables*

$$(3.1.4) \quad \left(\frac{1}{\text{Vol}(B_\epsilon)} \int_{B_\epsilon} n_{\mathbf{f}+z}(K) dz \right)^k$$

are uniformly integrable as $\epsilon \rightarrow 0$.

PROOF. Let $\varphi : \mathbb{C}^k \rightarrow \mathbb{R}$ be a continuous test function with compact support. By Lemma 3.1.2 with the notation $F = (\mathbf{f}, \dots, \mathbf{f})$ we have

$$(3.1.5) \quad \frac{1}{\text{Vol}(B_\epsilon^k)} \int_{\mathbb{C}^k} \varphi(x) |F'(x)|^2 \mathbf{1}_{B_\epsilon^k}(F(x)) dx = \frac{1}{\text{Vol}(B_\epsilon^k)} \int_{B_\epsilon^k} \varphi[F^{-1}(y)] dy.$$

If we replace φ by $\sup|\varphi|$ on the right, the expression becomes

$$(3.1.6) \quad \sup|\varphi| \left[\frac{1}{\text{Vol}(B_\epsilon)} \int_{B_\epsilon} n_{\mathbf{f}+z}(\text{supp } \varphi) dz \right]^k,$$

which is uniformly integrable by assumption. Now take expectations of (3.1.5), let $\epsilon \rightarrow 0$ and use uniform integrability. The right hand side, by the bounded convergence theorem and Lemma 3.1.3 converges to

$$(3.1.7) \quad \mathbf{E} \sum_{z \in \mathcal{Z}^k} \varphi(z_1, \dots, z_k) = \int \varphi d\mu_k.$$

The left hand side, by the Fubini argument, becomes

$$\begin{aligned} & \int \varphi(x) \frac{\mathbf{E} [|F'(x)|^2 \mathbf{1}(F(x) \in B_\epsilon^k)]}{\text{Vol}(B_\epsilon^k)} dx \\ &= \int \varphi(x) \frac{\mathbf{E} [|\mathbf{f}'(x_1) \cdots \mathbf{f}'(x_k)|^2 \mathbf{1}(\mathbf{f}(x_1), \dots, \mathbf{f}(x_k) \in B_\epsilon)]}{\text{Vol}(B_\epsilon)^k} dx \end{aligned}$$

completing the proof. \square

3.2. Exponential tail of the number of zeros

In order to extend Hammersley's formula (Theorem 3.1.1) beyond polynomials we need to control the moments of the number of zeros in a compact set. For this, we will use the following tail bound which is interesting in itself. It is a variant of Offord's estimate for GAFs.

THEOREM 3.2.1. *Let $\mathbf{f}(z)$ be a random analytic function on a domain D . Suppose that there exists $b, \delta > 0$ so that*

$$\mathbf{E} |\mathbf{f}(z)|^{\pm \delta} < b \quad \text{for all } z \in D.$$

Then in any compact $K \subset D$ the number $n_{\mathbf{f}}(K)$ of zeros has exponential tail: there are $c, \alpha > 0$ depending on b, δ, K only so that for all $\lambda > 0$ we have

$$P(\lambda < n_{\mathbf{f}}(K)) < ce^{-\alpha\lambda}.$$

The theorem hinges on the following lemma.

LEMMA 3.2.2. *Let X be a nonnegative random variable satisfying $\mathbf{E}(X^\delta) \leq b$ for some $\delta > 0$. Then for any event A with $p = P(A)$ we have*

$$\mathbf{E}(\log^+ X; A) \leq (1 + \log^+ b - \log p)p/\delta.$$

PROOF. It is convenient to use the standard representation $X = G^{-1}(U)$ where U is a uniform $[0, 1]$ random variable and G^{-1} is the ‘‘inverse’’ of $G(x) = \mathbf{P}(X \geq x)$ in the following sense:

$$G^{-1}(y) = \sup\{x : G(x) \geq y\}.$$

We will use the fact that this inverse satisfies $G(G^{-1}(y)) \geq y$, and so $G(X) \geq U$.

By Markov’s inequality $x^\delta G(x) \leq \mathbf{E}X^\delta \leq b$, and therefore

$$\delta \log^+ x \leq \log^+ b - \log G(x).$$

Let $q(X) = \mathbf{P}[A|X]$. We have $G(X) \geq U$, and therefore

$$\begin{aligned} \mathbf{E}[q(X)\delta \log^+ X] &\leq \mathbf{E}[q(X)(\log^+ b - \log G(X))] \\ &= p \log^+ b - \mathbf{E}[r(U)\log G(X)] \\ &\leq p \log^+ b - \mathbf{E}[r(U)\log U], \end{aligned}$$

where $r(u) = q(G^{-1}(u))$ is a function with values in $[0, 1]$ and total integral p over $[0, 1]$. Thus the last term can be written as

$$-\int_0^1 r(u)\log u \, du \leq -\int_0^p \log u \, du = p(1 - \log p),$$

where the inequality follows by rearranging the values of r within the given constraints so that large values of r correspond to large values of $-\log$. The claim follows. \square

PROOF. [Proof of Theorem 3.2.1] Let φ be a non-negative smooth function with compact support in D so that $\varphi \equiv 1$ on K . Then $\|\Delta\varphi\|_{L^1} < \infty$, and by (2.4.4) we have

$$n_{\mathbf{f}}(K) \leq \int_D \varphi(z) dn_{\mathbf{f}}(z) = \frac{1}{2\pi} \int_D \Delta\varphi(z) \log |\mathbf{f}(z)| \, dm(z).$$

We now take expectation of both sides on the event $A = \{n_{\mathbf{f}}(K) > \lambda\}$, and let $p = \mathbf{P}(A)$. Using Markov’s inequality and a Fubini argument we get

$$\begin{aligned} \lambda p &\leq \mathbf{E}[n_{\mathbf{f}}(K); A] \leq \frac{1}{2\pi} \int_D |\Delta\varphi(z)| \mathbf{E}(|\log |\mathbf{f}(z)||; A) \, dm(z) \\ &\leq \frac{1}{2\pi} \|\Delta\varphi\|_{L^1} \sup_{z \in D} \mathbf{E}(|\log |\mathbf{f}(z)||; A) \end{aligned}$$

Lemma 3.2.2 with $X = |\mathbf{f}(z)|^{\pm 1}$ provides the bound

$$\mathbf{E}(|\log |\mathbf{f}(z)||; A) \leq 2(1 + \log^+ b - \log p)p/\delta$$

and therefore

$$\mathbf{P}(\lambda < n_{\mathbf{f}}(K)) = p \leq e(b \vee 1) \exp(-\lambda\delta\pi/\|\Delta\varphi\|_{L^1}). \quad \square$$

3.3. Joint intensities for random analytic functions

The exponential tail estimate of the previous section allows us to extend Theorem 3.1.1 to random analytic functions with bounded moments. More precisely, we have the following theorem.

THEOREM 3.3.1 (Hammersley's formula for analytic functions). *Let \mathbf{f} be a random analytic function in a domain D satisfying $\mathbf{f} \neq 0$ a.s. and*

$$(3.3.1) \quad \mathbf{E}|\mathbf{f}(z)|^{\pm\delta} < b \quad \text{for some } b, \delta > 0 \text{ and all } z \in D.$$

Then we have the following weak limit*

$$(3.3.2) \quad \mu_k = \lim_{\epsilon \rightarrow 0} \frac{\mathbf{E} [|\mathbf{f}'(x_1) \cdots \mathbf{f}'(x_k)|^2 \mathbf{1}(\mathbf{f}(x_1), \dots, \mathbf{f}(x_k) \in B_\epsilon)]}{\text{Vol}(B_\epsilon)^k} dx_1 \cdots dx_k.$$

In particular, the limit exists. Note that if the density in (3.3.2) converges uniformly on compact subsets of \mathbb{C}^k to some function ρ , then μ_k has density ρ .

In applications, we apply the theorem to restrictions of \mathbf{f} to compact subsets of D , since condition (3.3.1) is usually satisfied only locally. As a first step in the proof of Theorem 3.3.1, we show that the moment condition (3.3.1) is inherited by randomly shifted versions of \mathbf{f} , that is, functions of the form $z \mapsto \mathbf{f}(z) + \epsilon U$ where $U \sim \text{uniform}(\mathbb{D})$. Clearly, it suffices to prove this pointwise.

LEMMA 3.3.2. *Let Z be a complex random variable, let $U \sim \text{uniform}(\mathbb{D})$ be independent, and let $\delta \in \mathbb{R}$. There exists $c_\delta > 0$ so that for all $\epsilon \in [0, 1]$:*

$$\mathbf{E}|Z + \epsilon U|^\delta \leq c_\delta (1 + \mathbf{E}|Z|^\delta).$$

PROOF. For $\delta \geq 0$ the claim follows from the inequality $|Z + \epsilon U|^\delta \leq 2^\delta (|Z|^\delta + |\epsilon U|^\delta)$.

Now let $\delta < 0$, and set $\eta = -\delta > 0$. If $3\epsilon < |Z + \epsilon U|$, then the triangle inequality gives $2\epsilon < |Z|$, and so $|Z + \epsilon U| > |Z|/2$. Thus

$$|Z + \epsilon U|^{-\eta} \mathbf{1}(|Z + \epsilon U| > 3\epsilon) \leq |Z/2|^{-\eta},$$

so we have

$$|Z + \epsilon U|^{-\eta} \leq |Z + \epsilon U|^{-\eta} \mathbf{1}(|Z + \epsilon U| \leq 3\epsilon) + |Z/2|^{-\eta}.$$

After taking expectations we get

$$\mathbf{E}|Z + \epsilon U|^{-\eta} \leq \int_{B_{3\epsilon}} |z|^{-\eta} \mathbf{P}(Z + \epsilon U \in dz) + \mathbf{E}|Z/2|^{-\eta}.$$

Given Z , the conditional probability that $Z + \epsilon U \in dz$ is $\mathbf{1}(z + Z \in \epsilon \mathbb{D}) / (\pi \epsilon^2)$. So the first term can be written as

$$\begin{aligned} \int_{B_{3\epsilon}} \mathbf{P}(Z \in z + \epsilon \mathbb{D}) \frac{|z|^{-\eta} dz}{\pi \epsilon^2} &\leq \mathbf{P}(|Z| < 4\epsilon) \int_{B_{3\epsilon}} \frac{|z|^{-\eta} dz}{\pi \epsilon^2} \\ &= c_\eta \mathbf{P}(|Z| < 4\epsilon) (4\epsilon)^{-\eta} \\ &\leq c_\eta \mathbf{E}|Z|^{-\eta}, \end{aligned}$$

the last inequality is Markov's. We conclude $\mathbf{E}|Z + \epsilon U|^{-\eta} \leq c'_\eta \mathbf{E}|Z|^{-\eta}$, as required. \square

PROOF. [Proof of Theorem 3.3.1] Let U be an independent $\text{uniform}(\mathbb{D})$ random variable, and let conditional expectation given \mathbf{f} mean that we are integrating over U .

By Lemma 3.1.4 it suffices to prove that

$$\left(\frac{1}{\text{Vol}(B_\epsilon)} \int_{B_\epsilon} n_{\mathbf{f}+z}(K) dz \right)^k = \mathbf{E}[n_{\mathbf{f}+\epsilon U}(K)|\mathbf{f}]^k$$

is uniformly integrable as $\epsilon \rightarrow 0$.

By Lemma 3.3.2, $\{\mathbf{f}(z) + \epsilon U : \epsilon < 1, z \in D\}$ uniformly satisfy the moment condition of Theorem 3.2.1, which then gives

$$(3.3.3) \quad \mathbf{P}(n_{\mathbf{f}+\epsilon U}(K) > \lambda) \leq c e^{-\alpha \lambda}$$

for some positive constants c, α which do not depend on ϵ . By Jensen's inequality for conditional expectations, we have

$$\mathbf{E} \left[\mathbf{E}[n_{\mathbf{f}+\epsilon U}(K)|\mathbf{f}]^{2k} \right] \leq \mathbf{E} \left[n_{\mathbf{f}+\epsilon U}(K)^{2k} \right] \leq c',$$

where c' can be chosen not to depend on ϵ because of (3.3.3). To finish, note that if X_ϵ is a collection of random variables, and for some $c > 0$, $\alpha > 1$ we have $\mathbf{E}|X_\epsilon|^\alpha < c$ for all ϵ , then the X_ϵ are uniformly integrable. \square

3.4. Joint intensities – The Gaussian case

For Gaussian analytic functions on compact sets with strictly positive definite covariance kernel the conditions of the general joint intensity Theorem 3.3.1 hold trivially.

The following exercise together with the last sentence in the statement of Theorem 3.3.1 gives a sufficient condition for Gaussian analytic functions to have k -point intensity functions (as opposed to merely k -point intensity measures).

EXERCISE 3.4.1. Assume that $\det[(K(z_i, z_j))_{1 \leq i, j \leq k}]$ does not vanish anywhere on a compact set $L \subset \mathbb{C}^k$. Show that

$$\lim_{\epsilon \rightarrow 0} \frac{\mathbf{E} [|\mathbf{f}'(x_1) \cdots \mathbf{f}'(x_k)|^2 \mathbf{1}(\mathbf{f}(x_1), \dots, \mathbf{f}(x_k) \in B_\epsilon)]}{\text{Vol}(B_\epsilon)^k} = \frac{\mathbf{E} [|\mathbf{f}'(x_1) \cdots \mathbf{f}'(x_k)|^2 \mid \mathbf{f}(x_1), \dots, \mathbf{f}(x_k) = 0]}{\pi^k \det[(K(z_i, z_j))_{1 \leq i, j \leq k}]}$$

uniformly on L .

COROLLARY 3.4.2 (Density formula for Gaussian analytic functions). *Let \mathbf{f} be a Gaussian analytic function on Λ with covariance kernel K . For $k \geq 1$, if $\det K(z_i, z_j)_{i, j \leq k}$ does not vanish anywhere on Λ , then the k -point intensity function ρ_k exists and is given by*

$$(3.4.1) \quad \rho_k(z_1, \dots, z_k) = \frac{\mathbf{E} [|\mathbf{f}'(z_1) \cdots \mathbf{f}'(z_k)|^2 \mid \mathbf{f}(z_1) = \dots = \mathbf{f}(z_k) = 0]}{\pi^k \det K(z_i, z_j)_{i, j \leq k}}.$$

Equivalently,

$$(3.4.2) \quad \rho_k(z_1, \dots, z_k) = \frac{\text{per}(C - \mathbf{B}\mathbf{A}^{-1}\mathbf{B}^*)}{\det(\pi\mathbf{A})},$$

where A, B, C are $k \times k$ matrices defined by

$$\begin{aligned} A(i, j) &= \mathbf{E} \left[\mathbf{f}(z_i) \bar{\mathbf{f}}(z_j) \right] \\ B(i, j) &= \mathbf{E} \left[\mathbf{f}'(z_i) \bar{\mathbf{f}}(z_j) \right] \\ C(i, j) &= \mathbf{E} \left[\mathbf{f}'(z_i) \bar{\mathbf{f}}'(z_j) \right]. \end{aligned}$$

(Recall Definition 2.1.5 of per.)

PROOF. [Proof of Corollary 3.4.2] Exercise 3.4.1 yields (3.4.1). Exercise 2.1.3 tells us that conditional on the event $\{\mathbf{f}(z_i) = 0, 1 \leq i \leq k\}$, the distribution of $(\mathbf{f}'(z_1), \dots, \mathbf{f}'(z_k))$ is again complex Gaussian with zero mean and covariance $C - BA^{-1}B^*$. Apply Wick formula (Lemma 2.1.7) to obtain (3.4.2). \square

3.4.1. Short-range repulsion. In chapter 1 we revealed that one of our motivations for studying zeros of random analytic functions is that they seem to model point processes with repulsion between points. Indeed, in Lemma 1.1.1 we saw that for a monic polynomial, the Jacobian determinant of the transformation from coefficients to roots is equal to $\prod |z_i - z_j|^2$, which clearly shows that under some fairly general assumptions on the distribution of coefficients of a random polynomial, the zeros must exhibit repulsive behaviour. However the phenomenon of short-range negative correlations of zeros is so ubiquitous and important that we would like to give another simple explanation.

Since for a Poisson process (or a finite collection of independent points) on the plane we have

$$\mathbf{P}[\text{there are two points in } D(z, \epsilon)] \asymp \epsilon^4,$$

by local repulsion of a point process we mean that this probability is $o(\epsilon^4)$ (typically $O(\epsilon^6)$). For point processes on the real line, the analogous quantity is $o(\epsilon^2)$ (typically $O(\epsilon^3)$). The following exercise shows that this is indeed true for a wide class of random smooth real-valued functions on \mathbb{R} .

EXERCISE 3.4.3. Let $\mathbf{f}: \mathbb{R} \rightarrow \mathbb{R}$ be a random smooth function. Fix $x \in \mathbb{R}$.

(i) Let $\mathbf{g}(y) = a + by + cy^2$, where $a = \mathbf{f}(x)$, $b = \mathbf{f}'(x)$, $c = \frac{1}{2}\mathbf{f}''(x)$. Show under fairly general conditions on \mathbf{f} that

$$\mathbf{P}[\mathbf{f} \text{ has at least two zeros in } (x - \epsilon, x + \epsilon)] = \mathbf{P}[\mathbf{g} \text{ has two zeros in } (-\epsilon, \epsilon)] + O(\epsilon^3).$$

For definiteness, we count zeros with multiplicity.

(ii) The graph of \mathbf{g} is a parabola whose shape is determined by c , whose “tip” has the horizontal location equal to $\xi := -\frac{b}{2c}$ and vertical location equal to $\eta := a - \frac{b^2}{4c^2}$. Condition on c . For \mathbf{g} to have two zeros in $(-\epsilon, \epsilon)$, we must have $\xi \in (-\epsilon, \epsilon)$ and $|\eta| < |c|\epsilon^2$. Deduce that

$$\mathbf{P}[\mathbf{g} \text{ has two zeros in } (-\epsilon, \epsilon)] \asymp \epsilon^3 \chi_{(a,b)}(0, 0) \mathbf{E} \left[c^2 \mid a = b = 0 \right],$$

where $\chi_{(a,b)}$ is the density of (a, b) (which we assume to exist at $(0, 0)$).

(As in the integral-geometric proof of Buffon’s needle problem 2.4.6, we can see this pictorially as follows. Imagine the parabola to be fixed in the plane and the origin of the coordinate axes to be random. Then the event under consideration is that the origin should fall in a $2\epsilon \times |c|\epsilon^2$ rectangle centered at the tip, and that is the source of the ϵ^3 factor).

(iii) Work out some sufficient conditions so that this probability is indeed $O(\epsilon^3)$. Adapt the proof to the case when $\mathbf{f}: \mathbb{C} \rightarrow \mathbb{C}$.

3.5. Fluctuation behaviour of the zeros

Let \mathbf{f} be one of the GAFs described in equations (2.3.4), (2.3.5) and (2.3.6) on one of the three domains, which we denote by Λ . Fix a function $\varphi : \Lambda \rightarrow \mathbb{R}$. Assume $\varphi \in C_c^2(\Lambda)$. We define

$$\mathcal{Z}_L(\varphi) = \sum_{z \in \mathbf{f}_L^{-1}(0)} \varphi(z).$$

Then we know that $\mathbf{E}[\mathcal{Z}_L(\varphi)] = \frac{1}{4\pi} \int_{\Lambda} \varphi(z) \Delta \log K_L(z, z) dm(z)$. Now we list a few results about the fluctuation behaviour of $\mathcal{Z}_L(\varphi)$.

Extending a result of Forrester and Honner (23) who dealt with the planar case only, Sodin and Tsirelson (80) showed that for all the three canonical models, for $\varphi \in C_c^2(\Lambda)$,

$$(3.5.1) \quad \text{Var}[\mathcal{Z}_L(\varphi)] = \frac{\kappa}{L} \|\Delta^* \varphi\|_{L^2(m^*)}^2 + o\left(\frac{1}{L}\right), \quad \text{as } L \rightarrow \infty.$$

Here κ is a numerical constant that is the same for all the three models while m^* and Δ^* are the invariant measure and invariant laplacian on Λ , normalized as follows

$$dm^*(z) = \begin{cases} \frac{1}{2\pi} dm(z) & \Lambda = \mathbb{C}, \\ \frac{1}{2\pi(1+|z|^2)^2} dm(z) & \Lambda = \mathbb{S}^2, \\ \frac{1}{2\pi(1-|z|^2)^2} dm(z) & \Lambda = \mathbb{D}, \end{cases} \quad \Delta^* = \begin{cases} \Delta & \Lambda = \mathbb{C}, \\ (1+|z|^2)^2 \Delta & \Lambda = \mathbb{S}^2, \\ (1-|z|^2)^2 \Delta & \Lambda = \mathbb{D}. \end{cases}$$

The most remarkable feature of (3.5.1) is that the variance decreases to zero as L increases! The dependence on the second derivative of φ is also novel. These reinforce our intuition of the zero set as rigid and lattice-like. Given the decrease in variance, it is surprising that asymptotic normality holds.

Asymptotic Normality [Sodin and Tsirelson] For each of the three models, for $\varphi \in C_c^2(\Lambda)$, as $L \rightarrow \infty$,

$$\sqrt{L}(\mathcal{Z}_L(\varphi) - \mathbf{E}[\mathcal{Z}_L(\varphi)]) \xrightarrow{d} N\left(0, \kappa \|\Delta^* \varphi\|_{L^2(m^*)}^2\right),$$

where κ is a constant that is the same for all the three geometries.

Sodin and Tsirelson do not use the asymptotics of the variance in proving asymptotic normality. For details consult (80). Here we content ourselves with a derivation of the variance of $\mathcal{Z}_L(\varphi)$ for $\varphi \in C_c^2(\Lambda)$. Write (from Edelman-Kostlan formula)

$$\mathcal{Z}_L(\varphi) - \mathbf{E}[\mathcal{Z}_L(\varphi)] = \int_{\Lambda} \Delta \varphi(z) \log |\hat{\mathbf{f}}(z)| \frac{dm(z)}{2\pi},$$

where $\hat{\mathbf{f}}(z) = \frac{\mathbf{f}_L(z)}{\sqrt{K_L(z, z)}}$. We usually omit the subscript L on the GAF \mathbf{f} for simplicity of notation. Then we get (justify the exchange of expectation and integral using Fubini's theorem as in (2.4.6))

$$(3.5.2) \quad \text{Var}[\mathcal{Z}_L(\varphi)] = \int_{\Lambda^2} \Delta \varphi(z) \Delta \varphi(w) \mathbf{E}[\log |\hat{\mathbf{f}}(z)| \log |\hat{\mathbf{f}}(w)|] \frac{dm(z)}{2\pi} \frac{dm(w)}{2\pi}.$$

For any z, w fixed, $(\hat{\mathbf{f}}(z), \hat{\mathbf{f}}(w))$ has a joint complex Gaussian distribution with mean zero and marginal variances equal to 1 and $\mathbf{E}[\hat{\mathbf{f}}_L(z)\overline{\hat{\mathbf{f}}_L(w)}] = \theta(z, w)^L$ where

$$\theta(z, w) = \frac{K_1(z, w)}{\sqrt{K_1(z, z)}\sqrt{K_1(w, w)}} = \begin{cases} e^{z\bar{w} - \frac{1}{2}|z|^2 - \frac{1}{2}|w|^2} & \Lambda = \mathbb{C}, \\ \frac{1+z\bar{w}}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}} & \Lambda = \mathbb{S}^2, \\ \frac{\sqrt{1-|z|^2}\sqrt{1-|w|^2}}{1-z\bar{w}} & \Lambda = \mathbb{D}. \end{cases}$$

It is also easy to see that $|\hat{\mathbf{f}}_L|$ is a random function invariant under the isometries of Λ .

It is more convenient to write (3.5.2) in terms of the invariant measure and the invariant Laplacian as

$$(3.5.3) \quad \text{Var}[\mathcal{Z}_L(\varphi)] = \int_{\Lambda^2} \Delta^* \varphi(z) \Delta^* \varphi(w) \mathbf{E}[\log|\hat{\mathbf{f}}(z)| \log|\hat{\mathbf{f}}(w)|] dm^*(z) dm^*(w).$$

Observe that $\mathbf{E}[\log|\hat{\mathbf{f}}(z)|]$ is a constant and hence integrating it against $\Delta^* \varphi$ yields zero. Therefore we can rewrite (3.5.3) as

$$(3.5.4) \quad \text{Var}[\mathcal{Z}_L(\varphi)] = \int_{\Lambda^2} \Delta^* \varphi(z) \Delta^* \varphi(w) \rho_L(z, w) dm^*(z) dm^*(w),$$

where $\rho_L(z, w) = \text{Cov}(\log|\hat{\mathbf{f}}_L(z)|, \log|\hat{\mathbf{f}}_L(w)|)$. From Lemma 3.5.2, we have in particular that $\rho_L(z, w) \geq 0$ and that

$$(3.5.5) \quad \frac{1}{4}|\theta(z, w)|^{2L} \leq \rho_L(z, w) \leq \frac{1}{2}|\theta(z, w)|^{2L}.$$

Now, write the right hand side of (3.5.4) as a sum of three integrals (A_L to be chosen appropriately later)

- (1) $I_1 := \int \mathbf{1}_{\{\rho_L(z, w) \leq A_L\}} \Delta^* \varphi(z) \Delta^* \varphi(w) \rho_L(z, w) dm^*(z) dm^*(w)$.
- (2) $I_2 := \int \mathbf{1}_{\{\rho_L(z, w) > A_L\}} (\Delta^* \varphi(z) - \Delta^* \varphi(w)) \Delta^* \varphi(w) \rho_L(z, w) dm^*(z) dm^*(w)$.
- (3) $I_3 := \int \mathbf{1}_{\{\rho_L(z, w) > A_L\}} (\Delta^* \varphi(w))^2 \rho_L(z, w) dm^*(z) dm^*(w)$.

It is evident that

$$(3.5.6) \quad |I_1| \leq A_L \|\Delta^* \varphi\|_{L^1(m^*)}^2.$$

To bound the second integral, first note that by uniform continuity of $\Delta^* \varphi$, for all z, w , we have $|\Delta^* \varphi(z) - \Delta^* \varphi(w)| \leq \epsilon(|\theta(z, w)|^2)$ where $\epsilon(t) \searrow 0$ as $t \nearrow 1$. From the bounds (3.5.5)

$$(3.5.7) \quad |I_2| \leq C(\varphi) \epsilon\left((2A_L)^{\frac{1}{L}}\right) \int \mathbf{1}_{\{\rho_L(z, 0) > A_L\}} \rho_L(z, 0) dm^*(z).$$

The third integral may be evaluated exactly as follows.

$$(3.5.8) \quad I_3 = \|\Delta^* \varphi\|_{L^2(m^*)}^2 \int \mathbf{1}_{\{\rho_L(z, 0) > A_L\}} \rho_L(z, 0) dm^*(z).$$

Again we used the invariance of ρ_L under isometries of Λ . Choose $A_L = L^{-2}$. Then, by (3.5.6), we see that $I_1 = O(L^{-2})$. Further, $A_L^{1/L} \rightarrow 1$, whence $\epsilon((2A_L)^{1/L}) \rightarrow 0$ and thus by (3.5.7) and (3.5.8) we get $I_2 = o(1)I_3$. Thus we conclude that

$$\text{Var}(\mathcal{Z}_L(\varphi)) = I_3(1 + o(1)) + O(L^{-2}),$$

and the remaining task is to compute

$$\int \mathbf{1}_{\{\rho_L(z, 0) > A_L\}} \rho_L(z, 0) dm^*(z).$$

Invoking Lemma 3.5.2 again, we write

$$\rho_L(z, 0) = \sum_{m=1}^{\infty} |\theta(z, 0)|^{2Lm} (4m^2)^{-1}.$$

We show the details in the planar case when $|\theta(z, 0)|^2 = e^{-|z|^2}$. The other two cases are dealt with in an almost identical manner. From (3.5.5),

$$\left\{ |z|^2 \leq \frac{2\log L - \log 4}{Lm} \right\} \subset \{ \rho_L(z, 0) > L^{-2} \} \subset \left\{ |z|^2 \leq \frac{2\log L - \log 2}{Lm} \right\}$$

and hence,

$$\left| \int \mathbf{1}_{\{\rho_L(z, 0) > A_L\}} \rho_L(z, 0) dm^*(z) - \int \mathbf{1}_{\{|z|^2 \leq \frac{2\log L}{L}\}} \rho_L(z, 0) dm^*(z) \right| \leq \frac{c \log L}{L^2 m^2}.$$

Thus,

$$\begin{aligned} I_3 &= \sum_{m=1}^{\infty} \frac{1}{4m^2} \int \mathbf{1}_{\{|z|^2 \leq \frac{2\log L}{L}\}} e^{-Lm|z|^2} dm^*(z) + O(L^{-2}) \\ &= \sum_{m=1}^{\infty} \frac{\pi}{4m^2} \int_0^{(2\log L)/L} e^{-Lmx} dx + O(L^{-2}) \\ &= \sum_{m=1}^{\infty} \frac{\pi}{4Lm^3} (1 - e^{-2m \log L}) + O(L^{-2}) \\ &= \frac{\pi}{4L} \zeta(3) + O(L^{-2}). \end{aligned}$$

Combine this with (3.5.8) to see that

$$\text{Var}[\mathcal{Z}_L(\varphi)] = \frac{\pi \zeta(3)}{4L} \|\Delta^* \varphi\|_{L^2(m^*)}^2 + o(L^{-1}).$$

REMARK 3.5.1. Remember that m^* has a non-standard normalization, for instance in the planar case, written in terms of the usual Lebesgue measure, the variance is

$$\frac{\pi \zeta(3)}{16L} \|\Delta^* \varphi\|_{L^2(m)}^2 + o(L^{-1}).$$

It remains to prove the following lemma.

LEMMA 3.5.2. *If (a, b) is complex Gaussian with $\mathbf{E}[a\bar{a}] = \mathbf{E}[b\bar{b}] = 1$ and $\mathbf{E}[a\bar{b}] = \theta$, then*

$$\text{Cov}(\log |a|, \log |b|) = \sum_{m=1}^{\infty} \frac{|\theta|^{2m}}{4m^2}.$$

PROOF. It is well known that the Laguerre polynomials

$$L_n(x) := \frac{e^x}{n!} \frac{d^n}{dx^n} (x^n e^{-x}),$$

for $n \geq 0$, form an orthonormal basis for $L^2(\mathbb{R}_+, e^{-x} dx)$. To see this, integrate by parts to obtain,

$$\int_{\mathbb{R}_+} L_n(x) L_m(x) e^{-x} dx = \int_{\mathbb{R}_+} \frac{(-x)^n}{n!} \frac{d^n L_m(x)}{dx^n} e^{-x} dx.$$

Thus if $n > m$, we get zero, since L_m is clearly a polynomial of degree m . If $n = m$, then $\frac{d^n L_m(x)}{dx^n}$ is $n!$ times the leading coefficient of L_n . From the definition of L_n , the leading coefficient is $\frac{(-1)^n}{n!}$, when we see that $\{L_n\}$ is an orthonormal basis. (Further

details can be found in (2) or any other book on special functions and orthogonal polynomials).

Now, $\log x$ is in $L^p(\mathbb{R}_+, e^{-x} dx)$ for any $p > 0$, and therefore we can write the expansion

$$\log x = \sum_{n=0}^{\infty} C_n L_n(x),$$

in the L^2 sense. The coefficients can be explicitly computed as follows. C_0 can be computed in terms of Euler's constant γ , but will be irrelevant to us. For $n \geq 1$,

$$\begin{aligned} C_n &= \int_0^{\infty} \log(x) \frac{e^x}{n!} \frac{d^n}{dx^n} (x^n e^{-x}) e^{-x} dx \\ &= \frac{(-1)^n}{n!} \int_0^{\infty} \frac{d^n \log(x)}{dx^n} x^n e^{-x} dx \\ &= \frac{(-1)^n}{n!} \int_0^{\infty} \frac{(-1)^{n-1} (n-1)!}{x^n} x^n e^{-x} dx \\ &= -\frac{1}{n}. \end{aligned}$$

If a is a standard complex Gaussian, then $|a|^2$ has density $e^{-x} dx$ on \mathbb{R}_+ . Thus, we can write (with $C_n = -\frac{1}{n}$, for $n \geq 1$)

$$\log |a| = \sum_{n=0}^{\infty} \frac{C_n}{2} L_n(|a|^2), \quad \log |b| = \sum_{n=0}^{\infty} \frac{C_n}{2} L_n(|b|^2).$$

These expansions are valid in L^2 of the probability space on which a, b are defined. From this we firstly deduce that $\mathbf{E}[\log |a|] = \frac{C_0}{2}$. Hence

$$(3.5.9) \quad \text{Cov}(\log |a|, \log |b|) = \sum_{(n,m) \neq (0,0)} \frac{C_n C_m}{4} \mathbf{E}[L_n(|a|^2) L_m(|b|^2)].$$

To compute $\mathbf{E}[L_n(|a|^2) L_m(|b|^2)]$, we use the following explicit expression for Laguerre polynomials, that follows easily from the definition (alternately, it suffices to prove that the polynomials given here are orthonormal).

$$L_n(x) = \sum_{k=0}^n \frac{(-1)^k n!}{k! k! (n-k)!} x^k.$$

Therefore

$$\mathbf{E}[L_n(|a|^2) L_m(|b|^2)] = \sum_{k=0}^n \sum_{\ell=0}^m \frac{(-1)^{k+\ell} n! m!}{(k!)^2 (n-k)! (\ell!)^2 (m-\ell)!} \mathbf{E}[|a|^{2k} |b|^{2\ell}].$$

Now from the Lemma 2.1.7, we get

$$\mathbf{E}[|a|^{2k} |b|^{2\ell}] = \sum_{r=0}^{k \wedge \ell} \binom{k}{r}^2 \binom{\ell}{r}^2 (r!)^2 (k-r)! (\ell-r)! |\theta|^{2r}.$$

To get this from Lemma 2.1.7, consider the Gaussian vector $(a, \dots, a, b, \dots, b)$, with k many "a"s and ℓ many "b"s. Then group the permutations in the expansion of the permanent, according to the number of $1 \leq i \leq k$ such that $\pi_i \geq k+1$. A permutation with r such indices i gives the term $|\theta|^{2r}$.

Thus we get

$$\begin{aligned}
\mathbf{E}[L_n(|a|^2)L_m(|b|^2)] &= \sum_{k=0}^n \sum_{\ell=0}^m \sum_{r=0}^{k \wedge \ell} \frac{(-1)^{k+\ell} n!m!|\theta|^{2r}}{(n-k)!(m-\ell)!(k-r)!(\ell-r)!(r!)^2} \\
&= \sum_{r \geq 0} \frac{n!m!|\theta|^{2r}}{(r!)^2} \sum_{k=r}^n \sum_{\ell=r}^m \frac{(-1)^{k+\ell}}{(n-k)!(m-\ell)!(k-r)!(\ell-r)!} \\
&= \sum_{r \geq 0} \frac{n!m!|\theta|^{2r}}{(r!)^2} \frac{(1-1)^{n-r}}{(n-r)!} \frac{(1-1)^{m-r}}{(m-r)!}.
\end{aligned}$$

Thus the only term that does not vanish is the one with $m = r = n$. Thus we have

$$\mathbf{E}[L_n(|a|^2)L_m(|b|^2)] = |\theta|^{2n} \delta_{n,m}.$$

Thus from (3.5.9), and using $C_n = -\frac{1}{n}$ for $n \geq 1$, it follows that

$$\text{Cov}(\log|a|, \log|b|) = \sum_{n \geq 1} \frac{|\theta|^{2n}}{4n^2}.$$

□

Determinantal Point Processes

4.1. Motivation

In this chapter we move away from zeros of random analytic functions and study a different class of point processes known as **determinantal point processes**. These arise surprisingly often, in random matrix theory, combinatorics and physics, as our representative list of examples in section 4.3 will show. Many examples were already known before Macchi introduced the general notion in 1975. To motivate the definition, we remind the reader that in quantum mechanics, a physical quantity, say the position of an electron, is represented by a complex valued function (the *wave function*) ψ such that $\int |\psi|^2 = 1$. Then $|\psi|^2$ gives the probability density function of the position. Now consider n individual wave functions ψ_1, \dots, ψ_n on Λ . The most obvious way to construct an n -particle wave function out of the ψ_i s is to consider

$$(\psi_1 \otimes \dots \otimes \psi_n)(x_1, \dots, x_n) = \prod_{i=1}^n \psi_i(x_i),$$

which is tantamount to making the individual positions be independent random variables. This does not capture the physical reality, for electrons repel, and moreover the particles are indistinguishable. For this reason, physicists symmetrize or anti-symmetrize the wave-function $\psi_1 \otimes \dots \otimes \psi_n$, either of which leads to a symmetrization of the probability density. We shall consider anti-symmetrization here. Symmetrizing $\psi_1 \otimes \dots \otimes \psi_n$ would lead to **permanental point processes**, which are studied in section 4.9.

For particles with repulsion (“fermions”), one should anti-symmetrize and this yields the wave function

$$\frac{1}{\sqrt{n!}} \sum_{\pi \in \mathbb{S}_n} \text{sgn}(\pi) \prod_{i=1}^n \psi_{\pi_i}(x_i) = \frac{1}{\sqrt{n!}} \det(\psi_j(x_i))_{i,j \leq n}.$$

If $\{\psi_i\}$ is orthonormal, then the absolute square of this wave function is a probability density, for, integrating $\prod_{i=1}^n \psi_{\pi_i}(x_i)$ against $\prod_{i=1}^n \bar{\psi}_{\sigma_i}(x_i)$ gives zero unless $\pi = \sigma$. Thus we get the probability density on Λ^n

$$\frac{1}{n!} \det(\psi_j(x_i)) \det(\bar{\psi}_i(x_j)) = \frac{1}{n!} \det(\mathbb{K}(x_i, x_j))_{i,j \leq n},$$

where $\mathbb{K}(x, y) = \sum_{i=1}^n \psi_i(x) \bar{\psi}_i(y)$. Note that the probability density vanishes whenever $x_i = x_j$ for some $i \neq j$ which indicates that the points tend to “repel”.

There is one more step required. If we want to define analogous point processes with infinitely many points, or even to effectively study *local* properties of the finite ones, we need to have the joint intensities (definition 1.2.2). Here a fortuitous simplification occurs which is at the very heart of the virtues of a determinantal point

process. It is that one can explicitly integrate out some of the variables and get the joint intensities! We leave this as an exercise, whose purpose is to motivate the definition that follows. Read the proof of Lemma 4.5.1 for a solution to this exercise.

EXERCISE 4.1.1. (1) Let ψ_k , $1 \leq k \leq n$, be an orthonormal set of functions in $L^2(\Lambda, \mu)$. Set $\mathbb{K}(x, y) = \sum_{k=1}^n \psi_k(x) \overline{\psi_k(y)}$. From the identity $\int_{\Lambda} \mathbb{K}(x, y) \mathbb{K}(y, z) d\mu(y) = \mathbb{K}(x, z)$, show that for $1 \leq m \leq n$,

$$\int_{\Lambda} \det(\mathbb{K}(x_i, x_j))_{i, j \leq m} d\mu(x_m) = (n - m + 1) \det(\mathbb{K}(x_i, x_j))_{i, j \leq m-1}.$$

(2) Consider a random vector in Λ^n with density $\frac{1}{n!} \det(\mathbb{K}(x_i, x_j))_{i, j \leq n}$. Erase the labels and regard it as a point process on Λ with n points. Deduce that the joint intensities of this point process are given by

$$(4.1.1) \quad \rho_k(x_1, \dots, x_k) = \det(\mathbb{K}(x_i, x_j))_{i, j \leq k}.$$

4.2. Definitions

We now proceed with the formal definitions. Let Λ be a locally compact Polish space and μ , a Radon measure on Λ (see section 1.2 for definitions). Let $\mathbb{K}(x, y) : \Lambda^2 \rightarrow \mathbb{C}$ be a measurable function.

DEFINITION 4.2.1. A point process \mathcal{X} on Λ is said to be a **determinantal process** with kernel \mathbb{K} if it is simple and its joint intensities with respect to the measure μ satisfy

$$(4.2.1) \quad \rho_k(x_1, \dots, x_k) = \det(\mathbb{K}(x_i, x_j))_{1 \leq i, j \leq k},$$

for every $k \geq 1$ and $x_1, \dots, x_k \in \Lambda$.

Note that no claim is made about the existence or uniqueness of a determinantal point process for a given kernel \mathbb{K} . The existence issue will be addressed in section 4.5, and the uniqueness question is resolved in Lemma 4.2.6. We make a few preliminary observations that restrict the kernels \mathbb{K} that we may allow.

- (1) Definition 4.2.1 says that the first intensity (with respect to μ) is $\mathbb{K}(x, x)$. If μ has no atoms, then $\mu \otimes \mu(\{x, x\} : x \in \Lambda)$ is zero. Thus for a general measurable function \mathbb{K} , which is only defined almost everywhere on Λ^2 , it does not even make sense to speak of the function $\mathbb{K}(x, x)$. Similarly, for k -point intensity to make sense at (x_1, \dots, x_k) , we shall need $\mathbb{K}(x_i, x_j)$ to be well defined for every pair (x_i, x_j) , and it is not always true that the set of such (x_1, \dots, x_k) has positive $\mu^{\otimes k}$ measure in Λ^k .
- (2) According to definition 1.2.2, the k^{th} joint intensity must be locally integrable on Λ^k with respect to $\mu^{\otimes k}$ (locally integrable means that the integral is finite on compact subsets of Λ^k). Thus for definition 4.2.1 to make sense, we must have $\det(\mathbb{K}(x_i, x_j))_{i, j \leq k}$ to be locally integrable on Λ^k .
- (3) Joint intensities are non-negative. Thus, determinant of $(\mathbb{K}(x_i, x_j))_{i, j \leq n}$ must not be negative.
- (4) Suppose \mathcal{X} is determinantal with kernel \mathbb{K} with respect to the background measure μ . If $h : \Lambda \rightarrow \mathbb{C}$ is such that $1/h$ is locally in $L^2(\mu)$, then we may define the kernel $\mathbb{K}_h(x, y) = h(x)\overline{h(y)}\mathbb{K}(x, y)$ and $d\mu_h(x) = \frac{1}{|h(x)|^2} d\mu(x)$. It is

easy to see that \mathcal{X} is determinantal also with kernel \mathbb{K}_h with respect to the measure μ_h , because

$$\det(\mathbb{K}_h(x_i, x_j))_{i,j \leq k} \prod_{j=1}^k d\mu_h(x_j) = \det(\mathbb{K}(x_i, x_j))_{i,j \leq k} \prod_{j=1}^k d\mu(x_j).$$

Thus we have some freedom in changing the measure and the kernel together. In all our examples we may take μ to be the Lebesgue measure (when Λ is a subset of \mathbb{R}^d) or the counting measure (when Λ is finite or countable). However we shall sometimes choose μ to be something naturally associated to the underlying space (eg., Gaussian measure when Λ is the complex plane),

Naturally, the first two problems do not arise if \mathbb{K} is continuous. However, it is natural to work under a much less restrictive assumption on the kernel and that is what we explore next.

4.2.1. Integral kernels. Consider a kernel \mathbb{K} that is locally square integrable on Λ^2 . This means that for any compact $D \subset \Lambda$, we have

$$(4.2.2) \quad \int_{D^2} |\mathbb{K}(x, y)|^2 d\mu(x) d\mu(y) < \infty.$$

Then, we may use \mathbb{K} as an integral kernel to define an associated integral operator as

$$(4.2.3) \quad \mathcal{X}f(x) = \int_{\Lambda} \mathbb{K}(x, y)f(y) d\mu(y) \quad \text{for a.e. } x \in \Lambda$$

for functions $f \in L^2(\Lambda, \mu)$ that vanish μ -a.e. outside a compact subset of Λ . For a compact set D , the **restriction** of \mathcal{X} to D is the bounded linear operator \mathcal{X}_D on $L^2(D, \mu)$ defined by

$$(4.2.4) \quad \mathcal{X}_D f(x) = \int_{\Lambda} \mathbb{K}(x, y)f(y) d\mu(y) \quad \text{for a.e. } x \in D.$$

By Cauchy-Schwarz inequality, the operator norm of \mathcal{X}_D is bounded by the square root of the integral in (4.2.2), which shows boundedness. In fact, \mathcal{X}_D is a compact operator, because it can be approximated in operator norm by operators with finite dimensional range (to see this, approximate \mathbb{K} by simple functions). Readers not familiar with the notion of compact operators may consult the book (72). In particular, see exercise 13 of chapter 4 and Theorem 4.25 therein. The fact we need about compact operators is this:

The spectrum is discrete, 0 is the only possible accumulation point and every non-zero eigenvalue has finite multiplicity.

We now make one additional assumption that \mathbb{K} is Hermitian, that is,

$$(4.2.5) \quad \mathbb{K}(x, y) = \overline{\mathbb{K}(y, x)} \quad \text{for every } x, y \in \Lambda.$$

Equivalently, we may say that \mathcal{X}_D is a self-adjoint operator for any D (compact subset of Λ). The spectral theorem for self-adjoint operators, together with the compactness of \mathcal{X}_D , yields the following fact.

$L^2(D, \mu)$ has an orthonormal basis $\{\varphi_j^D\}$ of eigenfunctions of \mathcal{X}_D . The corresponding eigenvalues $\{\lambda_j^D\}$ have finite multiplicity (except possibly the zero eigenvalue)

and the only possible accumulation point of the eigenvalues is 0. We say that \mathcal{K}_D is of **trace class** if

$$(4.2.6) \quad \sum_j |\lambda_j^D| < \infty.$$

If \mathcal{K}_D is of trace class for every compact subset D , then we say that \mathcal{K} is **locally of trace class**. The following lemma relates the kernel to the eigenfunctions and eigenvalues.

LEMMA 4.2.2. *Let \mathbb{K} be a Hermitian kernel in $L^2(\Lambda^2, \mu \otimes \mu)$. Assume that the associated integral operator \mathcal{K} is of trace class. Then, there exists $\Lambda_1 \subset \Lambda$ with $\mu(\Lambda \setminus \Lambda_1) = 0$ such that the following hold.*

(1) *The series*

$$(4.2.7) \quad \sum_j \lambda_j \varphi_j(x) \bar{\varphi}_j(y)$$

converges absolutely for $x, y \in \Lambda_1$.

(2) *For each $x \in \Lambda_1$, the series in (4.2.7) converges in $L^2(\Lambda)$ as a function of y . The same is true with x and y interchanged. In addition, the series converges in $L^2(\Lambda \times \Lambda)$ as a function of (x, y) .*

(3) *The resulting sum in (4.2.7) is equal to $\mathbb{K}(x, y)$ for a.e. $(x, y) \in \Lambda^2$.*

(4) *Redefine \mathbb{K} by the series (4.2.7). Then the function*

$$(4.2.8) \quad (x_1, \dots, x_k) \rightarrow \prod_{j=1}^k \mathbb{K}(x_j, x_{j+1}), \quad \text{where } k+1 = 1,$$

is well-defined a.s. on Λ^k with respect to $\mu^{\otimes k}$. Moreover the resulting function is integrable on Λ^k .

PROOF. (1) By assumption of trace class,

$$\int_{\Lambda} \left(\sum_j |\lambda_j| |\varphi_j(x)|^2 \right) d\mu(x) = \sum_j |\lambda_j|$$

is finite. This shows that the series $\sum |\lambda_j| |\varphi_j(x)|^2$ converges in $L^2(\Lambda)$ and also that it converges pointwise for every $x \in \Lambda_1$ for some $\Lambda_1 \subset \Lambda$ with $\mu(\Lambda \setminus \Lambda_1) = 0$. By Cauchy-Schwarz inequality,

$$(4.2.9) \quad \left| \sum_{j \geq N} \lambda_j \varphi_j(x) \bar{\varphi}_j(y) \right|^2 \leq \left(\sum_{j \geq N} |\lambda_j| |\varphi_j(x)|^2 \right) \left(\sum_{j \geq N} |\lambda_j| |\varphi_j(y)|^2 \right).$$

Hence, if $x, y \in \Lambda_1$, then the series

$$(4.2.10) \quad \sum_j \lambda_j \varphi_j(x) \bar{\varphi}_j(y)$$

converges absolutely.

(2) For fixed $x \in \Lambda_1$, we see from (4.2.9) that

$$\int \left| \sum_{j \geq N} \lambda_j \varphi_j(x) \bar{\varphi}_j(y) \right|^2 d\mu(y) \leq \left(\sum_j |\lambda_j| |\varphi_j(x)|^2 \right) \sum_{j \geq N} |\lambda_j|$$

which implies that $\sum \lambda_j \varphi_j(x) \bar{\varphi}_j(y)$ is Cauchy in $L^2(\Lambda)$. Obviously we may interchange the roles of x and y . Further, again using (4.2.9) we get

$$\begin{aligned} \int_{\Lambda^2} \left| \sum_{j \geq N} \lambda_j \varphi_j(x) \bar{\varphi}_j(y) \right|^2 d\mu(y) d\mu(x) &\leq \left(\int_{\Lambda} \sum_{j \geq N} |\lambda_j| |\varphi_j(x)|^2 d\mu(x) \right)^2 \\ &\leq \left(\sum_{j \geq N} |\lambda_j| \right)^2. \end{aligned}$$

This proves that as a function of (x, y) , the series is Cauchy in $L^2(\Lambda^2)$.

- (3) Let $f \in L^2(\Lambda, \mu)$. Write f in terms of the orthonormal basis $\{\varphi_j\}$ to get for any $x \in \Lambda_1$,

$$\begin{aligned} \mathcal{K}f(x) &= \sum_j \left(\int_{\Lambda} f(y) \bar{\varphi}_j(y) d\mu(y) \right) \lambda_j \varphi_j(x) \\ &= \int_{\Lambda} \left[\sum_j \lambda_j \varphi_j(x) \bar{\varphi}_j(y) \right] f(y) d\mu(y) \end{aligned}$$

where the interchange of integral and sum is justified by the $L^2(\Lambda)$ convergence of the series $y \rightarrow \sum \lambda_j \varphi_j(x) \bar{\varphi}_j(y)$ for each $x \in \Lambda_1$. But then $\sum \lambda_j \varphi_j(x) \bar{\varphi}_j(y)$ must be a kernel for the integral operator \mathcal{K} that also has kernel \mathbb{K} . This implies that we must have

$$(4.2.11) \quad \sum_j \lambda_j \varphi_j(x) \bar{\varphi}_j(y) = \mathbb{K}(x, y)$$

for $\mu \otimes \mu$ -a.e. (x, y)

- (4) Note that the redefinition changes \mathbb{K} only on a set of $\mu \otimes \mu$ measure zero. By part (1) it follows that $\mathbb{K}(x_1, x_2) \dots \mathbb{K}(x_k, x_1)$ is well-defined for any $x_1, \dots, x_k \in \Lambda_1$, hence a.e. in Λ^k . Integrability follows from (here $k+1$ should be interpreted as 1)

$$\begin{aligned} \int_{\Lambda^k} \prod_{i=1}^k |\mathbb{K}(x_i, x_{i+1})| d\mu(x_1) \dots d\mu(x_k) &\leq \sum_{j_1, \dots, j_k} \prod_{i=1}^k |\lambda_{j_i}| \prod_{i=1}^k \int_{\Lambda} \varphi_{j_i}(x_i) \bar{\varphi}_{j_{i-1}}(x_i) d\mu(x_i) \\ &\leq \sum_{j_i, i \leq k} \prod_{i=1}^k |\lambda_{j_i}|. \end{aligned}$$

The last line used Cauchy-Schwarz inequality. This sum is just $(\sum |\lambda_j|)^k$ and hence is finite. \square

Now we return to determinantal processes.

ASSUMPTION 4.2.3. We assume henceforth that the kernel $\mathbb{K} \in \Lambda^2 \rightarrow \mathbb{C}$ is locally square integrable and that the associated integral operator \mathcal{K} is Hermitian, non-negative definite and locally of trace class. In the notations that we have been using, this is equivalent to saying that

- $\mathbb{K}(x, y) = \overline{\mathbb{K}(y, x)}$ a.s. $(\mu \otimes \mu)$,
- $\det(\mathbb{K}(x_i, x_j))_{i, j \leq k} \geq 0$ a.s. $(\mu^{\otimes k})$,
- for any compact $D \subset \Lambda$, we have $\lambda_j^D \geq 0$ for every j and $\sum_j \lambda_j^D < \infty$.

By Lemma 4.2.2, we redefine \mathbb{K} on a set of measure zero so that

$$(4.2.12) \quad \mathbb{K}(x, y) = \sum_j \lambda_j^D \varphi_j^D(x) \overline{\varphi_j^D}(y)$$

for all $x, y \in D_1$, where $\mu(D \setminus D_1) = 0$. The series converges in all the senses stated in the lemma.

REMARK 4.2.4. A word about our assumptions on \mathbb{K} . There is no loss of generality in assuming that the kernel is locally in L^2 , for, the two-point intensity must be locally integrable on Λ^2 . Non-negativity of the determinants of \mathbb{K} is similarly forced upon us. However the Hermitian assumption is not essential and indeed, there are interesting examples of determinantal processes with non-Hermitian kernels (particularly those that involve “time” or “dynamics”, see (39) for some examples). Nevertheless, most of the general properties that we prove do not hold true for non-Hermitian kernels and we do not consider them in this book.

A particularly important case is when \mathcal{K}_Λ itself is a bounded operator with all non-zero eigenvalues equal to 1. In this case we say that \mathbb{K} is a **projection kernel**, since \mathcal{K} turns out to be the projection operator onto the closed span of its eigenfunctions. If the latter space is finite dimensional, we also say that \mathbb{K} is a **finite dimensional** or **finite rank** projection kernel.

REMARK 4.2.5. It is evident that if \mathcal{X} is a determinantal point process on Λ , then for any $D \subset \Lambda$, the point process $\mathcal{X} \cap D$ is also determinantal, its kernel being just the restriction of the original kernel to $D \times D$. If the kernel (more precisely, the associated operator) on Λ is locally of trace class, then the corresponding kernel on D is of trace class. The reader should keep this in mind as most of our theorems will be stated under the apparently less general assumption of a trace class kernel, but remain valid for $\mathcal{X} \cap D$ for determinantal processes with a *locally* trace class kernel.

Now consider a determinantal point process with kernel \mathbb{K} (with respect to a Radon measure μ) that satisfies assumption 4.2.3 (from now on, this will be tacitly assumed). By virtue of Lemma 4.2.2, the first three issues raised after definition 4.2.1 are taken care of. In other words, for such kernels, $\det(\mathbb{K}(x_i, x_j))_{i, j \leq k}$ is well-defined for μ -a.e. x_1, \dots, x_k , is non-negative and is locally integrable on Λ^k . All these are essential if they are to be joint intensities of a point process.

This in itself does not imply the existence of a point process with these joint intensities, of course. In fact Theorem 4.5.5 will show that not all kernels are kernels of determinantal processes. We show uniqueness (of determinantal point process for a given kernel) now. We must do this because, in general, specifying the joint intensity functions for a point process is not enough to specify the distribution of the process. However, as we noted in chapter remark 1.2.4, the joint intensities do determine the law of the point process if the number of points in any compact set has finite exponential moments. This we verify for determinantal processes in the following lemma.

LEMMA 4.2.6. *Assume that a point process \mathcal{X} with joint intensities as in (4.2.1) does exist. Then for any compact $D \subset \Lambda$, there exist constants $c(D), C(D)$ such that*

$$(4.2.13) \quad \mathbf{P}[\mathcal{X}(D) > k] \leq C(D)e^{-c(D)k}.$$

Therefore, for a given kernel \mathbb{K} , there exists at most one point process whose joint intensities are given by

$$(4.2.14) \quad \rho_k(x_1, \dots, x_k) = \det(\mathbb{K}(x_i, x_j))_{i,j \leq k}.$$

PROOF. We shall use Hadamard's inequality which states that for any $k \times k$ matrix $A = [\mathbf{v}_1 \dots \mathbf{v}_k]$,

$$(4.2.15) \quad \det(A^* A) \leq \prod_{i=1}^k \|\mathbf{v}_i\|^2 = \prod_{i=1}^k (A^* A)_{i,i}.$$

In our case, for μ -a.e. x_1, \dots, x_k , the matrix $(\mathbb{K}(x_i, x_j))_{i,j \leq k}$ is a non-negative definite matrix, whence it is of the form $A^* A$. By (1.2.3), for any compact Borel set $D \subset \Lambda$, we must have

$$\begin{aligned} \mathbf{E} \left[\binom{\mathcal{X}(D)}{k} k! \right] &= \int_{D^k} \det(\mathbb{K}(x_i, x_j))_{1 \leq i, j \leq k} d\mu(x_1) \dots d\mu(x_k) \\ &\leq \int_{D^k} \prod_{i=1}^k \mathbb{K}(x_i, x_i) d\mu(x_1) \dots d\mu(x_k) \\ &= \left(\int_D \mathbb{K}(x, x) d\mu(x) \right)^k. \end{aligned}$$

Set $\kappa(D) = \int_D \mathbb{K}(x, x) d\mu(x)$ (finite because D is compact) and deduce that for $s > 0$

$$\begin{aligned} \mathbf{E} \left[(1+s)^{\mathcal{X}(D)} \right] &= \sum_{k \geq 0} \mathbf{E} \left[\binom{\mathcal{X}(D)}{k} \right] s^k \\ &\leq \sum_{k \geq 0} \frac{\kappa(D)^k s^k}{k!} \\ &= e^{s\kappa(D)}. \end{aligned}$$

This shows that

$$\begin{aligned} \mathbf{P}[\mathcal{X}(D) > k] &\leq (1+s)^{-k} \mathbf{E} \left[(1+s)^{\mathcal{X}(D)} \right] \\ &\leq C(D) e^{-c(D)k} \end{aligned}$$

with $C(D) = e^{s\kappa(D)}$ and $c(D) = \log(1+s)$.

In particular, the distribution of $\mathcal{X}(D)$ is determined by its moments, which are in turn determined by the joint intensities. Since this is valid for every compact D , and Λ is locally compact, it follows that the distribution of \mathcal{X} is uniquely determined. \square

4.3. Examples of determinantal processes

In this section we give a slew of examples of determinantal point processes. Proving that any of these processes is determinantal is not trivial, and there is no single method that works for all examples. In view of this, we only list the examples here, and proceed to the general theory. Only for the first example do we give a proof in this chapter. Other proofs, where we do give them, are postponed to chapter 6. In Theorem 4.5.5 we shall see necessary and sufficient conditions on a Hermitian integral kernel for it to define a determinantal process. The point here is not to give

a list of kernels satisfying those conditions, but instead, to show how a motley collection of point processes that arise naturally in probability theory turn out to be determinantal.

4.3.1. Non-intersecting random walks. Karlin and McGregor (43) provide the first result we know related to determinantal processes. The following theorem has the fundamental calculation, but does not say what is a determinantal process here. That we describe as a corollary to the theorem.

THEOREM 4.3.1 (Karlin and McGregor (1959)). *Consider n i.i.d. Markov chains on \mathbb{Z} started from $i_1 < i_2 < \dots < i_n$ where all the i_j s are even. Let $P_{i,j}(s, s+t)$ be the t -step transition probabilities between times s and $s+t$ for each of the chains. The chains are not assumed to be time homogeneous.¹ Suppose that the one-step transition probabilities satisfy $P_{i,i+1}(t, t+1) + P_{i,i-1}(t, t+1) = 1$ for all t . Then the probability that at time t , the Markov chains are at $j_1 < j_2 < \dots < j_n$, and that no two of the chains intersect up to time t , is*

$$(4.3.1) \quad \det \begin{pmatrix} P_{i_1, j_1}(0, t) & \dots & P_{i_1, j_n}(0, t) \\ \dots & \dots & \dots \\ P_{i_n, j_1}(0, t) & \dots & P_{i_n, j_n}(0, t) \end{pmatrix}.$$

REMARK 4.3.2. Observe that if the Markov chains in Theorem 4.3.1 are simple symmetric random walks, then $P_{i,j}(0, t) = \frac{1}{2^t} \binom{t}{\frac{j-i+t}{2}}$ if $j-i+t$ is even and $P_{i,j}(0, t) = 0$ otherwise. In this case, scaling the matrix in (4.3.1) by a factor of 2^{nt} we obtain an expression for the number of non-intersecting paths between $i_1 < i_2 < \dots < i_n$ and $j_1 < j_2 < \dots < j_n$. This observation and further extensions are described by Gessel and Viennot in (28).

The Karlin-McGregor formula does express a certain nonintersection probability as a determinant. But where is a determinantal process in all this? The following corollary describes one special case of a far more general theorem in Johansson's paper (40). Below, the Markov chains are time homogeneous, and we use the notation $P_t(x, y)$ and $P_{x,y}(0, t)$ interchangeably.

COROLLARY 4.3.3. *Let X_p , $1 \leq p \leq n$ be independent time-homogeneous Markov chains with one-step transition probabilities $\{P_{i,j}\}$ satisfying $P_{i,i+1} + P_{i,i-1} = 1$. We shall assume that the random walk is reversible with respect to the measure π on \mathbb{Z} , so that $\pi(x)P_t(x, y) = \pi(y)P_t(y, x)$ for all $x, y \in \mathbb{Z}$ and $t \in \mathbb{Z}_+$. Condition on the event that for each $p \leq n$, the chain X_p is at location x_p at times 0 and $2t$, and that no two of the chains intersect in the duration from 0 to $2t$. Then the configuration of the particles midway, $\{X_p(t) : 1 \leq p \leq n\}$, is a determinantal point process on \mathbb{Z} with kernel*

$$\mathbb{K}(u, v) = \sum_{j=1}^n \psi_j(u) \overline{\psi}_j(v)$$

with respect to the background measure π on \mathbb{Z} . Here, $\psi_j : \mathbb{Z} \rightarrow \mathbb{R}$ are defined by

$$\psi_j(r) = \sum_{k=1}^n \left(A^{-\frac{1}{2}} \right)_{j,k} \frac{1}{\pi(r)} P_t(x_k, r)$$

¹A time homogeneous Markov chain is one for which the transition probability $\mathbf{P}[X_{t+1} = j \mid X_t = i]$ does not depend on t .

with $A_{j,k} = \frac{1}{\pi(x_k)} P_{2t}(x_j, x_k)$.

REMARK 4.3.4. \mathbb{K} is indeed a Hermitian projection kernel since ψ_j are orthonormal in $L^2(\pi)$. To see this, use reversibility and Markov property to see that

$$\sum_{r \in \mathbb{Z}} \frac{P_t(x, r)}{\pi(r)} \frac{P_t(y, r)}{\pi(r)} \pi(r) = \frac{1}{\pi(y)} \sum_{r \in \mathbb{Z}} P_t(x, r) P_t(r, y) = \frac{P_{2t}(x, y)}{\pi(y)}$$

which shows that A is the Gram matrix (in $L^2(\pi)$) of the functions $P_t(x_p, \cdot) \pi(\cdot)^{-1}$, $p \leq n$. Therefore, ψ_j are orthonormal.

Now we give Karlin and McGregor's original proof of Theorem 4.3.1. A second proof is outlined in the exercise at the end of this section.

Proof of Theorem 4.3.1. Let us define for $\vec{i} = (i_1, \dots, i_n)$ and $\vec{j} = (j_1, \dots, j_n)$,

$$(4.3.2) \quad \Gamma(\vec{i}, \vec{j}, t) = \{\text{Paths } \vec{\gamma}(s) : \gamma_k(0) = i_k \text{ and } \gamma_k(t) = j_k \forall k \in [n]\}$$

where $[n] = \{1, 2, \dots, n\}$, so that

$$(4.3.3) \quad \mathbf{P}(\vec{\gamma}) = \prod_{k=1}^n \prod_{s=0}^{t-1} P_{\gamma_k(s), \gamma_k(s+1)}(s, s+1).$$

Given $\vec{\gamma} \in \Gamma(\vec{i}, \vec{j}, t)$, write $\text{coin}(\vec{\gamma})$ to denote the number of times the trajectories γ_k coincide:

$$(4.3.4) \quad \text{coin}(\vec{\gamma}) = \#\{(\alpha, \beta, s) : 0 < s < t, \alpha \neq \beta, \text{ and } \gamma_\alpha(s) = \gamma_\beta(s)\}.$$

Now given a permutation $\sigma \in S_n$, let us write $\sigma(\vec{j}) = (j_{\sigma(1)}, \dots, j_{\sigma(n)})$ and introduce the notation

$$\begin{aligned} \Gamma_+(\vec{i}, \vec{j}, t) &= \{\vec{\gamma} \in \Gamma(\vec{i}, \sigma(\vec{j}), t) : \text{coin}(\vec{\gamma}) > 0, \text{ sign}(\sigma) = 1\}. \\ \Gamma_-(\vec{i}, \vec{j}, t) &= \{\vec{\gamma} \in \Gamma(\vec{i}, \sigma(\vec{j}), t) : \text{sign}(\sigma) = -1\}. \\ \Gamma_0(\vec{i}, \vec{j}, t) &= \{\vec{\gamma} \in \Gamma(\vec{i}, \sigma(\vec{j}), t) : \text{coin}(\vec{\gamma}) = 0\}. \end{aligned}$$

Observe that if $\text{coin}(\vec{\gamma})$ is zero, then σ is the identity permutation. Thus the determinant of interest can be decomposed as follows:

$$\begin{aligned} \det \begin{pmatrix} P_{i_1, j_1}(0, t) & \dots & P_{i_1, j_n}(0, t) \\ \dots & \dots & \dots \\ P_{i_n, j_1}(0, t) & \dots & P_{i_n, j_n}(0, t) \end{pmatrix} &= \sum_{\sigma \in S_n} \text{sign}(\sigma) \sum_{\vec{\gamma} \in \Gamma(\vec{i}, \sigma(\vec{j}), t)} \mathbf{P}(\vec{\gamma}) \\ &= \sum_{\vec{\gamma} \in \Gamma_0(\vec{i}, \vec{j}, t)} \mathbf{P}(\vec{\gamma}) + \sum_{\vec{\gamma} \in \Gamma_+(\vec{i}, \vec{j}, t)} \mathbf{P}(\vec{\gamma}) - \sum_{\vec{\gamma} \in \Gamma_-(\vec{i}, \vec{j}, t)} \mathbf{P}(\vec{\gamma}). \end{aligned}$$

We will construct a bijective and measure-preserving mapping Φ between $\Gamma_+(\vec{i}, \vec{j}, t)$ and $\Gamma_-(\vec{i}, \vec{j}, t)$. Let $\vec{\gamma} \in \Gamma_+(\vec{i}, \vec{j}, t)$. By construction, at least two of the particles whose trajectories are defined by $\vec{\gamma}$ must be coincident at some time. Let t' be the first such time, and in case multiple pairs of particles are coincident at t' , choose the smallest coincident pair (k, ℓ) according to the dictionary order. Now imagine switching the labels of particles k and ℓ at time t' , and define $\Phi(\vec{\gamma})$ to be the resulting set of trajectories. By construction, it is clear that Φ maps $\Gamma_+(\vec{i}, \vec{j}, t)$ bijectively onto $\Gamma_-(\vec{i}, \vec{j}, t)$ and $\mathbf{P}(\Phi(\vec{\gamma})) = \mathbf{P}(\vec{\gamma})$. Hence

$$(4.3.5) \quad \sum_{\vec{\gamma} \in \Gamma_+(\vec{i}, \vec{j}, t)} \mathbf{P}(\vec{\gamma}) - \sum_{\vec{\gamma} \in \Gamma_-(\vec{i}, \vec{j}, t)} \mathbf{P}(\vec{\gamma}) = 0$$

and we deduce that

$$\det \begin{pmatrix} P_{i_1, j_1}(0, t) & \cdots & P_{i_1, j_n}(0, t) \\ \cdots & \cdots & \cdots \\ P_{i_n, j_1}(0, t) & \cdots & P_{i_n, j_n}(0, t) \end{pmatrix} = \sum_{\tilde{\gamma} \in \Gamma_0(\vec{i}, \vec{j}, t)} \mathbf{P}(\tilde{\gamma}),$$

as claimed. \square

Now we derive the corollary.

Proof of Corollary 4.3.3. Fix integers $u_1 < u_2 < \dots < u_n$ and consider the conditioned Markov chains as in the statement of the corollary. The event that $\{X_p(t) : 1 \leq p \leq n\} = \{u_p : p \leq n\}$ is the same as the event $\{X_p(t) = u_p : p \leq n\}$, because of the non-crossing condition. By applying Theorem 4.3.1 three times (from time 0 to time t , from time t to $2t$ and from time 0 to $2t$), the probability distribution of the time- t locations is given by

$$\begin{aligned} \mathbf{P}[X_p(t) = u_p \ \forall p] &= \frac{\det(P_t(x_p, u_q))_{p, q \leq n} \det(P_t(u_p, x_q))_{p, q \leq n}}{\det(P_{2t}(x_p, x_q))} \\ &= \frac{\det\left(\frac{1}{\pi(u_q)} P_t(x_p, u_q)\right)_{p, q \leq n} \det\left(\frac{1}{\pi(x_q)} P_t(u_p, x_q)\right)_{p, q \leq n}}{\det\left(\frac{1}{\pi(x_q)} P_{2t}(x_p, x_q)\right)} \prod_{q=1}^n \pi(u_q) \\ &= \frac{\det\left(\frac{1}{\pi(u_q)} P_t(x_p, u_q)\right)_{p, q \leq n} \det\left(\frac{1}{\pi(u_p)} P_t(x_q, u_p)\right)_{p, q \leq n}}{\det\left(\frac{1}{\pi(x_q)} P_{2t}(x_p, x_q)\right)} \prod_{q=1}^n \pi(u_q). \end{aligned}$$

Let B, A be the $n \times n$ matrices with $B_{p, q} = \frac{1}{\pi(u_q)} P_t(x_p, u_q)$ and $A_{p, q} = \frac{1}{\pi(x_q)} P_{2t}(x_p, x_q)$ (this A is the same as in the statement of the corollary). Then the above equation may be written succinctly as

$$\begin{aligned} \mathbf{P}[X_p(t) = u_p \ \forall p] &= \det(B^t A^{-1} B) \prod_{q=1}^n \pi(u_q) \\ &= \det\left(\left(A^{-\frac{1}{2}} B\right)^t \left(A^{-\frac{1}{2}} B\right)\right) \prod_{q=1}^n \pi(u_q). \end{aligned}$$

Observe that $(A^{-\frac{1}{2}} B)_{r, q}$ is precisely what we defined as $\psi_r(u_q)$. This shows that

$$\begin{aligned} \mathbf{P}[X_p(t) = u_p \ \forall p] &= \det\left(\left(\psi_r(u_p)\right)_{p, r \leq n} \left(\psi_r(u_q)\right)_{r, q \leq n}\right) \prod_{q=1}^n \pi(u_q) \\ &= \det(\mathbb{K}(u_p, u_q))_{p, q \leq n} \prod_{q=1}^n \pi(u_q). \end{aligned}$$

This shows that the top intensities are that of a determinantal point process. As demonstrated in remark 4.3.4, \mathbb{K} is a Hermitian projection kernel on $L^2(\pi)$. Therefore, by exercise 4.1.1, we can integrate out variables one by one and show that k -point intensities for $k < n$ are also determinantal. Since there are only n points, for $k > n$ the joint intensities must be zero, as indeed they are because \mathbb{K} is a kernel of rank n . \square

An alternate proof of the Karlin-McGregor formula (Theorem 4.3.1) is outlined in the exercise that follows. We learned this proof from S.R.S. Varadhan (personal communication).

EXERCISE 4.3.5. Let X_p , $1 \leq p \leq n$, be i.i.d. Markov chains on \mathbb{Z} with transition probabilities satisfying $P_{i,i+1}(s, s+1) + P_{i,i-1}(s, s+1) = 1$. Let $X_p(0) = i_p$, where $i_1 < \dots < i_n$. Also fix integers $j_1 < \dots < j_p$. Then

- (1) Fix a permutation $\sigma \in S_n$. For $s \leq t$, define

$$(4.3.6) \quad M^\sigma(s) := \prod_{p=1}^n \mathbf{P}[X_p(t) = j_{\sigma(p)} \mid X_p(s)].$$

Then M^σ is a martingale with respect to the filtration generated by $\mathcal{F}_s := \{X_p(u) : u \leq s, 1 \leq p \leq n\}$. Hence $M := \sum_{\sigma \in S_n} \text{sgn}(\sigma) M^\sigma$ is also a martingale.

- (2) Let $\tau = \inf\{s : X_p(s) = X_q(s) \text{ for some } p \neq q\}$. Apply the optional stopping theorem to the martingale $M(\cdot)$ and the stopping time $\tau \wedge t$ to deduce the Karlin-McGregor formula.

4.3.2. Uniform spanning trees. Let G be a finite undirected connected graph and let E be the set of oriented edges (an undirected edge connecting vertices x, y in G appears in E with both orientations xy and yx). A **spanning tree** of G is a subgraph that is connected, has no cycles and contains every vertex of G . Let T be chosen uniformly at random from the set of spanning trees of G . For each directed edge $e = vw$, let $\chi^e : E \rightarrow \mathbb{R}$ be the function defined as $\chi^e := \mathbf{1}_{vw} - \mathbf{1}_{wv}$ denote the unit flow along e . Consider the measure on E that gives mass $1/2$ to each oriented edge and let $\ell^2(E)$ be the Hilbert space with inner products taken with respect to this measure. Thus χ^e has unit norm. Define the following subspaces of $\ell^2(E)$. For each vertex v , we call $\sum_w \chi^{vw}$ the “star at the vertex v ”. Similarly, for each oriented cycle e_1, \dots, e_n , we refer to $\sum_{i=1}^n \chi^{e_i}$ also as a cycle.

$$\begin{aligned} \mathbf{H} &= \{f : E \rightarrow \mathbb{R} : f(vw) = -f(wv) \forall v, w\} = \text{span}\{\chi^e\}. \\ \star &= \text{span}\left\{\sum_w \chi^{vw} : v \text{ is a vertex}\right\}. \\ \diamond &= \text{span}\left\{\sum_{i=1}^n \chi^{e_i} : e_1, \dots, e_n \text{ is an oriented cycle}\right\}. \end{aligned}$$

Any oriented cycle enters each vertex of G exactly as many times as it leaves the vertex. This shows that $\star \perp \diamond$. On the other hand, suppose $f \in \mathbf{H}$ is orthogonal to \star as well as \diamond . From orthogonality with \diamond , one can define a new function F on the vertex set of G by integrating f along oriented paths starting from a fixed vertex. Thus if u, v are adjacent vertices in the graph, then $F(u) - F(v) = f(vu)$. Orthogonality with \star shows that F is discrete harmonic, i.e., at any vertex v , it has the mean value property

$$(4.3.7) \quad F(v) = \frac{1}{\text{deg}(v)} \sum_{u \sim v} F(u)$$

where the sum is over all vertices adjacent to v . Therefore, F is constant and hence f must be identically zero. In fact, it is easy to see that $\mathbf{H} = \star \oplus \diamond$.

For $e \in E$, define $I^e := \mathcal{K}_\star \chi^e$, where \mathcal{K}_\star denotes orthogonal projection onto \star . Now, for each undirected edge in G , choose one of the two possible oriented representatives in E . Which orientation we choose does not matter, but once made, the choices will stay fixed throughout. Then for each pair of undirected edges e, f in G , define $\mathbb{K}(e, f) := (I^e, I^f)$, where on the right, we use the chosen orientations. Note that if we had chosen the opposite orientation for an edge e , then $\mathbb{K}(e, f)$ changes sign

for each $f \neq e$. This does not change the determinants that occur in (4.3.8) because, both a row and column change sign.

The following result was proved by Burton and Pemantle (11), who represented $\mathbb{K}(e, f)$ as the current flowing through f when a unit of current is sent from the tail to the head of e . The Hilbert space formulation above, as well as the proof of the Burton-Pemantle theorem presented in chapter 6 are from the paper of Benjamini, Lyons, Peres and Schramm (5).

THEOREM 4.3.6 (Burton and Pemantle). *The set of edges in the uniform spanning tree T forms a determinantal process with kernel \mathbb{K} with respect to counting measure on the set of unoriented edges. That is, for any (unoriented) edges e_1, \dots, e_k of G , we have*

$$(4.3.8) \quad \mathbf{P}[e_1, \dots, e_k \in T] = \det(\mathbb{K}(e_i, e_j))_{i,j \leq k}.$$

4.3.3. Uniform perfect matching of a planar graph. Let $R = \{(m, n) : 1 \leq m \leq M, 1 \leq n \leq N\}$ be a rectangular region in the lattice \mathbb{Z}^2 . A **perfect matching** of R is a subgraph of R such that each vertex of R is incident to exactly one edge in the subgraph. In other words, a perfect matching divides the vertices of R into pairs in such a way that each pair of vertices is connected by an edge in the graph. Assume that MN is even, so that perfect matchings do exist.

Colour the vertices of R in white and black in a chess board fashion. Then define a matrix K with complex entries whose rows and columns are labeled by vertices of R as follows.

$$K(x, y) := \begin{cases} 1 & \text{if } y \text{ is horizontally adjacent to } x. \\ i & \text{if } y \text{ is vertically adjacent to } x. \\ 0 & \text{in all other cases.} \end{cases}$$

K is a modification of the adjacency matrix of R and is known as the **Kastelyn matrix**.

From the set of all perfect matchings of R , pick one uniformly at random and denote it by \mathcal{P} . We state without proof the following theorem of Richard Kenyon (46).

THEOREM 4.3.7 (Kenyon). *Let $(w_1, b_1), \dots, (w_k, b_k)$ be distinct edges of the graph R , where w_i are white vertices and b_i are black. Then*

$$\mathbf{P}[(w_i, b_i) \in \mathcal{P}, 1 \leq i \leq k] = \left| \det(K^{-1}(w_i, b_j))_{i,j \leq k} \right|.$$

Kenyon proved this in greater generality for planar bipartite graphs. Note the absolute value on the determinant, the matrix K is not positive definite. More accurately what we have here is a ‘‘Pfaffian process’’, a close relative of the determinantal process. The interested reader may consult the survey (47) for more on the subject of perfect matchings.

4.3.4. Gaussian unitary ensemble. The most well known example of a determinantal point process is the **Gaussian unitary ensemble**, introduced by Wigner in his statistical approach to energy levels in heavy nuclei. Mehta’s wonderful book (56) is the standard reference for this and most other random matrix examples that we shall see later.

THEOREM 4.3.8 (Wigner). *Let A be an $n \times n$ matrix with i.i.d. standard complex Gaussian entries and set $H = \frac{A+A^*}{\sqrt{2}}$. Then, the set of eigenvalues of H form a*

determinantal point process on \mathbb{R} with kernel

$$\mathbb{K}_n(x, y) = \sum_{k=0}^{n-1} H_k(x)H_k(y).$$

with respect to the Gaussian measure $d\mu(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx$ on \mathbb{R} . Here $H_k(\cdot)$ are Hermite polynomials, obtained by applying Gram-Schmidt orthogonalization procedure to $\{1, x, x^2, \dots\}$ in $L^2(\mathbb{R}, \mu)$. In particular, $\int H_k \overline{H}_\ell d\mu = \delta_{k,\ell}$. Equivalently, the vector of eigenvalues (put in a uniformly chosen random order) has joint density

$$\frac{1}{(2\pi)^{\frac{n}{2}} \prod_{j=1}^n j!} \exp \left\{ -\frac{1}{2} \sum_{k=1}^n \lambda_k^2 \right\} \prod_{i < j} (\lambda_i - \lambda_j)^2$$

with respect to Lebesgue measure on \mathbb{R}^n .

The associated operator \mathcal{K} is easily seen to be the projection from $L^2(\mathbb{R}, \mu)$ onto $\text{span}\{H_k : 0 \leq k \leq n-1\}$. We will not go into the proof of Wigner's result but direct the interested reader to chapter 3 of (56) or chapter 1 of (25).

4.3.5. Sine kernel process. Now we define a translation invariant determinantal point process on the real line that arises as the limit of very many interesting point processes. For example, the Gaussian unitary ensembles, when scaled appropriately, converge to this point process.

The sine kernel process is the determinantal point process on \mathbb{R} with the kernel

$$\mathbb{K}(x, y) = \frac{\sin \pi(x-y)}{\pi(x-y)}.$$

This kernel is not square integrable on \mathbb{R}^2 , but only on compact subsets thereof. Thus the associated operator \mathcal{K} is not of trace class but locally of trace class. What is \mathcal{K} ? Clearly, it is a convolution operator because, $\mathbb{K}(x, y)$ depends only on $x-y$, and hence is more simply represented in the Fourier domain. We define the Fourier transform \hat{f} by

$$\hat{f}(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-itx} dx, \quad \text{for } f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}),$$

and by extension to all of L^2 . It is well-known that $f \rightarrow \hat{f}$ is a unitary transformation of $L^2(\mathbb{R})$. Returning to the sine kernel, write $\mathbb{K}(x, y)$ as $\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(x-y)u} du$ to see that

$$\begin{aligned} \int_{\mathbb{R}} \mathbb{K}(x, y) f(y) dy &= \int_{\mathbb{R}} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(x-y)u} du \right) f(y) dy \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \hat{f}(u) e^{ixu} du \\ &= (\mathbf{1}_{[-\pi, \pi]} \hat{f})^\wedge(-x). \end{aligned}$$

Fourier inversion formula says that $\hat{f}^\wedge(-x) = f(x)$. Thus the integral operator \mathcal{K} is the operator in L^2 described as follows: Apply Fourier transform, multiply by the indicator function of $[-\pi, \pi]$, then apply the inverse Fourier transform. Since Fourier transform is a unitary transformation, another way to put it is that \mathcal{K} is the

projector onto the space of functions in $L^2(\mathbb{R})$ whose Fourier transform is supported in $[-\pi, \pi]$.

4.3.6. Circular Unitary Ensemble. Let $\mathcal{U}(n)$ denote the group of $n \times n$ unitary matrices. There is a unique Borel probability measure on $\mathcal{U}(n)$ that is invariant under left multiplication by unitary matrices. This measure is also invariant under right multiplication by unitary matrices, as well as under inversion. This measure is called **Haar measure**. See Theorem 5.14 of the book (72) for a proof of the existence and uniqueness of Haar measure on compact topological groups (which $\mathcal{U}(n)$ is, of course).

Dyson (20) introduced the **circular unitary ensemble** which is the set of eigenvalues of a random unitary matrix sampled from the Haar measure on $\mathcal{U}(n)$. The measure induced on eigenvalues was, however, known from Weyl.

THEOREM 4.3.9 (Weyl, Dyson). *With U as above, let $\{e^{i\theta_j} : 1 \leq j \leq n\}$ be the set of its eigenvalues. The counting measure of eigenvalues is a the determinantal point process on S^1 with kernel*

$$\mathbb{K}(e^{i\theta}, e^{i\varphi}) = \frac{1}{2\pi} \sum_{k=0}^{n-1} e^{ik\theta - ik\varphi}.$$

with respect to Lebesgue measure on S^1 (with total measure 2π). Equivalently, the vector of eigenvalues (in uniform random order) has density

$$\frac{1}{n!(2\pi)^n} \prod_{j < k} |e^{i\theta_j} - e^{i\theta_k}|^2$$

with respect to Lebesgue measure on $(S^1)^n$.

The associated operator to \mathbb{K} is the projection operator from $L^2(S^1)$ onto the subspace spanned by $\{e^{ik\theta} : 0 \leq k \leq n-1\}$.

4.3.7. Ginibre ensemble. Ginibre (29) introduced three ensembles of matrices with i.i.d. Gaussian entries without imposing a Hermitian condition. In the three cases, the Gaussians were real, complex or quaternion. Here we consider the complex case. This example is an important one, as the eigenvalues are in the complex plane and are similar to, yet different from zeros of Gaussian analytic functions.

THEOREM 4.3.10 (Ginibre (1965)). *Let M be an $n \times n$ matrix with i.i.d. standard complex Gaussian entries. Then the eigenvalues of M form a determinantal point process on the complex plane with kernel*

$$(4.3.9) \quad \mathbb{K}_n(z, w) = \sum_{k=0}^{n-1} \frac{(z\bar{w})^k}{k!}$$

with respect to the background measure $\frac{1}{\pi} e^{-|z|^2} dm(z)$. Equivalently, the vector of eigenvalues (in uniform random order) has density

$$(4.3.10) \quad \frac{1}{\pi^n \prod_{k=1}^n k!} e^{-\sum_{k=1}^n |z_k|^2} \prod_{i < j} |z_i - z_j|^2$$

with respect to Lebesgue measure on \mathbb{C}^n .

The projection operator associated to the kernel projects from $L^2(\mathbb{C}, \mu)$ onto the span of $\{z^k : 0 \leq k \leq n-1\}$. As $n \rightarrow \infty$, this converges to the projection onto the space of all entire functions in $L^2(\mu)$. The kernel is

$$(4.3.11) \quad \mathbb{K}(z, w) = e^{z\bar{w}},$$

and the associated determinantal point process is invariant in distribution under isometries of the plane. Hence it will be instructive to compare the properties of this determinantal process with the corresponding properties of the zeros of the planar Gaussian analytic function (2.3.4).

4.3.8. Spherical Ensemble. In this section we discuss a random matrix ensemble introduced in (51). The determinantal process it gives rise to, and related point processes were studied earlier by Caillol (12) and by Forrester, Jancovici and Madore (24), without this connection to random matrices.

THEOREM 4.3.11 (Krishnapur). *Let A, B be independent $n \times n$ random matrices with i.i.d. standard complex Gaussian entries. Then the eigenvalues of $A^{-1}B$ form a determinantal point process on the complex plane with kernel*

$$(4.3.12) \quad \mathbb{K}(z, w) = (1 + z\bar{w})^{n-1}$$

with respect to the background measure $\frac{n}{\pi(1+|z|^2)^{n+1}} dm(z)$. Equivalently, one may say that the vector of eigenvalues (in uniform order) has density

$$(4.3.13) \quad \frac{1}{n!} \left(\frac{n}{\pi}\right)^n \prod_{k=1}^n \binom{n-1}{k} \prod_{k=1}^n \frac{1}{(1+|z_k|^2)^{n+1}} \prod_{i<j} |z_i - z_j|^2$$

with respect to Lebesgue measure on \mathbb{C}^n .

REMARK 4.3.12. These eigenvalues are best thought of as points on the two dimensional sphere \mathbb{S}^2 , using stereographic projection from the plane. A simple calculation shows that the density of the vector of points (w.r.t. the n -fold product of the area measure on the sphere) is simply

$$\text{Const.} \prod_{i<j} \|P_i - P_j\|_{\mathbb{R}^3}^2,$$

where $\|\cdot\|_{\mathbb{R}^3}$ is the Euclidean norm on \mathbb{R}^3 . From this it is evident that the point process is invariant in distribution under isometries of the sphere. Note also the similarity of this density to that of the circular unitary ensemble.

4.3.9. Truncated unitary matrices. Let U be a matrix drawn from the Haar distribution on $\mathcal{U}(N+m)$. Partition U as

$$(4.3.14) \quad U = \begin{bmatrix} A & C^* \\ B & D \end{bmatrix}$$

where A has size $N \times N$. Życzkowski and Sommers (88) found the exact distribution of eigenvalues of A . Incidentally, permuting the rows or columns of U does not change its distribution, by the invariance of Haar measure. Therefore, any $N \times N$ submatrix of U - not necessarily a principal submatrix - has the same eigenvalue distribution as A .

THEOREM 4.3.13 (Życzkowski and Sommers(2000)). *The eigenvalues of A form a determinantal point process on \mathbb{D} with kernel*

$$(4.3.15) \quad \mathbb{K}(z, w) = \sum_{k=0}^{N-1} \frac{(m+1)\dots(m+k)}{k!} (z\bar{w})^k$$

with respect to the background measure $d\mu(z) = \frac{m}{\pi}(1-|z|^2)^{m-1}dm(z)$ on \mathbb{D} .

Equivalently, we may say that the vector of eigenvalues (in uniform random order) has density

$$(4.3.16) \quad \frac{1}{N!} \left(\frac{m}{\pi}\right)^{N-1} \prod_{k=0}^{N-1} \binom{m+k}{k} \prod_{k=1}^N (1-|z_k|^2)^{m-1} \prod_{i<j} |z_i - z_j|^2$$

with respect to Lebesgue measure on \mathbb{D}^N .

4.3.10. Zero set of a hyperbolic GAF. Recall the hyperbolic GAFs from (2.3.6). When $L = 1$, we have the i.i.d. power series $\mathbf{f}(z) := \sum_{n=0}^{\infty} a_n z^n$ where a_n are i.i.d. standard complex normals. This defines a random analytic function in the unit disk almost surely. Peres and Virág (68) discovered the following result which is of central importance to us, as it connects the two topics in the title of this book. We give a proof in chapter 5.

THEOREM 4.3.14 (Peres and Virág). *The zero set of \mathbf{f}_1 is a determinantal process in the disk with the **Bergman kernel***

$$(4.3.17) \quad \mathbb{K}(z, w) = \frac{1}{\pi(1-z\bar{w})^2} = \frac{1}{\pi} \sum_{k=0}^{\infty} (k+1)(z\bar{w})^k,$$

with respect to Lebesgue measure in the unit disk.

The Bergman kernel of a domain $\Lambda \subset \mathbb{C}$ is the projection kernel from $L^2(\Lambda, \pi^{-1}dm)$ onto the subspace of holomorphic functions. For the unit disk this is easily seen to be given by (4.3.17).

4.3.11. Singular points of matrix-valued GAFs. When $m = 1$ in (4.3.15), we get precisely the truncation of the Bergman kernel of (4.3.17) to the first N summands (it was just whimsical that the factor of π^{-1} was absorbed into the kernel in one case and into the background measure in the other). We may also let $N \rightarrow \infty$ in (4.3.15) for any real number $m > 0$ to get the kernels

$$\mathbb{K}_m(z, w) = \frac{1}{(1-z\bar{w})^{m+1}}$$

for $z, w \in \mathbb{D}$. In view of example 4.3.10 which is the case $L = 1$ in (2.3.6), one might expect that the zero set of the canonical hyperbolic GAF for any $L > 0$ in (2.3.6), is determinantal with kernel \mathbb{K}_m with respect to $d\mu_m(z) = \frac{m}{\pi}(1-|z|^2)^{m-1}dm(z)$ on \mathbb{D} where we must take $m = L$ so as to match the first intensity. This is *false*, for except the case $L = 1$, none of the GAFs in (2.3.6) have a determinantal zero set, as we shall see in chapter 5. The correct generalization of Theorem 4.3.14 is the following result from ((?)), which however makes sense only for integer values of m . The determinantal processes featured here were studied earlier by Jancovici and Tellez (37) but without the connection to zeros of analytic functions.

THEOREM 4.3.15 (Krishnapur). *Let G_k , $k \geq 0$, be i.i.d. $m \times m$ matrices, each with i.i.d. standard complex Gaussian entries. Then for each $m \geq 1$, the singular points of $G_0 + zG_1 + z^2G_2 + \dots$, that is to say, the zeros of $\det(G_0 + zG_1 + z^2G_2 + \dots)$, form*

a *determinantal point process on the unit disk, with kernel \mathbb{K}_m with respect to the measure μ_m .*

The kernels here are projection kernels onto the subspace of holomorphic functions in $L^2(\mathbb{D}, \mu_m)$.

4.4. How to generate determinantal processes

As we shall see in Theorem 4.5.3, the most general determinantal processes with Hermitian kernels (we do not consider non-Hermitian kernels in this book) are mixtures of *determinantal projection processes*. Mixture means a convex combination of measures. Projection determinantal processes are determinantal processes whose kernel \mathbb{K}_H defines a projection operator \mathcal{K}_H to a subspace $H \subset L^2(\Lambda, \mu)$ or, equivalently, $\mathbb{K}_H(x, y) = \sum \varphi_k(x) \overline{\varphi_k(y)}$ where $\{\varphi_k\}$ is any orthonormal basis for H .

LEMMA 4.4.1. *Suppose \mathcal{X} is a determinantal projection process on Λ , with kernel $\mathbb{K}(x, y) = \sum_{k=1}^n \varphi_k(x) \overline{\varphi_k(y)}$ where $\{\varphi_k : 1 \leq k \leq n\}$ is a finite orthonormal set in $L^2(\Lambda)$. Then the number of points in \mathcal{X} is equal to n , almost surely.*

PROOF. The conditions imply that the matrix

$$(\mathbb{K}(x_i, x_j))_{1 \leq i, j \leq k} = (\varphi_j(x_i))_{\substack{i \leq k \\ j \leq n}} (\overline{\varphi_i(x_j)})_{\substack{i \leq n \\ j \leq k}}$$

has rank at most n for any $k \geq 1$. From (1.2.3), we see that $\mathbf{E} \left[\binom{\mathcal{X}(\Lambda)}{k} \right] = 0$ for $k > n$. This shows that $\mathcal{X}(\Lambda) \leq n$ almost surely. However, the first intensity $\rho_1(x) = \mathbb{K}(x, x)$, which implies that

$$\begin{aligned} \mathbf{E}[\mathcal{X}(\Lambda)] &= \int_{\Lambda} \mathbb{K}(x, x) d\mu(x) \\ &= \sum_{k=1}^n \int_{\Lambda} |\varphi_k(x)|^2 d\mu(x) \\ &= n. \end{aligned}$$

Therefore $\mathcal{X}(\Lambda) = n$, almost surely. \square

Despite the fact that determinantal processes arise naturally and many important statistics can be computed, the standard definition 4.2.1 is lacking in direct probabilistic intuition. Below we present an algorithm that is somewhat more natural from a probabilist's point of view, and can also be used for modelling determinantal processes.

In the discrete case (i.e., if μ is an atomic measure), the projection operator \mathcal{K}_H can be applied to the delta function at a point, where,

$$(4.4.1) \quad \delta_x(y) := \begin{cases} \frac{1}{\mu\{x\}} & \text{if } y = x. \\ 0 & \text{otherwise.} \end{cases}$$

Then, $\mathcal{K}_H \delta_x(\cdot) = \mathbb{K}(\cdot, x)$. In the general case we *define* $\mathcal{K}_H \delta_x := \mathbb{K}(\cdot, x)$. Nevertheless, in greater generality than the discrete setting, $\mathcal{K}_H \delta_x$ does have an independent meaning as the “closest thing to δ_x in the Hilbert space H ”. For, if the evaluation $f \rightarrow f(x)$ is a bounded linear functional on H , then this functional is represented by inner product with $\mathcal{K}_H \delta_x$. In physics, $\mathcal{K}_H \delta_x$ is often called a coherent state.

Since \mathbb{K} is a projection kernel, we have the reproducing property $\int \mathbb{K}(x, y)\mathbb{K}(y, z)d\mu(y) = \mathbb{K}(x, z)$, which implies that $\mathbb{K}(x, x) = \|\mathcal{K}_H\delta_x\|^2$. Now, let $\|\cdot\|$ denote the norm of $L^2(\mu)$. The intensity measure of the process is given by

$$(4.4.2) \quad d\mu_H(x) = \rho_1(x)d\mu(x) = \|\mathcal{K}_H\delta_x\|^2 d\mu(x).$$

When μ is supported on countably many points, we have $\|\mathcal{K}_H\delta_x\| = \text{dist}(\delta_x, H^\perp)$ (where \perp denotes orthogonal complement), giving a natural interpretation of the intensity ρ_1 .

Note that $\mu_H(\Lambda) = \dim(H)$, so $\mu_H/\dim(H)$ is a probability measure on Λ . We construct the determinantal process as follows. Start with $n = \dim(H)$, and $H_n = H$.

ALGORITHM 4.4.2.

- If $n = 0$, stop.
- Pick a random point X_n from the probability measure $\frac{1}{n}\mu_{H_n}$.
- Let $H_{n-1} \subset H_n$ be the orthogonal complement of the function $\mathcal{K}_{H_n}\delta_{X_n}$ in H_n . In the discrete case (or if evaluations, $f \rightarrow f(x)$, are bounded linear functionals for all $x \in \Lambda$), then $H_{n-1} = \{f \in H_n : f(X_n) = 0\}$. Note that $\dim(H_{n-1}) = n - 1$ a.s.
- Decrease n by 1 and iterate.

PROPOSITION 4.4.3. *The points (X_1, \dots, X_n) constructed by Algorithm 4.4.2 are distributed as a uniform random ordering of the points in a determinantal process \mathcal{X} with kernel \mathbb{K} .*

PROOF. Construct the random vector (X_1, \dots, X_n) using the algorithm. We want to find its density at $(x_1, \dots, x_n) \in \Lambda^n$ (where x_i are distinct).

Let $\psi_j = \mathcal{K}_H\delta_{x_j}$. Projecting to H_j is equivalent to first projecting to H and then to H_j , and it is easy to check that $\mathcal{K}_{H_j}\delta_{x_j} = \mathcal{K}_{H_j}\psi_j$. Thus, by (4.4.2), the density of the random vector (X_1, \dots, X_n) equals

$$p_n(x_1, \dots, x_n) = \prod_{j=1}^n \frac{\|\mathcal{K}_{H_j}\psi_j\|^2}{j}.$$

Note that $H_j = H \cap \langle \psi_{j+1}, \dots, \psi_n \rangle^\perp$, and therefore $V = \prod_{j=1}^n \|\mathcal{K}_{H_j}\psi_j\|$ is exactly the repeated ‘‘base times height’’ formula for the volume of the parallelepiped determined by the vectors ψ_1, \dots, ψ_n in the finite-dimensional vector space $H \subset L^2(\Lambda)$. It is well-known that V^2 equals the determinant of the *Gram matrix* whose i, j entry is given by the scalar product of ψ_i, ψ_j . But $\int \psi_i \bar{\psi}_j d\mu = \mathbb{K}(x_i, x_j)$ by definition of $\mathcal{K}_H\delta_{x_j}$. Thus, we get

$$p_n(x_1, \dots, x_n) = \frac{1}{n!} \det(\mathbb{K}(x_i, x_j)).$$

The set $\{X_1, \dots, X_n\}$, viewed as a point process, has the n -point joint intensity

$$(4.4.3) \quad \sum_{\pi \in S_n} p_n(x_{\pi_1}, \dots, x_{\pi_n}) = n! p_n(x_1, \dots, x_n),$$

which agrees with that of the determinantal process \mathcal{X} . By Lemma 4.4.1 the claim follows. \square

EXAMPLE 4.4.4 (Uniform spanning trees). We continue the discussion of Example 4.3.2. Let G_{n+1} be an undirected graph on $n + 1$ vertices. For every edge e , the effective resistance of e is the current that flows through e , when a total of one unit of current is sent from the tail of e to the tip of e . It is given by $R(e) = (I^e, I^e)$. To use our algorithm to choose a uniform spanning tree, proceed as follows:

- If $n = 0$, stop.
- Take X_n to be a random edge, chosen so that $\mathbf{P}(X_n = e_i) = \frac{1}{n}R(e_i)$.
- Construct G_n from G_{n+1} by contracting the edge X_n (which means that we identify the end points), and update the effective resistances $\{R(e)\}$.
- Decrease n by one and iterate.

For sampling uniform spanning trees, more efficient algorithms are known, but for general determinantal processes, the above procedure is the most efficient we are aware of. In fact, recently Scardicchio, Zachary and Torquato (73) have implemented this algorithm with various additional tricks and found it to be quite efficient in practice.

4.5. Existence and basic properties

In this section we present necessary and sufficient conditions for the existence of a determinantal point process with a specified Hermitian kernel and then study the basic properties of such processes. Here is a brief guide to the theorems presented in this section.

We start with the Lemma 4.5.1 which shows that when \mathbb{K} is a finite dimensional projection kernel, then a determinantal process with kernel \mathbb{K} does exist. In fact, Proposition 4.4.3 already showed this, but we give another proof. Then, in Theorem 4.5.3 we show that a determinantal point process with a trace-class kernel is a mixture of projection determinantal processes. In the process, we also prove the existence of a determinantal point process for kernels that are dominated by a finite dimensional projection. These are put together in Theorem 4.5.5 to show that a necessary and sufficient condition for a locally trace-class operator to define a determinantal point process is that its spectrum must be contained in $[0, 1]$.

LEMMA 4.5.1. *Suppose $\{\varphi_k\}_{k=1}^n$ is an orthonormal set in $L^2(\Lambda)$. Then there exists a determinantal process with kernel $\mathbb{K}(x, y) = \sum_{k=1}^n \varphi_k(x)\overline{\varphi_k}(y)$.*

PROOF. For any x_1, \dots, x_n we have $(\mathbb{K}(x_i, x_j))_{1 \leq i, j \leq n} = A A^*$, where $A_{i,k} = \varphi_k(x_i)$. Therefore, $\det(\mathbb{K}(x_i, x_j))$ is non-negative. Moreover,

$$\begin{aligned} \int_{\Lambda^n} \det(\mathbb{K}(x_i, x_j))_{i,j} d\mu(x_1) \dots d\mu(x_n) &= \int_{\Lambda^n} \det(\varphi_j(x_i))_{i,j} \det(\overline{\varphi}_i(x_j))_{i,j} d\mu(x_1) \dots d\mu(x_n) \\ &= \int_{\Lambda^n} \sum_{\pi, \tau \in S_n} \operatorname{sgn}(\pi \tau) \prod_{k=1}^n \varphi_{\pi(k)}(x_k) \overline{\varphi}_{\tau(k)}(x_k) \prod_{k=1}^n d\mu(x_k). \end{aligned}$$

In the sum, if $\pi(k) \neq \tau(k)$, then $\int_{\Lambda} \varphi_{\pi(k)}(x_k) \overline{\varphi}_{\tau(k)}(x_k) dx_k = 0$, and when $\pi(k) = \tau(k)$, this integral is 1. Thus, only the terms with $\pi = \tau$ contribute. We get

$$\int_{\Lambda^n} \det(\mathbb{K}(x_i, x_j))_{1 \leq i, j \leq n} d\mu(x_1) \dots d\mu(x_n) = n!,$$

which along with the non-negativity of $\det(\mathbb{K}(x_i, x_j))_{1 \leq i, j \leq n}$ shows that $\frac{1}{n!} \det(\mathbb{K}(x_i, x_j))_{1 \leq i, j \leq n}$ is a probability density on Λ^n . If we look at the resulting random variable as a set of unlabelled points in Λ , we get the desired n -point joint intensity ρ_n . Lower joint intensities are obtained by integrating over some of the x_i s:

$$(4.5.1) \quad \rho_k(x_1, \dots, x_k) = \frac{1}{(n-k)!} \int_{\Lambda^{n-k}} \rho_n(x_1, \dots, x_n) \prod_{j>k} d\mu(x_j).$$

We caution that (4.5.1) is valid only for a point process that has n points almost surely. In general, there is no way to get lower joint intensities from higher ones.

We now show how to get ρ_{n-1} . The others can be got in exactly the same manner, or inductively. Set $k = n - 1$ in (4.5.1) and expand $\rho_n(x_1, \dots, x_n) = \det(\mathbb{K}(x_i, x_j))_{1 \leq i, j \leq n}$ as we did before to get

$$\begin{aligned} \rho_{n-1}(x_1, \dots, x_{n-1}) &= \int_{\Lambda} \rho_n(x_1, \dots, x_n) d\mu(x_n) \\ &= \sum_{\pi, \tau} \operatorname{sgn}(\pi) \operatorname{sgn}(\tau) \prod_{k=1}^{n-1} \varphi_{\pi(k)}(x_k) \bar{\varphi}_{\tau(k)}(x_k) \int_{\Lambda} \varphi_{\pi(n)}(x_n) \bar{\varphi}_{\tau(n)}(x_n) d\mu(x_n). \end{aligned}$$

If $\pi(n) \neq \tau(n)$, the integral vanishes. And if $\pi(n) = \tau(n) = j$, π and τ map $\{1, \dots, n-1\}$ to $\{1, 2, \dots, n\} - \{j\}$. Preserving the ordering of both the latter set, π and τ may be regarded as permutations, say $\tilde{\pi}$ and $\tilde{\tau}$, of the set $\{1, 2, \dots, n-1\}$ in the obvious way. Evidently, $\operatorname{sgn}(\tilde{\pi}) \operatorname{sgn}(\tilde{\tau}) = \operatorname{sgn}(\pi) \operatorname{sgn}(\tau)$, because $\pi(n) = \tau(n)$. This gives us

$$\rho_{n-1}(x_1, \dots, x_{n-1}) = \sum_{j=1}^n \det(\varphi_k(x_i))_{1 \leq i \leq n-1, k \neq j} \det(\bar{\varphi}_k(x_i))_{k \neq j, 1 \leq i \leq n-1}.$$

We must show that this quantity is equal to $\det(\mathbb{K}(x_i, x_j))_{i, j \leq n-1}$. For this note that

$$(\mathbb{K}(x_i, x_j))_{i, j \leq n-1} = (\varphi_k(x_i))_{1 \leq i \leq n-1, k \leq n} (\bar{\varphi}_k(x_i))_{k \leq n, i \leq n-1},$$

and apply the Cauchy-Binet formula. Recall that for matrices A, B of orders $m \times n$ and $n \times m$ respectively, where $n \geq m$, the Cauchy-Binet formula says

$$(4.5.2) \quad \det(AB) = \sum_{1 \leq i_1, \dots, i_m \leq n} \det(A[i_1, \dots, i_m]) \det(B[i_1, \dots, i_m]),$$

where we let $A[i_1, \dots, i_m]$ stand for the matrix formed by taking the columns numbered i_1, \dots, i_m and $B[i_1, \dots, i_m]$ for the matrix formed by the corresponding rows of B . This completes the proof. \square

Note that the last step of the proof, where we applied Cauchy-Binet formula, is essentially the solution to exercise 4.1.1.

EXERCISE 4.5.2. The *Janossy density* of a point process \mathcal{X} is defined to be

$$(4.5.3) \quad \mathcal{J}_k(z_1, \dots, z_k) = \lim_{\epsilon \rightarrow 0} \frac{\mathbf{P}(\mathcal{X} \text{ has exactly } k \text{ points, one in each of } B(z_j, \epsilon))}{\prod_{j=1}^k \mu(B(z_j, \epsilon))}.$$

(There is a definition by integrals, analogous to that for joint intensities, but we restrict ourselves to the above definition).

Given $S \subset [n] \stackrel{\text{def}}{=} \{1, \dots, n\}$ write $\lambda^S \stackrel{\text{def}}{=} \prod_{j \in S} \lambda_j$. Let \mathcal{X} be a point process containing no more than $n < \infty$ points with Janossy densities

$$(4.5.4) \quad \mathcal{J}_k(z_1, \dots, z_k) = \frac{(1 - \lambda)^{[n]}}{Z_k} \det(\mathbb{L}(z_i, z_j))_{1 \leq i, j \leq k}$$

for $1 \leq k \leq n$ where $\mathbb{L}(z, w) = \sum_{j=1}^n \frac{\lambda_j}{1 - \lambda_j} \varphi_j(z) \bar{\varphi}_j(w)$ and $Z_k = \sum_{S \subset [n], |S|=k} \lambda^S (1 - \lambda)^{S^c}$. Suppose that Z_k is also the probability that \mathcal{X} contains exactly k points. Check that \mathcal{X} is a determinantal process with kernel $\mathbb{K}(z, w) = \sum_{j=1}^n \lambda_j \varphi_j(z) \bar{\varphi}_j(w)$.

Now we state the main theorems. The most common application of the following theorem is to describe the behavior of a determinantal process already restricted to a subset.

THEOREM 4.5.3. *Suppose \mathcal{X} is a determinantal process with a Hermitian, non-negative definite, trace-class kernel \mathbb{K} . Write*

$$\mathbb{K}(x, y) = \sum_{k=1}^n \lambda_k \varphi_k(x) \bar{\varphi}_k(y),$$

where $\{\varphi_k\}$ is an orthonormal set of eigenfunctions of the integral operator \mathcal{K} with eigenvalues $\lambda_k \in [0, 1]$. (Here $n = \infty$ is allowed). Let I_k , $1 \leq k \leq n$ be independent random variables with $I_k \sim \text{Bernoulli}(\lambda_k)$. Set

$$\mathbb{K}_I(x, y) = \sum_{k=1}^n I_k \varphi_k(x) \bar{\varphi}_k(y).$$

\mathbb{K}_I is a random analogue of the kernel \mathbb{K} . Let \mathcal{X}_I be the determinantal process with kernel \mathbb{K}_I (i.e., first choose the I_k 's and then independently sample a discrete set that is determinantal with kernel \mathbb{K}_I). Then

$$(4.5.5) \quad \mathcal{X} \stackrel{d}{=} \mathcal{X}_I.$$

In particular, the total number of points in the process \mathcal{X} has the distribution of a sum of independent Bernoulli(λ_k) random variables.

REMARK 4.5.4. In many examples the kernel \mathbb{K} defines a projection operator, i.e., $\lambda_k = 1$ for all k . Then $I_k = 1$ for all k , almost surely, and the theorem is trivial. Nevertheless, the theorem has interesting consequences when applied to the restriction of the process \mathcal{X} to D for any compact subset $D \subset \Lambda$, as already mentioned in remark 4.2.5.

Proof of Theorem 4.5.3. First assume that \mathcal{K} is a finite-dimensional operator:

$$\mathbb{K}(x, y) = \sum_{k=1}^n \lambda_k \varphi_k(x) \bar{\varphi}_k(y)$$

for some finite n . We show that the processes on the left and right side of (4.5.5) have the same joint intensities. By Lemma 4.2.6, this implies that these processes have the same distribution.

Note that the process \mathcal{X}_I exists by Lemma 4.5.1. For $m > n$, the m -point joint intensities of both \mathcal{X} and \mathcal{X}_I are clearly zero. Now consider $m \leq n$ and $x_1, \dots, x_m \in \Lambda$. We claim that:

$$(4.5.6) \quad \mathbf{E} \left[\det \left(\mathbb{K}_I(x_i, x_j) \right)_{1 \leq i, j \leq m} \right] = \det \left(\mathbb{K}(x_i, x_j) \right)_{1 \leq i, j \leq m}.$$

To prove (4.5.6), note that

$$(4.5.7) \quad \left(\mathbb{K}_I(x_i, x_j) \right)_{1 \leq i, j \leq m} = A B,$$

where A is the $m \times n$ matrix with $A_{i,k} = I_k \varphi_k(x_i)$ and B is the $n \times m$ matrix with $B_{k,j} = \bar{\varphi}_k(x_j)$.

For A, B of orders $m \times n$ and $n \times m$ recall that the Cauchy-Binet formula says

$$\det(AB) = \sum_{1 \leq i_1, \dots, i_m \leq n} \det(A[i_1, \dots, i_m]) \det(B\{i_1, \dots, i_m\}),$$

where $A[i_1, \dots, i_m]$ stand for the matrix formed by taking the columns numbered i_1, \dots, i_m and $B\{i_1, \dots, i_m\}$ for the matrix formed by the corresponding rows of B .

Apply this to A, B defined above and take expectations. Observe that $B\{i_1, \dots, i_m\}$ is non-random and

$$\mathbf{E}[\det(A[i_1, \dots, i_m])] = \det(C[i_1, \dots, i_m])$$

where C is the $m \times n$ matrix $C_{i,k} = \lambda_k \varphi_k(x_i)$. Since the determinant involves products of entries, independence of I_k s is being used crucially. Now, applying the Cauchy-Binet formula in the reverse direction to C and B , we obtain (4.5.6) and hence also (4.5.5). Given $\{I_k\}_{k \geq 1}$, Lemma 4.5.1 shows that the process \mathcal{X}_I is well defined and Lemma 4.4.1 shows that \mathcal{X}_I has $\sum_k I_k$ points, almost surely. Therefore,

$$\mathcal{X}(\Lambda) \stackrel{d}{=} \sum_k I_k.$$

So far we assumed that the operator \mathcal{K} determined by the kernel \mathbb{K} is finite dimensional. Now suppose \mathcal{K} is a general trace class operator. Then $\sum_k \lambda_k < \infty$ and hence, almost surely, $\sum_k I_k < \infty$. By Lemma 4.5.1 again, \mathcal{X}_I exists and (4.5.7) is valid by the same reasoning. Taking expectations and observing that the summands in the Cauchy-Binet formula (for the matrices A and B at hand) are non-negative, we obtain

$$\mathbf{E} \left[\det(\mathbb{K}_I(x_i, x_j))_{1 \leq i, j \leq m} \right] = \sum_{1 \leq i_1, \dots, i_m} \det(C[i_1, \dots, i_m]) \det(B\{i_1, \dots, i_m\}),$$

where C is the same as before. To conclude that the right hand side is $\det(\mathbb{K}(x_i, x_j))_{1 \leq i, j \leq m}$, we first apply the Cauchy-Binet formula to the finite approximation $(\mathbb{K}_N(x_i, x_j))_{1 \leq i, j \leq m}$, where $\mathbb{K}_N(x, y) = \sum_{k=1}^N \lambda_k \varphi_k(x) \bar{\varphi}_k(y)$. Use Lemma 4.2.2 to see for μ -a.e. $x, y \in \Lambda$ that $\mathbb{K}_N(x, y)$ converges to $\mathbb{K}(x, y)$ as $N \rightarrow \infty$. Hence, for μ -a.e. $x_1, \dots, x_m \in \Lambda$, we have

$$\mathbf{E} \left[\det(\mathbb{K}_I(x_i, x_j))_{1 \leq i, j \leq m} \right] = \det(\mathbb{K}(x_i, x_j))_{1 \leq i, j \leq m},$$

as was required to show. (In short, the proof for the infinite case is exactly the same as before, only we cautiously avoided applying Cauchy-Binet formula to the product of two infinite rectangular matrices). \square

Now we give a probabilistic proof of the following criterion for a Hermitian integral kernel to define a determinantal process.

THEOREM 4.5.5 (Macchi (55), Soshnikov (83)). *Let \mathbb{K} determine a self-adjoint integral operator \mathcal{K} on $L^2(\Lambda)$ that is locally trace class. Then \mathbb{K} defines a determinantal process on Λ if and only if the spectrum of \mathcal{K} is contained in $[0, 1]$.*

PROOF. If Λ is compact and \mathcal{K} is of trace class, then \mathcal{K} has point spectrum and we may write

$$(4.5.8) \quad \mathbb{K}(x, y) = \sum_k \lambda_k \varphi_k(x) \bar{\varphi}_k(y)$$

where $\{\varphi_k\}$ is an orthonormal set in $L^2(\Lambda, \mu)$ and $\lambda_k \geq 0$ and $\sum \lambda_k < \infty$.

In general, it suffices to construct the point process restricted to an arbitrary compact subset of Λ with kernel, the restriction of \mathbb{K} to the compact subset. What is more, the spectrum of \mathcal{K} is contained in $[0, 1]$ if, and only if, the eigenvalues of \mathcal{K} restricted to any compact set are in $[0, 1]$. Thus we may assume that Λ is compact, that \mathcal{K} is of trace class and that (4.5.8) holds.

Sufficiency: If \mathcal{K} is a projection operator, this is precisely Lemma 4.5.1. If the eigenvalues are $\{\lambda_k\}$, with $\lambda_k \leq 1$, then as in the proof of Theorem 4.5.3, we construct the process \mathcal{X}_I . The proof there shows that \mathcal{X}_I is determinantal with kernel \mathbb{K} .

Necessity: Suppose that \mathcal{X} is determinantal with kernel \mathbb{K} . Since the joint intensities of \mathcal{X} are non-negative, \mathbb{K} must be non-negative definite. Now suppose that the largest eigenvalue of \mathcal{K} is $\lambda > 1$. Let \mathcal{X}_1 be the process obtained by first sampling \mathcal{X} and then independently deleting each point of \mathcal{X} with probability $1 - \frac{1}{\lambda}$. Computing the joint intensities shows that \mathcal{X}_1 is determinantal with kernel $\frac{1}{\lambda}\mathbb{K}$.

Now \mathcal{X} has finitely many points (we assumed that \mathcal{X} is trace class) and $\lambda > 1$. Hence, $\mathbf{P}[\mathcal{X}_1(\Lambda) = 0] > 0$. However, $\frac{1}{\lambda}\mathbb{K}$ has all eigenvalues in $[0, 1]$, with at least one eigenvalue equal to 1, whence by Theorem 4.5.3, $\mathbf{P}[\mathcal{X}_1(\Lambda) \geq 1] = 1$, a contradiction. \square

EXAMPLE 4.5.6 (Non-measurability of the Bernoullis). A natural question that arises from Theorem 4.5.3 is whether, given a realization of the determinantal process \mathcal{X} , we can determine the values of the I_k 's. This is not always possible, i.e., the I_k 's are not measurable w.r.t. the process \mathcal{X} in general.

Consider the graph G with vertices $\{a, b, c, d\}$ and edges $e_1 = (a, b), e_2 = (b, c), e_3 = (c, d), e_4 = (d, a), e_5 = (a, c)$. By the Burton-Pemantle Theorem (11) (see Example 4.3.2), the edge-set of a uniformly chosen spanning tree of G is a determinantal process. In this case, the kernel restricted to the set $D = \{e_1, e_2, e_3\}$ is turns out to be

$$(\mathbb{K}(e_i, e_j))_{1 \leq i, j \leq 3} = \frac{1}{8} \begin{pmatrix} 5 & -3 & -1 \\ -3 & 5 & -1 \\ -1 & -1 & -1 \end{pmatrix}.$$

This matrix has eigenvalues $\frac{1}{8}, \frac{7-\sqrt{17}}{16}, \frac{7+\sqrt{17}}{16}$. But G has eight spanning trees, and hence, all measurable events have probabilities that are multiples of $\frac{1}{8}$, it follows that the Bernoullis cannot be measurable.

Theorem 4.5.3 gives us the distribution of the number of points $\mathcal{X}(D)$ in any subset of Λ . Given several regions D_1, \dots, D_r , can we find the joint distribution of $\mathcal{X}(D_1), \dots, \mathcal{X}(D_r)$? It seems that a simple probabilistic description of the joint distribution exists only in the special case when D_i 's are related as follows.

DEFINITION 4.5.7. Let \mathbb{K} be a standard integral kernel and \mathcal{K} the associated integral operator acting on $L^2(\Lambda)$. We say that the subsets D_1, \dots, D_r of Λ are **simultaneously observable** if the following happens. Let $D = \cup_i D_i$.

There is an orthogonal basis $\{\varphi_k\}$ of $L^2(D)$ consisting of eigenfunctions of \mathcal{K}_D such that for each $i \leq r$, the set $\{\varphi_k|_{D_i}\}$ of the restricted functions is an orthogonal basis of $L^2(D_i)$ consisting of eigenfunctions of \mathcal{K}_{D_i} .

The motivation for this terminology comes from quantum mechanics, where two physical quantities can be simultaneously measured if the corresponding operators commute. Commuting is the same as having common eigenfunctions, of course.

EXAMPLE 4.5.8. Consider the infinite Ginibre process described under example 4.3.7 which is determinantal on the complex plane with kernel

$$(4.5.9) \quad \mathbb{K}(z, w) = \sum_{k=0}^{\infty} \frac{(z\bar{w})^k}{k!}$$

with respect to $d\mu(z) = \pi^{-1} \exp\{-|z|^2\} dm(z)$. Then if $S = \{z : r < |z| < R\}$ is an annulus centered at zero, then

$$\begin{aligned} \int_S \mathbb{K}(z, w) w^k d\mu(w) &= \int_S \frac{z^k |w|^{2k}}{k!} \frac{1}{\pi} e^{-|w|^2} dm(w) \\ &= \frac{z^k}{k!} \int_{r^2}^{R^2} t^k e^{-t} dt, \end{aligned}$$

which shows that $\{z^k : k \geq 0\}$ is an orthogonal basis of eigenfunctions of \mathcal{K}_S . Thus, if D_i are arbitrary annuli centered at the origin, then they satisfy the conditions of definition 4.5.7. The interesting examples we know are all of this kind, based on the orthogonality of $\{z^k\}$ on any annulus centered at the origin. We shall say more about these processes in section 4.7.

PROPOSITION 4.5.9. *Under the assumptions of Theorem 4.5.3, let $D_i \subset \Lambda$, $1 \leq i \leq r$ be mutually disjoint and simultaneously observable. Let \mathbf{e}_i be the standard basis vectors in \mathbb{R}^r . Denote by φ_k , the common eigenfunctions of \mathbb{K} on the D_i 's and by $\lambda_{k,i}$ the corresponding eigenvalues. Then $\lambda_k := \sum_i \lambda_{k,i}$ are the eigenvalues of $\mathcal{K}_{\cup D_i}$ and hence $\lambda_k \leq 1$. Then*

$$(4.5.10) \quad (\mathcal{X}(D_1), \dots, \mathcal{X}(D_r)) \stackrel{d}{=} \sum_k (\xi_{k,1}, \dots, \xi_{k,r}),$$

where $\vec{\xi}_k = (\xi_{k,1}, \dots, \xi_{k,r})$ are independent for different values of k , with $\mathbf{P}(\vec{\xi}_k = \mathbf{e}_i) = \lambda_{k,i}$ for $1 \leq i \leq r$ and $\mathbf{P}(\vec{\xi}_k = \mathbf{0}) = 1 - \lambda_k$. In words, $(\mathcal{X}(D_1), \dots, \mathcal{X}(D_r))$ has the same distribution as the vector of counts in r cells, if we pick n balls and assign the k^{th} ball to the i^{th} cell with probability $\lambda_{k,i}$ (there may be a positive probability of not assigning it to any of the cells).

PROOF. At first we make the following assumptions.

- (1) $\cup_i D_i = \Lambda$.
- (2) \mathbb{K} defines a finite dimensional projection operator. That is, $\mathbb{K}(x, y) = \sum_{k=1}^n \varphi_k(x) \overline{\varphi}_k(y)$ for $x, y \in \Lambda$, and $\{\varphi_k\}$ is an orthonormal set in $L^2(\Lambda)$ and $n < \infty$.

By our assumption, $\{\varphi_k|_{D_i}\}$ is an orthogonal (but not orthonormal) basis of $L^2(D_i)$ of eigenfunctions of \mathcal{K}_{D_i} , for $1 \leq i \leq r$. Thus for $x, y \in D_i$, we may write

$$(4.5.11) \quad \mathbb{K}(x, y) = \sum_k \lambda_{k,i} \frac{\varphi_k(x) \overline{\varphi}_k(y)}{\int_{D_i} |\varphi_k|^2 d\mu}.$$

Comparing with the expansion of \mathbb{K} on Λ , we see that $\lambda_{k,i} = \int_{D_i} |\varphi_k|^2 d\mu$.

We write

$$(4.5.12) \quad \begin{pmatrix} \mathbb{K}(x_1, x_1) & \dots & \mathbb{K}(x_1, x_n) \\ \dots & \dots & \dots \\ \mathbb{K}(x_n, x_1) & \dots & \mathbb{K}(x_n, x_n) \end{pmatrix} = \begin{pmatrix} \varphi_1(x_1) & \dots & \varphi_n(x_1) \\ \dots & \dots & \dots \\ \varphi_1(x_n) & \dots & \varphi_n(x_n) \end{pmatrix} \begin{pmatrix} \overline{\varphi}_1(x_1) & \dots & \overline{\varphi}_1(x_n) \\ \dots & \dots & \dots \\ \overline{\varphi}_n(x_1) & \dots & \overline{\varphi}_n(x_n) \end{pmatrix}.$$

In particular,

$$(4.5.13) \quad \det(\mathbb{K}(x_i, x_j))_{1 \leq i, j \leq n} = \left(\sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^n \varphi_{\sigma_i}(x_i) \right) \left(\sum_{\tau \in S_n} \text{sgn}(\tau) \prod_{i=1}^n \overline{\varphi}_{\tau_i}(x_i) \right).$$

Now if k_i are non-negative integers with $\sum_i k_i = n$, note that

$$\{\mathcal{X}(D_i) \geq k_i \text{ for all } 1 \leq i \leq r\} = \{\mathcal{X}(D_i) = k_i \text{ for all } 1 \leq i \leq r\},$$

since by Lemma 4.4.1, a determinantal process whose kernel defines a rank- n projection operator has exactly n points, almost surely. Thus, we have

$$\begin{aligned} \mathbf{P}[\mathcal{X}(D_i) = k_i \text{ for all } 1 \leq i \leq r] &= \mathbf{E} \left[\prod_{i=1}^r \binom{\mathcal{X}(D_i)}{k_i} \right] \\ &= \frac{1}{k_1! \cdots k_r!} \int_{\prod_{i=1}^r D_i^{k_i}} \det(\mathbb{K}(x_k, x_\ell))_{1 \leq k, \ell \leq n} d\mu(x_1) \cdots d\mu(x_n) \\ &= \frac{1}{k_1! \cdots k_r!} \int_{\prod_{i=1}^r D_i^{k_i}} \sum_{\sigma, \tau} \text{sgn}(\sigma) \text{sgn}(\tau) \prod_{m=1}^n \varphi_{\sigma_m}(x_m) \bar{\varphi}_{\tau_m}(x_m) d\mu(x_1) \cdots d\mu(x_n). \end{aligned}$$

Any term with $\sigma \neq \tau$ vanishes upon integrating. Indeed, if $\sigma(m) \neq \tau(m)$ for some m , then

$$\int_{D_{j(m)}} \varphi_{\sigma_m}(x_m) \bar{\varphi}_{\tau_m}(x_m) d\mu(x_m) = 0$$

where $j(m)$ is the index for which

$$k_1 + \cdots + k_{j(m)-1} < m \leq k_1 + \cdots + k_{j(m)}.$$

Therefore,

$$\begin{aligned} \mathbf{E} \left[\prod_{i=1}^r \binom{\mathcal{X}(D_i)}{k_i} \right] &= \frac{1}{k_1! \cdots k_r!} \sum_{\sigma} \prod_{m=1}^n \int_{D_{j(m)}} |\varphi_{\sigma_m}(x)|^2 dx \\ &= \frac{1}{k_1! \cdots k_r!} \sum_{\sigma} \prod_{m=1}^n \lambda_{\sigma_m, j(m)}. \end{aligned}$$

Now consider (4.5.10) and set $M_i = \sum_k \xi_{k,i}$ for $1 \leq i \leq r$. We want $\mathbf{P}[M_j = k_j, j \leq r]$. This problem is the same as putting n balls into r cells, where the probability for the j^{th} ball to fall in cell i is $\lambda_{j,i}$. To have k_i balls in cell i for each i , we first take a permutation σ of $\{1, 2, \dots, n\}$ and then put the σ_m^{th} ball into cell $j(m)$ if $k_1 + \cdots + k_{j(m)-1} < m \leq k_1 + \cdots + k_{j(m)}$. However, this counts each assignment of balls $\prod_{i=1}^r k_i!$ times. This implies that

$$\mathbf{P}[M_1 = k_1, \dots, M_r = k_r] = \frac{1}{k_1! \cdots k_r!} \sum_{\sigma} \prod_{m=1}^n \lambda_{\sigma_m, j(m)}.$$

Thus,

$$(4.5.14) \quad (\mathcal{X}(D_1), \dots, \mathcal{X}(D_r)) \stackrel{d}{=} (M_1, \dots, M_r),$$

which is precisely what we wanted to show. It remains to deal with the two assumptions that we made at the beginning.

First, if the kernel does not define a projection, apply Theorem 4.5.3 to write \mathcal{X} as a mixture of determinantal projection processes. Since the eigenfunctions of these projection kernels are a subset of eigenfunctions of the original kernel, D_i are simultaneously observable for each projection kernel that appears in the mixture

also. Applying (4.5.14) to each component in the mixture we obtain the theorem. Applying the theorem for these projection determinantal processes, we get the theorem for the original determinantal processes.

Second, if $\cup_i D_i \neq \Lambda$, we could restrict the point process to $\cup_i D_i$ to obtain a determinantal process whose kernel may or may not be a projection. The theorem thus follows. \square

4.6. Central limit theorems

As an application of Theorem 4.5.3 we can derive the following central limit theorem for determinantal processes due to Costin and Lebowitz (15) in case of the sine kernel, and due to Soshnikov (84) for general determinantal processes.

THEOREM 4.6.1. *Let \mathcal{X}_n be a sequence of determinantal processes on Λ_n with kernels \mathbb{K}_n . Let D_n be a Borel subset of Λ_n such that $\mathcal{X}_n(D_n)$ is finite almost surely and such that $\text{Var}(\mathcal{X}_n(D_n)) \rightarrow \infty$ as $n \rightarrow \infty$.*

Then

$$(4.6.1) \quad \frac{\mathcal{X}_n(D_n) - \mathbf{E}[\mathcal{X}_n(D_n)]}{\sqrt{\text{Var}(\mathcal{X}_n(D_n))}} \xrightarrow{d} N(0, 1).$$

PROOF. By Theorem 4.5.3, $\mathcal{X}_n(D_n)$ has the same distribution as a sum of independent Bernoullis with parameters being the eigenvalues of the integral operators associated with \mathbb{K}_n restricted to D_n . A straightforward application of Lindeberg-Feller CLT for triangular arrays gives the result. \square

REMARK 4.6.2. Earlier proofs of results of the kind of Theorem 4.6.1 ((15), (84)) used the moment generating function for particle counts. Indeed, one standard way to prove central limit theorems (including the Lindeberg-Feller theorem) uses generating functions. The advantage of this proof is that the reason for the validity of the CLT is more transparent and a repetition of well known computations are avoided. Moreover, by applying the classical theory of sums of independent variables, local limit theorems, large deviation principles and extreme value asymptotics follow without any extra effort.

4.7. Radially symmetric processes on the complex plane

Proposition 4.5.9 implies that when a determinantal process with kernel \mathbb{K} has the form $\mathbb{K}(z, w) = \sum_k c_k (z\bar{w})^k$, with respect to a radially symmetric measure μ , then the set of absolute values of the points of the process, has the same distribution as a set of independent random variables. More precisely, we have

THEOREM 4.7.1. *Let \mathcal{X} be a determinantal process with kernel \mathbb{K} with respect to a radially symmetric measure μ on \mathbb{C} . Assume $\mu(\mathbb{C}) < \infty$. Write $\mathbb{K}(z, w) = \sum_k \lambda_k a_k^2 (z\bar{w})^k$, where $a_k z^k$, $0 \leq k \leq n-1$ are the normalized eigenfunctions for \mathbb{K} . The following construction describes the distribution of $\{|z|^2 : z \in \mathcal{X}\}$.*

- Let Z be picked from the probability distribution $\frac{\mu}{\mu(\mathbb{C})}$, and let $\mathbf{Q}_0 = |Z|^2$.
- For $1 \leq k \leq n-1$ let \mathbf{Q}_k be an independent size-biased version of \mathbf{Q}_{k-1} (i.e., \mathbf{Q}_k has density $f_k(q) = \frac{a_k^2}{a_{k-1}^2} q$ with respect to the law of \mathbf{Q}_{k-1}).
- Consider the point process in which each point \mathbf{Q}_k , $0 \leq k \leq n-1$, is included with probability λ_k independently of everything else.

When μ has density $\varphi(|z|)$, then Q_k has density

$$(4.7.1) \quad \pi a_k^2 q^k \varphi(\sqrt{q}).$$

Theorem 4.7.1 (and its higher dimensional analogues) is the only kind of example that we know for interesting simultaneously observable counts.

PROOF. Let ν be the measure of the squared modulus of a point picked from μ . In particular, if μ has density $\varphi(|z|)$, then we have $d\nu(q) = \pi\varphi(\sqrt{q})dq$.

For $1 \leq i \leq r$, let D_i be mutually disjoint open annuli centered at 0 with inner and outer radii r_i and R_i respectively. Since the functions z^k are orthogonal on any annulus centered at zero, it follows that the D_i 's are simultaneously observable. To compute the eigenvalues, we integrate these functions against the restricted kernel; clearly, all terms but one cancel, and we get that for $z \in D_i$

$$\begin{aligned} z^k \lambda_{k,i} &= \int_{D_i} \lambda_k a_k^2 (z\bar{w})^k w^k d\mu(w), \quad \text{and so} \\ \lambda_{k,i} &= \lambda_k a_k^2 \int_{D_i} |w|^{2k} d\mu(w) \\ &= \lambda_k a_k^2 \int_{r_i^2}^{R_i^2} q^k d\nu(q). \end{aligned}$$

As r_i, R_i change, the last expression remains proportional to the probability that the k times size-biased random variable Q_k falls in (r_i^2, R_i^2) . When we set $(r_i, R_i) = (0, \infty)$, the result is λ_k because $a_k w^k$ has norm 1. Thus the constant of proportionality equals λ_k . The theorem now follows from Proposition 4.5.9. \square

EXAMPLE 4.7.2 (Ginibre ensemble revisited). Recall that the n^{th} Ginibre ensemble described in Example 4.3.7 is the determinantal process \mathbf{G}_n on \mathbb{C} with kernel $\mathbb{K}_n(z, w) = \sum_{k=0}^{n-1} \lambda_k a_k^2 (z\bar{w})^k$ with respect to the complex Gaussian measure $d\mu = \frac{1}{\pi} e^{-|z|^2} dm(z)$, where $a_k^2 = 1/k!$, and $\lambda_k = 1$. The modulus-squared of a complex Gaussian is a Gamma(1, 1) random variable, and its k -times size-biased version has Gamma($k+1, 1$) distribution (see (4.7.1)). Theorem 4.7.1 immediately yields the following result.

THEOREM 4.7.3 (Kostlan (50)). *The set of absolute values of the points of \mathbf{G}_n has the same distribution as $\{Y_1, \dots, Y_n\}$ where Y_i are independent and $Y_i^2 \sim \text{Gamma}(i, 1)$.*

All of the above holds for $n = \infty$ also (see example 4.5.8), in which case we have a determinantal process with kernel $e^{z\bar{w}}$ with respect to $d\mu = \frac{1}{\pi} e^{-|z|^2} dm(z)$. This case is also of interest as \mathbf{G}_∞ is a translation invariant process in the plane.

EXAMPLE 4.7.4 (Zero set of a Gaussian analytic function). Recall from example 4.3.10 that the zero set of $\mathbf{f}_1(z) := \sum_{n=0}^{\infty} a_n z^n$ is a determinantal process in the disk with the Bergman kernel

$$\mathbb{K}(z, w) = \frac{1}{\pi(1 - z\bar{w})^2} = \frac{1}{\pi} \sum_{k=0}^{\infty} (k+1)(z\bar{w})^k,$$

with respect to Lebesgue measure in the unit disk. This fact will be proved in chapter 5. Theorem 4.7.1 applies to this example, with $a_k^2 = (k+1)/\pi$ and $\lambda_k = 1$ (to make \mathbb{K} trace class, we first have to restrict it to the disk of radius $r < 1$ and let $r \rightarrow 1$). From

(4.7.1) we immediately see that Q_k has Beta($k + 1, 1$) distribution. Equivalently, we get the following.

THEOREM 4.7.5 (Peres and Virág (68)). *The set of absolute values of the points in the zero set of \mathbf{f}_1 has the same distribution as $\{U_1^{1/2}, U_2^{1/4}, U_3^{1/6}, \dots\}$ where U_i are i.i.d. uniform $[0, 1]$ random variables.*

We can of course consider the determinantal process with the truncated Bergman kernel $\mathbb{K}_n(z, w) = \frac{1}{\pi} \sum_{k=0}^{n-1} (k+1)(z\bar{w})^k$. The set of absolute values of this process has the same distribution as $\{U_k^{1/2k} : 1 \leq k \leq n\}$.

4.8. High powers of complex polynomial processes

Rains (70) showed that sufficiently high powers of eigenvalues of a random unitary matrix are independent.

THEOREM 4.8.1 (Rains (70)). *Let $\{z_1, \dots, z_n\}$ be the set of eigenvalues of a random unitary matrix chosen according to Haar measure on $\mathcal{U}(n)$. Then for every $k \geq n$, $\{z_1^k, \dots, z_n^k\}$ has the same distribution as a set of n points chosen independently according to uniform measure on the unit circle in the complex plane.*

In the following proposition, we point out that this theorem holds whenever the angular distribution of the points is a trigonometric polynomial.

PROPOSITION 4.8.2. *Let $(z_1, \dots, z_n) \in (S^1)^{\otimes n}$ have density $P(z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n)$ with respect to uniform measure on $(S^1)^{\otimes n}$, where P is a polynomial of degree d or less in each variable. Then for every $k > d$ the vector (z_1^k, \dots, z_n^k) has the distribution of n points chosen independently according to uniform measure on S^1 .*

PROOF. Fix $k > d$ and consider any joint moment of (z_1^k, \dots, z_n^k) ,

$$\mathbf{E} \left[\prod_{i=1}^n \left(z_i^{km_i} \bar{z}_i^{k\ell_i} \right) \right] = \int_{(S^1)^{\otimes n}} \prod_{i=1}^n \left(z_i^{km_i} \bar{z}_i^{k\ell_i} \right) P(z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n) d\lambda,$$

where λ denotes the uniform measure on $(S^1)^{\otimes n}$. If $m_i \neq \ell_i$ for some i then the integral vanishes. To see this, note that the average of a monomial over $(S^1)^{\otimes n}$ is either 1 or 0 depending on whether the exponent of every z_i matches that of \bar{z}_i . Suppose without loss of generality that $m_1 > \ell_1$. Thus, if P is written as a sum of monomials, in each term, we have an excess of z_1^k which cannot be matched by an equal power of \bar{z}_1 because P has degree less than k as a polynomial in \bar{z}_1 .

We conclude that the joint moments are zero unless $m_i = \ell_i$ for all i . If $m_i = \ell_i$ for all i , then the expectation equals 1. Thus, the joint moments of (z_1^k, \dots, z_n^k) are the same as those of n i.i.d. points chosen uniformly on the unit circle. This proves the proposition. \square

More generally, by conditioning on the absolute values we get the following.

COROLLARY 4.8.3. *Let ζ_1, \dots, ζ_k be complex random variables with distribution given by*

$$P(z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n) d\mu_1(z_1) \cdots d\mu_n(z_n),$$

where P is a polynomial of degree d or less in each variable, and the measures μ_i are radially symmetric. Then for every $k > d$, the angles $\arg(\zeta_1^k), \dots, \arg(\zeta_n^k)$ are independent, have uniform distribution on $[0, 2\pi]$, and are independent of the moduli $\{|\zeta_1|, \dots, |\zeta_n|\}$.

Corollary 4.8.3 applies to powers of points of determinantal processes with kernels of the form $\mathbb{K}(z, w) = \sum_{k=0}^d c_k (z\bar{w})^k$ with respect to a radially symmetric measure μ on the complex plane. Combining this observation with our earlier results on the independence of the absolute values of the points, we get the following result.

THEOREM 4.8.4. *Let $\mathcal{X} = \{z_1, \dots, z_n\}$ be a determinantal process on the complex plane with kernel $\mathbb{K}(z, w) = \sum_{\ell=0}^{d-1} c_\ell (z\bar{w})^\ell$ (satisfying $\#\{k \geq 0 : c_k \neq 0\} = n$) with respect to a radially symmetric measure μ . Then for every $\ell \geq d$, the set $\{z_1^\ell, \dots, z_n^\ell\}$ is distributed as a set of independent random variables with radially symmetric distribution.*

4.9. Permanental processes

In this section we introduce **permanental point processes**, to show the analogy and to contrast their properties with determinantal processes. Permanental processes are positively correlated, meaning that they have even more clumps and vacant regions than Poisson processes, see figure 1. In physics, they correspond *bosons* as opposed to determinantal processes that correspond to *fermions*.

DEFINITION 4.9.1. A point process \mathcal{X} on Λ is said to be a **permanental process** with kernel \mathbb{K} if its joint intensities satisfy:

$$(4.9.1) \quad \rho_k(x_1, \dots, x_k) = \text{per}(\mathbb{K}(x_i, x_j))_{1 \leq i, j \leq k},$$

for every $k \geq 1$ and $x_1, \dots, x_k \in \Lambda$.

We continue to make our standard assumptions on the kernel \mathbb{K} (Assumption 4.2.3). While analogous theorems to those for determinantal processes hold, as we shall show, probabilistically permanental processes are much simpler. Indeed, Proposition 4.9.2 characterizes all permanental point processes in terms of Poisson processes, an analogous probabilistic characterization of determinantal processes seems not to exist.

PROPOSITION 4.9.2. *Let F be a mean zero complex Gaussian process on Λ . Given F , let \mathcal{X} be a Poisson process in Λ with intensity $|F|^2$. Then \mathcal{X} is a permanental process with kernel $\mathbb{K}(x, y) = \mathbf{E}[F(x)\bar{F}(y)]$.*

PROOF. Given F , the joint intensities of \mathcal{X} are $\bar{\rho}_k(x_1, \dots, x_k) = \prod_{i=1}^k |F(x_i)|^2$. Hence it follows that the unconditional joint intensities of \mathcal{X} are $\rho_k(x_1, \dots, x_k) = \mathbf{E}[\prod_{i=1}^k |F(x_i)|^2]$. Now apply the Wick formula Lemma 2.1.7. \square

COROLLARY 4.9.3. *If \mathbb{K} determines a self-adjoint non-negative definite locally trace-class integral operator \mathcal{K} , then there exists a permanental process with kernel \mathbb{K} .*

PROOF. Proposition 4.9.2 shows existence. Uniqueness is proved analogously to Theorem 4.2.6. We just sketch the changes to be made in that proof. For a Hermitian positive definite matrix M , we have $|M_{i,j}|^2 \leq M_{i,i}M_{j,j}$ for any i, j . Therefore

$$\text{per}(M) \leq \sum_{\sigma} \prod_{k=1}^n |M_{k, \pi_k}| \leq n! \prod_{k=1}^n M_{k,k}.$$

Use this in place of inequality (4.2.15) for the determinant to deduce that for any compact set D

$$\mathbf{E}[(1+s)^{\mathcal{X}(D)}] \leq \sum_{k=0}^{\infty} \kappa(D)^k s^k$$

where $\kappa(D) = \int_D \mathbb{K}(x, x) d\mu(x)$. Thus the series converges for $|s| < \kappa(D)^{-1}$ and the moments of $\mathcal{X}(D)$ determine the distribution of $\mathcal{X}(D)$. This is exactly what we needed to show that the joint intensities determine the distribution of \mathcal{X} . \square

PROOF. Let $\{\varphi_k\}$ be an orthonormal set of eigenfunctions for \mathcal{K} with eigenvalues λ_k , and write $\mathbb{K}(x, y) = \sum_{k=1}^n \lambda_k \varphi_k(x) \overline{\varphi_k}(y)$, we allow $n = \infty$. Let a_k be independent standard complex Gaussian random variables and define $F(z) = \sum_{k=1}^n \sqrt{\lambda_k} a_k \varphi_k(z)$. Then F is a complex Gaussian process on Λ with covariance kernel \mathbb{K} , and we use Proposition 4.9.2 to construct the required permanental process. \square

Next we give the analogue of Theorem 4.5.3 for permanental processes. It is much simpler to prove thanks to Proposition 4.9.2. As always, densities on Λ^ℓ are expressed with respect to the background measure $\mu^{\otimes \ell}$.

THEOREM 4.9.4. (*Hough et al. (33)*) Suppose \mathcal{X} is a permanental process in Λ with a Hermitian, non-negative definite, trace-class kernel \mathbb{K} . Write $\mathbb{K}(x, y) = \sum_{k=1}^n \lambda_k \varphi_k(x) \overline{\varphi_k}(y)$, where φ_k are orthonormal eigenfunctions of \mathcal{K} with eigenvalues λ_k ($n = \infty$ is allowed). Let $\vec{\alpha} = (\alpha_1, \dots, \alpha_n)$, where α_i are non-negative integers such that $\ell = \ell(\vec{\alpha}) = \alpha_1 + \dots + \alpha_n < \infty$ and let $Z^{\vec{\alpha}}$ be the random vector in Λ^ℓ with density:

$$(4.9.2) \quad p_\ell(z_1, \dots, z_\ell) = \frac{1}{\ell! \alpha_1! \dots \alpha_n!} \left| \text{per} \begin{bmatrix} \{\varphi_1(z_1) \cdots \varphi_1(z_\ell)\} \alpha_1 \\ \cdots \cdots \cdots \\ \{\varphi_n(z_1) \cdots \varphi_n(z_\ell)\} \alpha_n \end{bmatrix} \right|^2,$$

where the notation $\{\varphi_i(z_1) \cdots \varphi_i(z_\ell)\} \alpha_i$ indicates that the row $\varphi_i(z_1) \cdots \varphi_i(z_\ell)$ is repeated α_i times. Let $\mathcal{X}^{\vec{\alpha}}$ be the point process determined by $Z^{\vec{\alpha}}$, i.e., $\mathcal{X}^{\vec{\alpha}}(D)$ is the number of $j \leq \ell$ such that $Z_j^{\vec{\alpha}} \in D$. Let $\gamma_1, \dots, \gamma_n$ be independent geometric random variables with $\mathcal{P}(\gamma_k = s) = \left(\frac{\lambda_k}{\lambda_k + 1}\right)^s \left(\frac{1}{\lambda_k + 1}\right)$, for $s = 0, 1, 2, \dots$. Then

$$\mathcal{X} \stackrel{d}{=} \mathcal{X}^{\vec{\gamma}},$$

where $\vec{\gamma} = (\gamma_1, \dots, \gamma_n)$. In particular, $\mathcal{X}(\Lambda)$ has the distribution of a sum of independent Geometric random variables with parameters $\frac{1}{\lambda_k + 1}$.

REMARK 4.9.5. The density given in (4.9.2) has physical significance. Interpreting the functions φ_k as eigenstates of a one-particle Hamiltonian, (4.9.2) gives the distribution for ℓ non-interacting bosons in a common potential given that α_i of them lie in the eigenstate φ_i . This density is the exact analogue of the density for the determinantal projection process with kernel whose eigenfunctions are (ψ_{i_j}) :

$$(4.9.3) \quad p(z_1, \dots, z_\ell) = \frac{1}{\ell!} \left| \det \begin{bmatrix} \varphi_{i_1}(z_1) & \cdots & \varphi_{i_1}(z_\ell) \\ \cdots & \cdots & \cdots \\ \varphi_{i_\ell}(z_1) & \cdots & \varphi_{i_\ell}(z_\ell) \end{bmatrix} \right|^2.$$

Physically, (4.9.3) gives the distribution for ℓ non-interacting fermions in a common potential given that one fermion lies in each of the eigenstates $\varphi_{i_1}, \dots, \varphi_{i_\ell}$. The fact that (4.9.3) vanishes if a row is repeated illustrates Pauli's exclusion principle, which states that multiple fermions cannot occupy the same eigenstate. See (27) for more details.

We shall make use of the following fact in proving Theorem 4.9.4.

FACT 4.9.6. Let \mathcal{Y} be a Poisson process on Λ with intensity measure ν . Assume that $\nu(\Lambda) < \infty$ and ν is absolutely continuous with respect to μ (the reference measure

of Λ). Let Y be the random vector of length $\mathscr{Y}(\Lambda)$ obtained from a uniform random ordering of the points of \mathscr{Y} . For $k \geq 1$, the law of Y on the event that $\mathscr{Y}(\Lambda) = k$ is a sub-probability measure on Λ^k with density

$$(4.9.4) \quad g_k(z_1, \dots, z_k) = \frac{1}{k!} \left[e^{-v(\Lambda)} \prod_{i=1}^k \frac{d\nu}{d\mu}(z_i) \right]$$

with respect to μ^k .

Proof of Theorem 4.9.4. We use Proposition 4.9.2 to construct \mathscr{X} as a Poisson process with intensity $|F|^2$ where $F(z) = \sum_{k=1}^n \sqrt{\lambda_k} a_k \varphi_k(z)$ and a_k are independent standard complex Gaussian random variables. Note that $\mathbf{E}(\mathscr{X}(\Lambda)) = \mathbf{E} \int_{\Lambda} |F(z)|^2 d\mu(z) = \sum_{k=1}^n \lambda_k < \infty$, hence \mathscr{X} contains a finite number of points a.s. Let X be the random vector obtained from a uniform random ordering of the points of \mathscr{X} . If we first condition on F , then by Fact 4.9.6 the joint density of the random vector X on the event $\{\mathscr{X}(\Lambda) = \ell\}$ is given by

$$j_{F,\ell}(z_1, \dots, z_\ell) = \frac{1}{\ell!} \left[e^{-\int_{\Lambda} |F(x)|^2 d\mu(x)} \prod_{i=1}^{\ell} |F(z_i)|^2 \right],$$

which is a sub-probability measure with total weight $\mathbf{P}[\mathscr{X}(\Lambda) = \ell | F]$. Integrating over the distribution of F we get that on the event $\{\mathscr{X}(\Lambda) = \ell\}$ the density of X is

$$(4.9.5) \quad \begin{aligned} j_{\ell}(z_1, \dots, z_{\ell}) &= \frac{1}{\ell!} \mathbf{E} \left[e^{-\int_{\Lambda} |F(x)|^2 d\mu(x)} \prod_{i=1}^{\ell} |F(z_i)|^2 \right] \\ &= \frac{1}{\ell!} \mathbf{E} \left[e^{-\sum_m \lambda_m |a_m|^2} \left| \prod_{i=1}^{\ell} \left(\sum_m \sqrt{\lambda_m} a_m \varphi_m(z_i) \right) \right|^2 \right], \end{aligned}$$

which is also a sub-probability measure with total weight $\mathbf{P}[\mathscr{X}(\Lambda) = \ell]$. We now expand the product inside the expectation (4.9.5) as a sum indexed by ordered set partitions (S_1, \dots, S_n) and (T_1, \dots, T_n) of $\{1, 2, \dots, \ell\}$. The set partitions corresponding to a summand q are constructed by letting S_k be the set of indices i for which q contains the term $\sqrt{\lambda_k} a_k \varphi_k(z_i)$ and T_k be the set of indices i for which q contains the term $\sqrt{\lambda_k} \bar{a}_k \bar{\varphi}_k(z_i)$. The summand corresponding to the partitions $(S_k), (T_k)$ is thus:

$$\mathbf{E} \left[e^{-\sum_m \lambda_m |a_m|^2} \left(\prod_k a_k^{|S_k|} \bar{a}_k^{|T_k|} \right) \left(\prod_k \prod_{i \in S_k} \lambda_k^{|S_k|/2} \varphi_k(z_i) \right) \left(\prod_k \prod_{i \in T_k} \lambda_k^{|T_k|/2} \bar{\varphi}_k(z_i) \right) \right],$$

which clearly vanishes unless $|S_k| = |T_k|$ for every k . Also note that for a standard complex normal random variable α ,

$$(4.9.6) \quad \mathbf{E} \left[e^{-\lambda |\alpha|^2} |\alpha|^{2\alpha} \right] = \frac{\alpha!}{(1 + \lambda)^{\alpha+1}}.$$

Therefore by fixing an ordered partition of the integer ℓ with n parts (some of the parts may be zero) and then summing over all ordered set partitions $(S_k), (T_k)$ with those sizes, we find that

$$(4.9.7) \quad j_{\ell}(z_1, \dots, z_{\ell}) = \frac{1}{\ell!} \sum_{(\alpha_1, \dots, \alpha_n): \sum \alpha_i = \ell} \left(\prod_{i=1}^n \frac{\lambda_i^{\alpha_i} \alpha_i!}{(1 + \lambda_i)^{\alpha_i+1}} \right) \left| \sum_{(S_m)_{m=1}^n: |S_m| = \alpha_m} \prod_{k=1}^n \prod_{i \in S_k} \varphi_k(z_i) \right|^2.$$

Now it is easy to see that

$$\sum_{(S_i): |S_i|=\alpha_i, k \geq 1, i \in S_k} \prod_{k \geq 1} \prod_{i \in S_k} \varphi_k(z_i) = \left(\prod_{i=1}^n \frac{1}{\alpha_i!} \right) \text{per} \begin{bmatrix} \{\varphi_1(z_1) & \cdots & \varphi_1(z_\ell)\} \alpha_1 \\ \cdots & \cdots & \cdots \\ \{\varphi_n(z_1) & \cdots & \varphi_n(z_\ell)\} \alpha_n \end{bmatrix}.$$

Recall that the geometric random variables γ_i in the statement of the theorem have the distributions $\mathbf{P}[\gamma_i = \alpha] = \frac{\lambda_i^\alpha}{(1+\lambda_i)^{1+\alpha}}$. Therefore we obtain

$$(4.9.8) \quad j_\ell(z_1, \dots, z_\ell) = \sum_{(\alpha_1, \dots, \alpha_n): \sum \alpha_i = \ell} \left(\frac{\prod_{i=1}^n \mathbf{P}[\gamma_i = \alpha_i]}{\ell! \prod_{i=1}^n \alpha_i!} \right) \left| \text{per} \begin{bmatrix} \{\varphi_1(z_1) & \cdots & \varphi_1(z_\ell)\} \alpha_1 \\ \cdots & \cdots & \cdots \\ \{\varphi_n(z_1) & \cdots & \varphi_n(z_\ell)\} \alpha_n \end{bmatrix} \right|^2.$$

Now we integrate (4.9.8) over all the variables z_i . Write the absolute square of the permanent on the right as

$$\text{per} \begin{bmatrix} \{\varphi_1(z_1) & \cdots & \varphi_1(z_\ell)\} \alpha_1 \\ \cdots & \cdots & \cdots \\ \{\varphi_n(z_1) & \cdots & \varphi_n(z_\ell)\} \alpha_n \end{bmatrix} \overline{\text{per}} \begin{bmatrix} \{\varphi_1(z_1) & \cdots & \varphi_1(z_\ell)\} \alpha_1 \\ \cdots & \cdots & \cdots \\ \{\varphi_n(z_1) & \cdots & \varphi_n(z_\ell)\} \alpha_n \end{bmatrix}$$

and expand these two factors over permutations π and σ of $\{1, 2, \dots, k\}$. Letting I_j denote the interval of integers $\{1 + \sum_{r=1}^{j-1} \alpha_r, \dots, \sum_{r=1}^j \alpha_r\}$ we get a sum of terms of the form

$$\left(\prod_{j=1}^n \prod_{i \in \pi^{-1}(I_j)} \varphi_j(z_i) \right) \left(\prod_{j=1}^n \prod_{i \in \sigma^{-1}(I_j)} \overline{\varphi_j(z_i)} \right).$$

By orthogonality of φ_j s, this term vanishes upon integration unless $\pi^{-1}(I_j) = \sigma^{-1}(I_j)$ for every $1 \leq j \leq n$. For a given π , there are $\prod_{j=1}^n \alpha_j!$ choices of σ that satisfy this. For each such σ , we get 1 upon integration over z_i s. Summing over all $\ell!$ choices for π , we get

$$\int_{\Lambda^k} j_\ell d\mu^\ell = \mathbf{P}[\mathcal{X}(\Lambda) = \ell] = \sum_{(\alpha_1, \dots, \alpha_n): \sum \alpha_i = \ell} \prod_{i=1}^n \mathbf{P}[\gamma_i = \alpha_i] = \mathbf{P} \left[\sum_{i=1}^n \gamma_i = \ell \right],$$

which proves the claim about the number of points in Λ . Thus by (4.9.8) \mathcal{X} is a mixture of the processes $\mathcal{X}^{\vec{\alpha}}(D)$, with weights given by $\prod_{i=1}^n \mathbf{P}[\gamma_i = \alpha_i]$, where $\vec{\alpha} = (\alpha_1, \dots, \alpha_n)$ with α_i being non-negative integers. This is what we wanted to prove. \square

The analogue of Proposition 4.5.9 is also true.

THEOREM 4.9.7. (*Hough et al. (33)*) *Under the assumptions of Theorem 4.9.4, suppose D_1, \dots, D_r are simultaneously observable as in definition 4.5.7. Denote by φ_k the common eigenfunctions of \mathbb{K} on the D_i 's and by $\lambda_{k,i}$ the corresponding eigenvalues. Then*

$$(4.9.9) \quad (\mathcal{X}(D_1), \dots, \mathcal{X}(D_r)) \stackrel{d}{=} \sum_k (\eta_{k,1}, \dots, \eta_{k,r}),$$

where $(\eta_{k,1}, \dots, \eta_{k,r})$ are independent for different values of k , for each k , the sum $\eta_k = \sum_i \eta_{k,i}$ has a geometric distribution with mean $\lambda_k := \sum_i \lambda_{k,i}$ and given $\sum_i \eta_{k,i} = \eta_k$,

$$(\eta_{k,1}, \dots, \eta_{k,r}) \stackrel{d}{=} \text{Multinomial} \left(\eta_k; \frac{\lambda_{k,1}}{\lambda_k}, \dots, \frac{\lambda_{k,r}}{\lambda_k} \right).$$

As before, we remark that this result is applicable to the restriction $\mathcal{X} \cap D$, for any Borel set $D \subset \Lambda$.

PROOF. Suppose D_1, \dots, D_r are simultaneously observable as in the statement of the theorem. Use Proposition 4.9.2 to write the restriction of \mathcal{X} to $\cup_{i=1}^r D_i$ as a Poisson process with intensity $|F(x)|^2$ where F is a Gaussian process with covariance kernel \mathbb{K} . Explicitly, $F(z) = \sum_k a_k \sqrt{\lambda_k} \varphi_k(z)$ for $z \in \cup_{i=1}^r D_i$, where a_k are i.i.d. standard complex Gaussians, i.e., the real and imaginary parts of a_k are i.i.d. $N(0, \frac{1}{2})$. Then given $\{a_k\}$, we know that $\mathcal{X}(D_i)$, $1 \leq i \leq r$ are independent Poisson($\int_{D_i} |F(x)|^2 d\mu(x)$). Writing $\int_{D_i} |F(x)|^2 d\mu(x) = \sum_k \lambda_{k,i} |a_k|^2$, we see that conditionally given $\{a_k\}$, the variables $\mathcal{X}(D_i)$ for $1 \leq i \leq r$ have the same distribution as $\sum_k (\eta_{k,1}, \dots, \eta_{k,r})$, where $\{\eta_{k,i}\}_{1 \leq i \leq r}$ are chosen by sampling η_k according to Poisson($\lambda_k |a_k|^2$) distribution and then assigning η_k points to the cells D_i multinomially with probabilities $\frac{\lambda_{k,i}}{\lambda_k}$.

Integrating over the randomness in $\{a_k\}$, we see that

$$\mathbf{P}(\eta_k = m) = \mathbf{E} \left[\frac{e^{-\lambda_k |a_k|^2} \lambda_k^m |a_k|^{2m}}{m!} \right] = \frac{\lambda_k^m}{(1 + \lambda_k)^{m+1}},$$

where we have used (4.9.6) to deduce the second equality. Hence η_k has a Geometric distribution with mean λ_k , and given η_k , the vector $(\eta_{k,1}, \dots, \eta_{k,r})$ is still Multinomial($\eta_k; \frac{\lambda_{k,1}}{\lambda_k}, \dots, \frac{\lambda_{k,r}}{\lambda_k}$). This completes the proof. \square

4.10. Notes

- The algorithm 4.4.3 was introduced in the survey paper (33). It was implemented in practice by Scardicchio, Zachary and Torquato in (73) who provide various enhancements to the algorithm for efficient implementation.
- One way to generalize the concept of determinantal and permanental processes is to consider point processes with joint intensities given by

$$(4.10.1) \quad \rho_n(x_1, \dots, x_n) = \det_\alpha(\mathbb{K}(x_i, x_j)) \stackrel{\text{def}}{=} \sum_{\pi \in S_n} \alpha^{n-\nu(\pi)} \prod_{i=1}^n \mathbb{K}(x_i, x_{\pi(i)}),$$

where $\nu(\pi)$ is the number of cycles in the permutation π .

Such point processes are called α -determinantal processes in Shirai and Takahashi (75). The values $\alpha = -1$ and $\alpha = +1$ correspond to determinantal and permanental processes, respectively. It is easy to check that the proof of Theorem 4.5.3 can be modified to get:

PROPOSITION 4.10.1. *Suppose there exists a point process \mathcal{X} on Λ with joint intensities given by (4.10.1), where \mathbb{K} is a Hermitian, non-negative definite, trace class kernel. Write*

$$(4.10.2) \quad \mathbb{K}(x, y) = \sum_{k=1}^n \lambda_k \varphi_k(x) \overline{\varphi_k(y)},$$

where $\{\varphi_k\}$ is an orthonormal set of eigenfunctions for the integral operator \mathcal{K} with eigenvalues λ_k . Then $\mathcal{X}(\Lambda)$ is:

- a sum of independent Binomial($-\frac{1}{\alpha}, -\alpha\lambda_k$) random variables, if $-\frac{1}{\alpha}$ is a positive integer.
- a sum of independent Negative Binomial($\frac{1}{\alpha}, \frac{1}{\alpha\lambda_k+1}$) random variables, if $\alpha > 0$.

In fact, if $-\frac{1}{\alpha}$ is a positive integer, this process is just a union of $-\frac{1}{\alpha}$ i.i.d. copies of the determinantal process with kernel $-\alpha\mathbb{K}$. Similarly, if $\frac{1}{\alpha}$ is a positive integer, this process is a union of $\frac{1}{\alpha}$ i.i.d. copies of the permanental process with kernel $\alpha\mathbb{K}$. More generally, the union of m i.i.d. copies of the process corresponding to α and kernel \mathbb{K} gives a process distributed according to αm and kernel $\frac{1}{m}\mathbb{K}$. If

\mathbb{K} is real, and defines a bounded, non-negative definite, symmetric, locally trace class integral operator \mathcal{K} on Λ , and the measure μ on Λ is non-atomic, then $\frac{2}{m}$ -determinantal processes also exist (75). For $\alpha > 0$, little is known about the existence of α -determinantal processes beyond these examples. Shirai and Takahashi (75) conjecture the following:

CONJECTURE 4.10.2. *If \mathbb{K} is a (real) kernel defining a self-adjoint, non-negative integral operator on $L^2(\Lambda)$ and $0 \leq \alpha \leq 2$, then the α -determinantal process with kernel \mathbb{K} exists. However, if $\alpha > 2$, then there exists such a kernel \mathbb{K} for which there is no corresponding α -determinantal process.*

We verify this conjecture for $\alpha > 4$. Let Λ be a discrete space consisting of three points, and consider the 3×3 matrix

$$\mathbb{K} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

It is easy to check that the eigenvalues of \mathbb{K} are 3, 3, 0 and

$$\det_\alpha(\mathbb{K}(i, j))_{1 \leq i, j \leq 3} = 2(4 - \alpha)(\alpha + 1),$$

which is negative for $\alpha > 4$. Since point processes must have non-negative joint intensities, we conclude that no α -determinantal processes with this kernel can exist for $\alpha > 4$.

4.11. Hints and solutions

Exercise 4.3.5 The first part is obvious. For the second, applying the optional sampling theorem as suggested in the exercise, we get

$$(4.11.1) \quad M(0) = \mathbf{E}[M(\tau)\mathbf{1}_{\tau \leq t}] + \mathbf{E}[M(t)\mathbf{1}_{\tau > t}].$$

Note that $M(s) = \det\left(P_{X_p(s), j_q}\right)_{p, q \leq n}$. In particular, $M(\tau)\mathbf{1}_{\tau \leq t} = 0$ because if $\tau \leq t$, then at time τ two rows of the matrix are equal. Also, $M(0) = \det\left(P_{i_p, j_q}(t)\right)_{p, q \leq n}$ is the right hand side of the Karlin-McGregor formula. On the other hand, $M(t)\mathbf{1}_{\tau > t}$ is precisely the indicator of the event that the random walks X_p arrive at locations j_p respectively at time t , and that no two walks meet up to and including time t . Thus $\mathbf{E}[M(t)\mathbf{1}_{\tau > t}]$ is the probability of the event asked for in the Karlin-McGregor formula. This completes the proof.

Exercise 4.5.2 Given $S \subset [n]$, write $\mathbb{K}_S(z, w) = \sum_{j \in S} \varphi_j(z)\overline{\varphi_j}(w)$. By assumption \mathcal{X} has conditional joint intensity functions

$$(4.11.2) \quad \begin{aligned} \rho_k(z_1, \dots, z_k \mid |\mathcal{X}| = k) &= \frac{(1 - \lambda)^{[n]}}{Z_k} \det(\mathbb{L}(z_i, z_j)_{1 \leq i, j \leq k}) \\ &= \frac{1}{Z_k} \sum_{S \subset [n]: |S|=k} \lambda^S (1 - \lambda)^{S^c} \det(\mathbb{K}_S(z_i, z_j)_{1 \leq i, j \leq k}). \end{aligned}$$

Since equation (4.11.2) gives the joint intensity for a point process containing k points a.s., we may integrate to obtain lower order joint intensity functions as in equation (4.5.1). Since each term $\det(\mathbb{K}_S(z_i, z_j)_{1 \leq i, j \leq k})$ gives the joint intensity function for a determinantal process containing exactly k points, we see immediately that for $\ell \leq k$:

$$\begin{aligned} \rho_\ell(z_1, \dots, z_\ell \mid |\mathcal{X}| = k) &= \frac{1}{Z_k} \sum_{S \subset [n]: |S|=k} \lambda^S (1 - \lambda)^{S^c} \det(\mathbb{K}_S(z_i, z_j)_{1 \leq i, j \leq \ell}) \\ &= \frac{1}{Z_k} \sum_{S \subset [n]: |S|=\ell} \det(\mathbb{K}_S(z_i, z_j)_{1 \leq i, j \leq \ell}) \sum_{\substack{\tilde{S} \subset [n] \\ \tilde{S} \supset S: |\tilde{S}|=k}} \lambda^{\tilde{S}} (1 - \lambda)^{\tilde{S}^c}. \end{aligned}$$

Considering now the probability that \mathcal{X} contains k points, we compute the unconditional joint intensity functions as follows:

$$(4.11.3) \quad \rho_\ell(z_1, \dots, z_\ell) = \sum_{k=\ell}^n \rho_\ell(z_1, \dots, z_\ell \mid |\mathcal{X}| = \ell) \mathbf{P}(|\mathcal{X}| = \ell)$$

$$(4.11.4) \quad = \sum_{k=\ell}^n \sum_{S \subset [n]; |S|=\ell} \det(\mathbb{K}_S(z_i, z_j)_{1 \leq i, j \leq \ell}) \sum_{\substack{S \subset [n] \\ \tilde{S} \supset S; |\tilde{S}|=k}} \lambda^{\tilde{S}} (1-\lambda)^{\tilde{S}^c}.$$

An easy inductive argument gives

$$(4.11.5) \quad \lambda^S = \sum_{\substack{\tilde{S} \subset [n] \\ S \supset \tilde{S}}} \lambda^{\tilde{S}} (1-\lambda)^{\tilde{S}^c}.$$

Combining equations (4.11.3) and (4.11.5) we deduce

$$(4.11.6) \quad \rho_\ell(z_1, \dots, z_\ell) = \sum_{S \subset [n]; |S|=\ell} \det(\mathbb{K}_S(z_i, z_j)_{1 \leq i, j \leq \ell})$$

$$(4.11.7) \quad = \det(\mathbb{K}(z_i, z_j)_{1 \leq i, j \leq \ell})$$

as claimed.

The Hyperbolic GAF

In chapter 4 we presented many eigenvalue ensembles in random matrix theory that happened to be determinantal point processes. In this chapter we return to Gaussian analytic functions and show that zeros of the i.i.d. power series ($L = 1$ in 5.1.1) form a determinantal process. Curiously enough, this is the only non-trivial example of a Gaussian analytic functions whose zero set is known to be determinantal! In particular, among the canonical models of zeros of Gaussian analytic functions introduced in chapter 2, all of which were treated on equal footing so far, the case $\Lambda = \mathbb{D}$, $L = 1$, is alone rather special. In this chapter we only consider the hyperbolic Gaussian analytic functions, and hence simply write \mathbf{f}_L for $\mathbf{f}_{\mathbb{D},L}$.

5.1. A determinantal formula

Let a_n be i.i.d. standard complex Gaussian random variables. Recall the hyperbolic Gaussian analytic function

$$(5.1.1) \quad \mathbf{f}_L(z) = \sum_{n=0}^{\infty} a_n \frac{\sqrt{L(L+1)} \dots (L+n-1)}{\sqrt{n!}} z^n$$

that converge on the unit disk and have zero sets invariant in distribution under isometries of the hyperbolic plane.

THEOREM 5.1.1 (Peres and Virág). *The joint intensity of zeros for the i.i.d. Gaussian power series $\mathbf{f}_1(z)$ in the unit disk exists, and satisfies*

$$(5.1.2) \quad \rho_n(z_1, \dots, z_n) = \pi^{-n} \det \left[\frac{1}{(1 - z_i \bar{z}_j)^2} \right]_{i,j}.$$

In view of this theorem, it is natural to ask whether the other canonical models of Gaussian analytic functions introduced in section 2.3 also have determinantal zero sets. The answer is no, because one can check that $\rho_2(z, w) > \rho_1(z)\rho_1(w)$ for the zero sets of these Gaussian analytic functions for sufficiently distant pairs of points z, w . Since a determinantal point process (with a hermitian kernel) has negative correlations (meaning $\rho_2(z, w) < \rho_1(z)\rho_1(w)$, $\forall z, w$), it follows that these zero sets are not determinantal. Figure 1 shows a picture of the two-point intensity for (5.1.1) for several values of L . Only in the case $L = 1$ is the relative intensity, $\frac{\rho_2(0,r)}{\rho_1(0)\rho_1(r)}$ bounded by 1 (By invariance, it is sufficient to consider $z = 0$ and $w = r > 0$).

EXERCISE 5.1.2. Show that for general L , the quantity $\frac{\rho_2(0,r)}{\rho_1(0)\rho_1(r)}$ is equal to (notation: $s = 1 - r^2$):

$$(5.1.3) \quad \frac{1 + (L^2 - 2L - 2)(s^L + s^{2+2L}) + (L+1)^2(s^{2L} + s^{2+L}) - 2L^2(s^{1+L} + s^{1+2L}) + s^{2+3L}}{(1 - s^L)^3}$$

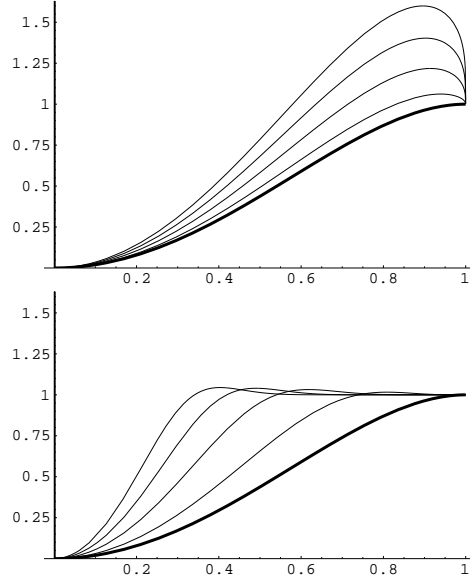


FIGURE 1. Relative intensity at $z = 0$ and $z = r$ as a function of r for $L = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$ and for $L = 1, 4, 9, 16, 25$.

In the case $L = 1$ it simplifies to $r^2(2 - r^2)$.

For every distance r , the correlation is minimal when $L = 1$ (see Figure 1). For all values of L different than 1, for small distance zeros are negatively correlated, while for large distances the correlation is positive. Since the points in a determinantal process are always negatively correlated (which is clear from the determinantal form of the pairwise joint intensity) it follows that none of the zero sets Z_L for $L \neq 1$ can be determinantal processes.

The fact that the zero set of \mathbf{f} is determinantal, allows us to apply all the machinery developed in Chapter 4 to this point process making certain computations especially easy. In particular, we apply Theorem 4.7.1 to determine the joint distribution of the moduli of the zeros and also the asymptotics of the “hole probability” that the disk $D(0, r)$ contains no zeros as $r \uparrow 1$.

5.1.1. Proof of determinantal formula. We now give the proof of Theorem 5.1.1, which relies on the i.i.d. nature of the coefficients of $\mathbf{f} = \mathbf{f}_{\mathbb{D}}$, Möbius invariance, Hammersley’s formula and an old identity relating permanents and determinants due to Borchardt (1855). Hammersley’s formula (3.4.1) provides us with an explicit expression for the joint intensity of the zero set $Z_{\mathbb{D},1}$, namely,

$$(5.1.4) \quad \rho_n(z_1, \dots, z_n) = \frac{\mathbf{E}\left(|\mathbf{f}'(z_1) \cdots \mathbf{f}'(z_n)|^2 \mid \mathbf{f}(z_1), \dots, \mathbf{f}(z_n) = 0\right)}{\pi^n \det(A)}$$

where A is the covariance matrix $A_{jk} = \mathbf{E}\mathbf{f}(z_j)\overline{\mathbf{f}(z_k)} = (1 - z_j\bar{z}_k)^{-1}$. The difficulty in applying this formula lies in understanding the conditional distribution of \mathbf{f} given that $\mathbf{f}(z_1) = \cdots = \mathbf{f}(z_k) = 0$. However, note that the conditional distribution of $\mathbf{f}(z)$ given that $\mathbf{f}(0) = 0$ is especially easy to understand. Conditioning $\mathbf{f}(0) = 0$ simply stipulates that $a_0 = 0$, and since the coefficients a_k are i.i.d., the conditional distribution of $\mathbf{f}(z)$

given $\mathbf{f}(0) = 0$ is the same as the *unconditional* distribution of $z\mathbf{f}(z)$. Amazingly, this simple observation generalizes, as explained in the following proposition:

PROPOSITION 5.1.3. *Let $\mathbf{f} = \mathbf{f}_1$ and $z_1, \dots, z_n \in \mathbb{D}$. The distribution of the random function*

$$(5.1.5) \quad T_{z_1}(z) \cdots T_{z_n}(z) \mathbf{f}(z),$$

where

$$(5.1.6) \quad T_\beta(z) = \frac{z - \beta}{1 - \bar{\beta}z}$$

denotes a Möbius transformation fixing the unit disk, is the same as the conditional distribution of $\mathbf{f}(z)$ given $\mathbf{f}(z_1) = \dots = \mathbf{f}(z_n) = 0$.

PROOF. We have already remarked that the assertion is clear when $n = 1$ and $z_1 = 0$. More generally, for $z_1 = \beta$, we claim that the random function $\tilde{\mathbf{f}} = \tau_\beta \cdot (\mathbf{f} \circ T_\beta)$ has the same distribution as \mathbf{f} , where

$$(5.1.7) \quad \tau_\beta(z) = \frac{(1 - |\beta|^2)^{1/2}}{1 - \bar{\beta}z}$$

satisfies $\tau_\beta^2(z) = T'_\beta(z)$. Indeed, we verify this assertion by computing:

$$\begin{aligned} \mathbf{E} \left[\tilde{\mathbf{f}}(z) \overline{\tilde{\mathbf{f}}(w)} \right] &= \mathbf{E} \left[\tau_\beta(z) \mathbf{f} \circ T_\beta(z) \overline{\tau_\beta(w) \mathbf{f} \circ T_\beta(w)} \right] \\ &= \tau_\beta(z) \overline{\tau_\beta(w)} \left(1 - T_\beta(z) \overline{T_\beta(w)} \right)^{-1} \\ &= \frac{1}{1 - z\bar{w}} \\ &= \mathbf{E}(\mathbf{f}(z) \overline{\mathbf{f}(w)}). \end{aligned}$$

Now, since $T_\beta(\beta) = 0$, from the formula

$$\tilde{\mathbf{f}}(z) = \tau_\beta(z) \sum_{k=0}^{\infty} a_k (T_\beta(z))^k$$

it is clear that the distribution of $T_\beta \cdot \tilde{\mathbf{f}}$ is identical to the conditional distribution of $\tilde{\mathbf{f}}$ given $\tilde{\mathbf{f}}(\beta) = 0$, whence the same must hold for \mathbf{f} in place of $\tilde{\mathbf{f}}$.

The proposition for $n > 1$ follows by induction: to go from n to $n + 1$, we must condition $(\mathbf{f} | \mathbf{f}(z_1) = \dots = \mathbf{f}(z_n) = 0)$ on $\mathbf{f}(z_{n+1}) = 0$. By the assumed identity for n points, this is equivalent to conditioning $(T_{z_1} \cdots T_{z_n} \cdot \mathbf{f})(z)$ on $\mathbf{f}(z_{n+1}) = 0$. It remains to observe that conditioning is a linear operator that commutes with multiplication by the deterministic functions T_{z_i} . Indeed, for two jointly complex Gaussian random vectors X, Y , the distribution of Y given $X = 0$ is the same as the distribution of Y with each entry projected to the orthocomplement (in L^2 of the underlying probability space) of the subspace spanned by the components X_i of X . Hence, applying the equality of distributions $(\mathbf{f}(z) | \mathbf{f}(z_{n+1}) = 0) \stackrel{d}{=} T_{z_{n+1}}(z) \mathbf{f}(z)$ completes the proof. \square

This result makes it easy to compute the joint distribution of $\mathbf{f}(z_k)$ given that $\mathbf{f}(z_1) = \dots = \mathbf{f}(z_n) = 0$, which is needed to apply Hammersley's formula. For fixed $z_1, \dots, z_n \in \mathbb{D}$ set

$$(5.1.8) \quad Y(z) = \prod_{j=1}^n T_{z_j}(z).$$

and observed that since $T_{z_k}(z_k) = 0$ and $T'_{z_k}(z_k) = 1/(1 - z_k \bar{z}_k)$:

$$(5.1.9) \quad Y'(z_k) = T'_{z_k}(z_k) \cdot \prod_{j:j \neq k} T_{z_j}(z_k) = \prod_{j=1}^n \frac{1}{1 - z_j \bar{z}_k} \cdot \prod_{j:j \neq k} (z_j - z_k)$$

for each $k \leq n$. Now from Proposition 5.1.3 we have:

COROLLARY 5.1.4. *Let $\mathbf{f} = \mathbf{f}_{\mathbb{D}}$ and $z_1, \dots, z_n \in \mathbb{D}$. The conditional joint distribution of the random variables $(\mathbf{f}'(z_k) : k = 1, \dots, n)$ given that $\mathbf{f}(z_1) = \dots = \mathbf{f}(z_n) = 0$, is the same as the unconditional joint distribution of $(Y'(z_k)\mathbf{f}(z_k) : k = 1, \dots, n)$.*

PROOF. The conditional distribution of \mathbf{f} given that $\mathbf{f}(z_j) = 0$ for $1 \leq j \leq n$, is the same as the unconditional distribution of $Y \cdot \mathbf{f}$. Since $Y(z_k) = 0$, the derivative of $Y(z)\mathbf{f}(z)$ at $z = z_k$ equals $Y'(z_k)\mathbf{f}(z_k)$. \square

It follows from Corollary 5.1.4 that

$$(5.1.10) \quad \mathbf{E}\left(|\mathbf{f}'(z_1) \cdots \mathbf{f}'(z_n)|^2 \mid \mathbf{f}(z_1), \dots, \mathbf{f}(z_n) = 0\right) = \mathbf{E}\left(|\mathbf{f}(z_1) \cdots \mathbf{f}(z_n)|^2\right) \prod_k |Y'(z_k)|^2.$$

and hence Hammersley's formula (3.4.1) now gives

$$(5.1.11) \quad \rho(z_1, \dots, z_n) = \frac{\mathbf{E}\left(|\mathbf{f}(z_1) \cdots \mathbf{f}(z_n)|^2\right) \prod_k |Y'(z_k)|^2}{\pi^n \det A}$$

where we recall that A is the covariance matrix $A_{jk} = \mathbf{E}(\mathbf{f}(z_j)\overline{\mathbf{f}(z_k)}) = (1 - z_j \bar{z}_k)^{-1}$. By Wick's formula 2.1.7 we know that $\text{per}(A) = \mathbf{E}\left(|\mathbf{f}(z_1) \cdots \mathbf{f}(z_n)|^2\right)$, and hence

$$(5.1.12) \quad \rho(z_1, \dots, z_n) = \frac{\text{per}(A) \prod_k |Y'(z_k)|^2}{\pi^n \det A}.$$

From the Cauchy determinant formula we obtain:

$$(5.1.13) \quad \det(A) = \prod_{k,j} \frac{1}{1 - z_j \bar{z}_k} \prod_{k < j} (z_j - z_k)(\bar{z}_j - \bar{z}_k).$$

Comparing this to (5.1.9), we see that

$$(5.1.14) \quad \det(A) = \prod_{k=1}^n |Y'(z_k)|.$$

To complete the proof we need to apply Borchardt's identity:

PROPOSITION 5.1.5 (Borchardt's identity). *Let x_i, y_i , $1 \leq i \leq n$, be complex numbers such that $x_i y_j \neq 1$ for any i, j . Then*

$$(5.1.15) \quad \text{per}\left(\frac{1}{1 - x_i y_j}\right)_{i,j \leq n} \det\left(\frac{1}{1 - x_i y_j}\right)_{i,j \leq n} = \det\left(\frac{1}{(1 - x_i y_j)^2}\right)_{i,j \leq n}.$$

We shall give a proof of this classical result in the next subsection. Borchardt's identity implies:

$$(5.1.16) \quad \text{per}(A) \det(A) = \det(M)$$

where $M_{jk} = (1 - z_j \bar{z}_k)^{-2}$. Now combining (5.1.12), (5.1.14) and (5.1.16) we deduce

$$(5.1.17) \quad \rho(z_1, \dots, z_n) = \pi^{-n} \det(M)$$

which proves Theorem 5.1.1.

5.1.2. Borchardt's identity. In this section we shall prove Borchardt's Identity (Proposition 5.1.5) dating back to 1855 (9). First, we prove an identity due to Carlitz and Levine (14) that gives Borchardt's identity as a special case. The proof we present is the original proof of Carlitz and Levine.

Notation: In this section, if A is a matrix with entries a_{ij} and m is any integer, then $A^{(m)}$ denotes the matrix whose entries are a_{ij}^m .

THEOREM 5.1.6 (Carlitz and Levine). *Let A be an $n \times n$ matrix with non-zero entries, such that the matrix $A^{(-1)} = (a_{ij}^{-1})$ has rank 2. Then*

$$(5.1.18) \quad \text{per}(A)\det(A) = \det\left(A^{(2)}\right).$$

Assuming this result, we easily obtain Borchardt's Identity.

Proof of Proposition 5.1.5. First assume that $x_i \neq 0 \forall i$. Then set $a_{ij} = \frac{1}{\frac{1}{x_i} - y_j}$ for $i, j \leq n$. Clearly $A^{(-1)}$ has rank 2. Thus (5.1.18) holds. Dividing both sides of the resulting equation by $\prod_{i=1}^n x_i$ gives (5.1.15). By continuity, this is valid even if some of the x_i s are equal to zero. \square

Proof of Theorem 5.1.6. Let A, B be arbitrary matrices and consider $\text{per}(A)\det(B)$. Expanding the permanent and the determinant over the permutation group \mathcal{S}_n , we get

$$\begin{aligned} \text{per}(A)\det(B) &= \sum_{\pi \in \mathcal{S}_n} \sum_{\sigma \in \mathcal{S}_n} \text{sgn}(\sigma) \prod_{k=1}^n a_{k\pi_k} b_{k\sigma_k} \\ &= \sum_{\pi} \sum_{\sigma} \text{sgn}(\sigma\pi) \prod_{k=1}^n a_{k\pi_k} b_{k\sigma(\pi_k)} \\ &= \sum_{\sigma} \text{sgn}(\sigma) \sum_{\pi} \text{sgn}(\pi) \prod_{k=1}^n a_{k\pi_k} b_{k\sigma(\pi_k)}. \end{aligned}$$

Now for a fixed σ let B_{σ} be the matrix obtained by permuting the columns of B according to the permutation σ . More precisely, $(B_{\sigma})_{ij} = b_{i\sigma_j}$. Let $*$ denote the Hadamard product, i.e., $(A * B)_{ij} = a_{ij}b_{ij}$. Then

$$\det(A * B_{\sigma}) = \sum_{\pi} \text{sgn}(\pi) \prod_{k=1}^n a_{k\pi_k} b_{k\sigma(\pi_k)}$$

is precisely the inner summand in the earlier expression for $\text{per}(A)\det(B)$. Thus we get

$$(5.1.19) \quad \text{per}(A)\det(B) = \sum_{\sigma} \text{sgn}(\sigma)\det(A * B_{\sigma}),$$

which is a formula due to Muir (see the book by Minc (57)).

When $B = A$, the summand on the right hand side of (5.1.19) corresponding to the identity permutation is precisely the term on the right hand side of (5.1.18)

We shall prove that under the assumptions of the theorem, when $B = A$, each term in (5.1.19) corresponding to a non-identity permutation separately vanishes. To see this, note that if $\sigma \neq \text{identity}$, then it has a cycle of length at least two. If it has a cycle of length exactly 2, say $\sigma(1) = \sigma^{-1}(1) = 2$, then the first and second columns of $A * A_{\sigma}$ are identical and thus $\det(A * A_{\sigma}) = 0$.

Now suppose that σ has a cycle of length greater than or equal to 3. Without loss of generality, assume that σ contains the cycle $(1, 2, \dots, k)$. Then we claim that

the matrix formed by the first k columns of $A * A_\sigma$ has rank at most $k - 1$, which obviously implies that $\det(A * A_\sigma) = 0$. To see this claim, note that the matrix formed by the first k columns of $A * A_\sigma$ is

$$\begin{pmatrix} a_{11}a_{12} & a_{12}a_{13} & \dots & a_{1k}a_{11} \\ a_{21}a_{22} & a_{22}a_{23} & \dots & a_{2k}a_{21} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1}a_{n2} & a_{n2}a_{n3} & \dots & a_{nk}a_{n1} \end{pmatrix}.$$

Factoring out $\prod_{j=1}^k a_{ij}$ from the i^{th} row does not affect the rank. The resulting matrix C has entries

$$c_{ij} = \prod_{r \neq j, j+1} a_{ir}^{-1},$$

where $k + 1$ is to be interpreted as 1.

Now by the assumptions on the rank of $A^{(-1)}$, we can assume without loss of generality that the column space of the matrix formed by the first k columns of $A^{(-1)}$ is spanned by the p^{th} and q^{th} columns. In symbols,

$$a_{ij}^{-1} = \alpha_j a_{ip}^{-1} + \beta_j a_{iq}^{-1},$$

for some constants $\alpha_j, \beta_j, 1 \leq j \leq k$. Thus

$$\begin{aligned} c_{ij} &= \prod_{r \neq j, j+1} (\alpha_r a_{ip}^{-1} + \beta_r a_{iq}^{-1}) \\ &= \sum_{m=0}^{k-2} \gamma_{i,m} a_{ip}^{-m} a_{iq}^{-(k-2-m)}, \end{aligned}$$

where $\gamma_{i,m}$ are constants. This means that the columns of C are spanned by the $k - 1$ vectors $\mathbf{v}_m, 0 \leq m \leq k - 2$, where $\mathbf{v}_m = a_{ip}^{-m} a_{iq}^{-(k-2-m)}$. This completes the proof of the theorem. \square

5.1.3. The number of zeros in a disk. Using results from Chapter 4, the determinant formula for the joint intensity of $Z_{\mathbb{D},1}$ allows us to determine the distribution of the number of zeros of \mathbf{f}_1 in a disk, and identify the law of the set of absolute values of the zeros.

COROLLARY 5.1.7. **(i)** *The number $N_r = |Z_{\mathbb{D},1} \cap B_r(0)|$ of zeros of $\mathbf{f}_{\mathbb{D}}$ in the disk of Euclidean radius r about 0, satisfies*

$$(5.1.20) \quad \mathbf{E}(1+s)^{N_r} = \prod_{k=1}^{\infty} (1+r^{2k}s)$$

for all real s . Thus N_r has the same distribution as $\sum_{k=1}^{\infty} X_k$ where $\{X_k\}$ is a sequence of independent $\{0,1\}$ -valued random variables with $\mathbf{P}(X_k = 1) = r^{2k}$.

(ii) *Moreover, the set of moduli $\{|z| : \mathbf{f}_1(z) = 0\}$ has the same law as the set $\{U_k^{1/(2k)}\}$, where U_1, U_2, \dots are i.i.d. random variables uniform in $[0,1]$.*

PROOF. Writing the Bergman kernel as $\mathbb{K}(z,w) = \sum_{k=1}^{\infty} k(z\bar{w})^{k-1}$, we see that this corollary is an immediate consequence of Theorem 4.7.1, where $a_k^2 = k$ and μ is the uniform measure in the unit disk. \square

From Corollary 5.1.7 we readily obtain the asymptotics of the hole probability $\mathbf{P}(N_r = 0)$. Furthermore, the infinite product in (5.1.20) occurs in one of Euler's partition identities, see (5.1.22), and this connection yields part (ii) of the next corollary. Observe that the normalization of hyperbolic area below differs from the one in chapter 2 by a factor of 4. This agrees with the usual convention, as remarked in the paragraph following (2.3.3).

COROLLARY 5.1.8. : (i) Let $h = 4\pi r^2/(1-r^2)$, the hyperbolic area of $B_r(0)$. As $r \uparrow 1$, we have

$$\mathbf{P}(N_r = 0) = \exp\left(\frac{-\pi h + o(h)}{24}\right) = \exp\left(\frac{-\pi^2 + o(1)}{12(1-r)}\right).$$

: (ii) The binomial moments of N_r equal

$$\mathbf{E}\binom{N_r}{k} = \frac{r^{k(k+1)}}{(1-r^2)(1-r^4)\cdots(1-r^{2k})}.$$

: (iii) The ratio $(N_r - \mu_r)/\sigma_r$ converges in law to standard normal as $r \uparrow 1$, where

$$\mu_r = \mathbf{E}N_r = \frac{r^2}{1-r^2}, \quad \text{and } \sigma_r^2 = \text{Var}N_r = \frac{r^2}{1-r^4}.$$

PROOF. (i) Corollary 5.1.7 implies that $\mathbf{P}(N_r = 0) = \prod_{k=1}^{\infty}(1-r^{2k})$ and the asymptotics for the right hand side are classical, see Newman (61), p. 19. For the reader's convenience we indicate the argument. Let $L = \log \mathbf{P}(N_r = 0) = \sum_{k=1}^{\infty} \log(1-r^{2k})$ which we compare to the integral

$$(5.1.21) \quad I = \int_1^{\infty} \log(1-r^{2k}) dk = \frac{1}{-2\log r} \int_{-2\log r}^{\infty} \log(1-e^{-x}) dx.$$

We have $I + \log(1-r^2) < L < I$, so $L = I + o(h)$. Since $-\log(1-e^{-x}) = \sum_{n=1}^{\infty} \frac{e^{-nx}}{n}$, the integral in (5.1.21) converges to $-\pi^2/6$. But $\frac{-1}{2\log r} = \frac{1/2+o(1)}{1-r} = \frac{h}{4\pi} + o(h)$, and we get $L = -\frac{\pi^2/12+o(1)}{1-r} = -\frac{\pi h}{24} + o(h)$, as claimed.

(ii) Let $q = r^2$. Theorem 5.1.7 implies that:

$$\begin{aligned} \mathbf{E} \sum_{k=0}^{\infty} \binom{N_r}{k} s^k &= \mathbf{E}(1+s)^{N_r} \\ &= \prod_{k=0}^{\infty} (1+q^k s). \end{aligned}$$

One of Euler's partition identities (see Pak (65), section 2.3.4) gives

$$(5.1.22) \quad \prod_{k=1}^{\infty} (1+q^k s) = \sum_{k=0}^{\infty} \frac{q^{\binom{k+1}{2}} s^k}{(1-q)\cdots(1-q^k)},$$

and the claim follows.

(iii) This result is obtained by applying Lindeberg's triangular array central limit theorem to the representation of N_r as the sum of independent random variables, as given in Corollary 5.1.7(i). \square

5.2. Law of large numbers

Our next result is a law of large numbers for the zero set Z_L . For $L = 1$, one could of course readily use Corollary 5.1.7 to prove the following proposition, although the conclusion would only be of convergence in probability and not almost surely. We give a more general argument which is valid for any $L > 0$.

PROPOSITION 5.2.1. *Let $L > 0$, and suppose that $\{\Lambda_h\}_{h>0}$ is an increasing family of Borel sets in \mathbb{D} , parameterized by hyperbolic area $h = A(\Lambda_h)$. Then the number $N(h) = |Z_L \cap \Lambda_h|$ of zeros of \mathbf{f}_L in Λ_h satisfies*

$$\lim_{h \rightarrow \infty} \frac{N(h)}{h} = \frac{L}{4\pi} \quad \text{a.s.}$$

We will use the following lemma in the proof.

LEMMA 5.2.2. *Let μ be a Borel measure on a metric space S , and assume that all balls of the same radius have the same measure. Let $\psi : [0, \infty) \rightarrow [0, \infty)$ be a non-increasing function. Let $A \subset S$ be a Borel set, and let $B = B_R(x)$ be a ball centered at $x \in S$ with $\mu(A) = \mu(B_R(x))$. Then for all $y \in S$*

$$\int_A \psi(\text{dist}(y, z)) d\mu(z) \leq \int_B \psi(\text{dist}(x, z)) d\mu(z).$$

PROOF. It suffices to check this claim for indicator functions $\psi(s) = \mathbf{1}_{\{s \leq r\}}$. In this case, the inequality reduces to

$$\mu(A \cap B_r(y)) \leq \mu(B_R(x) \cap B_r(x)),$$

which is clearly true both for $r \leq R$ and for $r > R$. \square

Proof of Proposition 5.2.1. Write $\Lambda = \Lambda_h$. The density of zeros with respect to hyperbolic measure is $L/4\pi$ (recall the difference by a factor of 4 in normalization of hyperbolic measure). Hence we get

$$\mathbf{E}N(h) = \int_{\Lambda} \rho_1(z) dm(z) = \frac{L}{4\pi} h.$$

Let $Q(z, w) = \rho_2(z, w)/(\rho_1(z)\rho_1(w))$. Then by formula (5.1.3) we have

$$Q(0, w) - 1 \leq C(1 - |w|^2)^L.$$

we denote the right hand side by $\psi(0, w)$ and extend ψ to \mathbb{D}^2 so that it only depends on hyperbolic distance.

$$\begin{aligned} \mathbf{E}(N(h)(N(h) - 1)) - (\mathbf{E}N(h))^2 &= \int_{\Lambda} \int_{\Lambda} (\rho_2(z, w) - \rho_1(z)\rho_1(w)) dm(w) dm(z) \\ &= \int_{\Lambda} \int_{\Lambda} (Q(z, w) - 1) \rho_1(w) dm(w) \rho_1(z) dm(z) \\ &\leq \int_{\Lambda} \int_{\Lambda} \psi(z, w) \rho_1(w) dm(w) \rho_1(z) dm(z) \end{aligned}$$

Let $B_R(0)$ be a ball with hyperbolic area $h = 4\pi R^2/(1 - R^2)$. Note that $\rho_1(w) dm(w)$ is constant times the hyperbolic area element, so we may use Lemma 5.2.2 to bound

the inner integral by

$$\begin{aligned} \int_{B_R(0)} \psi(0, w) \rho_1(w) dm(w) &= c \int_0^R (1-r^2)^L (1-r^2)^{-2} r dr \\ &= \frac{c}{2} \int_S^1 s^{L-2} ds \end{aligned}$$

with $S = 1 - R^2$. Thus we get

$$(5.2.1) \quad \text{Var } N(h) = \mathbf{E}N(h) + \mathbf{E}(N(h)(N(h) - 1)) - (\mathbf{E}N(h))^2 \leq \frac{hL}{4\pi} + \frac{chL}{8\pi} \int_S^1 s^{L-2} ds.$$

For $L > 1$ this is integrable, so $\text{Var } N(h) \leq O(h)$. For $L < 1$ we can bound the right hand side of (5.2.1) by $O(hS^{L-1}) = O(h^{2-L})$. Thus in both cases, as well as when $L = 1$ (see Corollary 5.1.8(iii)), we have

$$\text{Var } N(h) \leq c(\mathbf{E}N(h))^{2-\beta}$$

with $\beta = L \wedge 1 > 0$. For $\eta > 1/\beta$, we find that

$$Y_k = \frac{N(k^\eta) - \mathbf{E}N(k^\eta)}{\mathbf{E}N(k^\eta)}$$

satisfies $\mathbf{E}Y_k^2 = O(k^{-\eta\beta})$, whence $\mathbf{E}\sum_k Y_k^2 < \infty$, so $Y_k \rightarrow 0$ a.s. Now, given h satisfying $(k-1)^\eta < h \leq k^\eta$ monotonicity implies that

$$(5.2.2) \quad \frac{N(k^\eta)}{\mathbf{E}N(k^\eta)} \frac{\mathbf{E}N(k^\eta)}{\mathbf{E}N((k-1)^\eta)} > \frac{N(h)}{\mathbf{E}N(h)} > \frac{N((k-1)^\eta)}{\mathbf{E}N((k-1)^\eta)} \frac{\mathbf{E}N((k-1)^\eta)}{\mathbf{E}N(k^\eta)}.$$

Since the left and right hand sides of equation 5.2.2 converge to 1 a.s., we deduce that $\frac{N(h)}{\mathbf{E}N(h)}$ converges to 1 a.s. as well, and the result follows. \square

5.3. Reconstruction from the zero set

Next we show that with probability one we can recover $|\mathbf{f}_L|$ from its zero set, Z_L . The following theorem gives a recipe for reconstructing $|\mathbf{f}_L(0)|$, almost surely. Translation invariance then implies that $|\mathbf{f}_L|$ can be reconstructed from Z_L on a dense subset of \mathbb{C} a.s., and hence by continuity Z_L determines $|\mathbf{f}_L|$ with probability one. Note that this result holds for arbitrary $L > 0$, and does not depend on the determinantal formula which only holds for $L = 1$.

THEOREM 5.3.1. **(i)** *Let $L > 0$. Consider the random function \mathbf{f}_L , and order its zero set Z_L in increasing absolute value, as $\{z_k\}_{k=1}^\infty$. Then*

$$(5.3.1) \quad |\mathbf{f}_L(0)| = c_L \prod_{k=1}^\infty e^{L/(2k)} |z_k| \quad \text{a.s.}$$

where $c_L = e^{(L-\gamma-\gamma L)/2} L^{-L/2}$ and $\gamma = \lim_n \left(\sum_{k=1}^n \frac{1}{k} - \log n \right)$ is Euler's constant.

(ii) *More generally, given $\zeta \in \mathbb{D}$, let $\{\zeta_k\}_{k=1}^\infty$ be Z_L , ordered in increasing hyperbolic distance from ζ . Then*

$$(5.3.2) \quad |\mathbf{f}_L(\zeta)| = c_L (1 - |\zeta|^2)^{-L/2} \prod_{k=1}^\infty e^{L/(2k)} \left| \frac{\zeta_k - \zeta}{1 - \bar{\zeta}\zeta_k} \right|.$$

Thus the analytic function $\mathbf{f}_L(z)$ is determined by its zero set, up to multiplication by a constant of modulus 1.

The main step in the proof of Theorem 5.3.1 is the following.

PROPOSITION 5.3.2. Let $c'_L = e^{L/2 - \gamma/2}$. We have

$$|\mathbf{f}_L(0)| = c'_L \lim_{r \rightarrow 1} (1 - r^2)^{-L/2} \prod_{\substack{z \in Z_L \\ |z| < r}} |z| \quad a.s.$$

We first need a simple lemma.

LEMMA 5.3.3. If X, Y are jointly complex Gaussian with variance 1, then for some absolute constant c we have

$$(5.3.3) \quad \left| \text{Cov}(\log|X|, \log|Y|) \right| \leq c |\mathbf{E}(X\bar{Y})|.$$

PROOF. Since $|\mathbf{E}(X\bar{Y})| \leq 1$, lemma 3.5.2, implies that:

$$(5.3.4) \quad \left| \text{Cov}(\log|X|, \log|Y|) \right| \leq |\mathbf{E}(X\bar{Y})| \sum_{m=1}^{\infty} \frac{1}{4m^2} \leq c |\mathbf{E}(X\bar{Y})|$$

□

Proof of Proposition 5.3.2. Assume that $\mathbf{f} = \mathbf{f}_L$ has no zeros at 0 or on the circle of radius r . Then Jensen's formula (Ahlfors (1), Section 5.3.1) gives

$$\log|\mathbf{f}(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log|\mathbf{f}(re^{i\alpha})| d\alpha + \sum_{z \in Z, |z| < r} \log \frac{|z|}{r},$$

where $Z = Z_L$. Let $|\mathbf{f}(re^{i\alpha})|^2 = \sigma_r^2 Y$, where

$$\sigma_r^2 = \text{Var}\mathbf{f}(re^{i\alpha}) = (1 - r^2)^{-L}$$

and Y is an exponential random variable with mean 1. We have

$$\mathbf{E} \log|\mathbf{f}(re^{i\alpha})| = \frac{\log \sigma_r^2 + \mathbf{E} \log Y}{2} = \frac{-L \log(1 - r^2) - \gamma}{2},$$

where the second equality follows from the integral formula for Euler's constant

$$\gamma = - \int_0^{\infty} e^{-x} \log x dx.$$

Introduce the notation

$$g_r(\alpha) = \log|\mathbf{f}(e^{i\alpha} r)| + \frac{L \log(1 - r^2) + \gamma}{2}$$

so that the distribution of $g_r(\alpha)$ does not depend on r and α , and $\mathbf{E}g_r(\alpha) = 0$. Let

$$L_r = \frac{1}{2\pi} \int_0^{2\pi} g_r(\alpha) d\alpha.$$

We first prove that $L_r \rightarrow 0$ a.s. over a suitable deterministic sequence $r_n \uparrow 1$. We compute:

$$\text{Var } L_r = \mathbf{E} \left(\frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} g_r(\alpha) g_r(\beta) d\beta d\alpha \right).$$

Since the above is absolutely integrable, we can exchange integral and expected value to get

$$\text{Var } L_r = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} \mathbf{E}(g_r(\alpha)g_r(\beta)) d\beta d\alpha = \frac{1}{2\pi} \int_0^{2\pi} \mathbf{E}(g_r(\alpha)g_r(0)) d\alpha.$$

where the second equality follows from rotational invariance. By Lemma 5.3.3, we have

$$\mathbf{E}(g_r(\alpha)g_r(0)) \leq c \frac{|\mathbf{E}(\mathbf{f}(re^{i\alpha})\overline{\mathbf{f}(r)})|}{\text{Var } \mathbf{f}(r)} = c \left| \frac{1-r^2}{1-r^2e^{i\alpha}} \right|^L.$$

Let $\epsilon = 1 - r^2 < 1/2$. Then for $\alpha \in [0, \pi]$ we can bound

$$|1 - r^2e^{i\alpha}| \geq \begin{cases} \epsilon & |\alpha| \leq \epsilon \\ 2r \sin \frac{\alpha}{2} \geq \frac{\alpha}{2} & \epsilon < \alpha < \pi/2 \\ 1 & \pi/2 \leq \alpha \leq \pi, \end{cases}$$

which gives

$$\frac{1}{c2\epsilon^L} \text{Var } L_r \leq \int_0^\pi \frac{d\alpha}{|1 - r^2e^{i\alpha}|^L} \leq \epsilon^{1-L} + \frac{1}{2} \int_\epsilon^{\pi/2} \frac{d\alpha}{\alpha^L} + \pi/2 \leq \begin{cases} c' & L < 1 \\ c' |\log \epsilon| & L = 1 \\ c'e^{1-L} & L > 1. \end{cases}$$

By Chebyshev's inequality and the Borel-Cantelli lemma, this shows that, as $r \rightarrow 1$ over the sequence $r_n = 1 - n^{-(1\nu(1/L)+\delta)}$, we have a.s. $L_{r_n} \rightarrow 0$ and

$$\sum_{z \in \mathcal{Z}, |z| < r} \log \frac{|z|}{r} - \frac{L \log(1-r^2) + \gamma}{2} \rightarrow \log |\mathbf{f}(0)|,$$

or, exponentiating:

$$(5.3.5) \quad e^{-\gamma/2} (1-r^2)^{-L/2} \prod_{\substack{z \in \mathcal{Z}_L \\ |z| < r}} \frac{|z|}{r} \rightarrow |\mathbf{f}(0)|.$$

Since the product is monotone decreasing and the ratio $(1-r_n^2)/(1-r_{n+1}^2)$ converges to 1, it follows that the limit is the same over every sequence $r_n \rightarrow 1$ a.s.

Finally, by the law of large numbers (Proposition 5.2.1), the number of zeros N_r in the ball of Euclidean radius r satisfies

$$(5.3.6) \quad N_r = \frac{r^2 L}{1-r^2} (1 + o(1)) = \frac{L + o(1)}{1-r^2} \quad \text{a.s.},$$

whence

$$r^{N_r} = \exp(N_r \log r) = e^{-L/2 + o(1)} \quad \text{a.s.}$$

Multiplying this with (5.3.5) yields the claim. \square

Proof of Theorem 5.3.1. (i) By the law of large numbers for N_r (see also (5.3.6)),

$$(5.3.7) \quad \sum_{|z_k| \leq r} \frac{1}{k} = \gamma + \log N_r + o(1) = \gamma + \log L - \log(1-r^2) + o(1).$$

Multiplying by $L/2$ and exponentiating, we get that

$$(5.3.8) \quad \prod_{|z_k| \leq r} e^{L/(2k)} = e^{\gamma L/2} L^{L/2} (1-r^2)^{-L/2} (1 + o(1)).$$

In conjunction with Proposition 5.3.2, this yields (5.3.1).

(ii) Let $\mathbf{f} = \mathbf{f}_L$ and

$$T(z) = \frac{z - \zeta}{1 - \bar{\zeta}z}.$$

By (5.4.5), \mathbf{f} has the same law as

$$(5.3.9) \quad \tilde{\mathbf{f}} = (T')^{L/2} \cdot (\mathbf{f} \circ T).$$

Now $T'(\zeta) = (1 - |\zeta|^2)^{-1}$. Therefore

$$|\tilde{\mathbf{f}}(\zeta)| = (1 - |\zeta|^2)^{-L/2} |\mathbf{f}(0)| = c_L \prod_{k=1}^{\infty} e^{L/(2k)} |z_k| \quad \text{a.s.},$$

where $\{z_k\}$ are the zeros of \mathbf{f} in increasing modulus. If $T(\zeta_k) = z_k$ then $\{\zeta_k\}$ are the zeros of $\tilde{\mathbf{f}}$ in increasing hyperbolic distance from ζ . We conclude that

$$|\tilde{\mathbf{f}}(\zeta)| = c_L (1 - |\zeta|^2)^{-L/2} \prod_{k=1}^{\infty} e^{L/(2k)} |T(\zeta_k)| \quad \text{a.s.}$$

□

5.3.1. Reconstruction under conditioning. For our study of the dynamics of zeros in Chapter 8, section 8.1.1, we will need a reconstruction formula for $|\mathbf{f}'_L(0)|$ when we condition that $0 \in Z_L$. The method is to show that if we condition \mathbf{f}_L so that $0 \in Z_L$, then the distribution of $\mathbf{f}_L(z)/z$ is mutually absolutely continuous to the unconditional distribution of \mathbf{f}_L . It is important to note that the distribution of \mathbf{f}_L given that its value is zero at 0 is *different* from the conditional distribution of \mathbf{f}_L given that its zero set has a point at 0. In particular, in the second case the conditional distribution of the coefficient a_1 is not Gaussian. The reason for this is that the two ways of conditioning are defined by the limits as $\epsilon \rightarrow 0$ of two different conditional distributions. In the first case, we condition on $|\mathbf{f}_{\mathbb{D}}(0)| < \epsilon$. In the second, we condition on $\mathbf{f}_{\mathbb{D}}$ having a zero in the disk $B_\epsilon(0)$ of radius ϵ about 0; the latter conditioning affects the distribution of a_1 .

We wish to approximate \mathbf{f}_L by its linearization near the origin. The first part of the following lemma, valid for general GAFs, is the same as Lemma 2.4.2 but the second part is a slight extension of it.

LEMMA 5.3.4. *Let $\mathbf{f}(z) = a_0 + a_1z + \dots$ be a Gaussian analytic function. Assume that a_0 is nonconstant. Let A_ϵ denote the event that the number of zeros of $\mathbf{f}(z)$ in the disk B_ϵ about 0, differs from the number of zeros of $\mathbf{h}(z) = a_0 + a_1z$ in B_ϵ .*

(i) *For all $\delta > 0$ there is $c > 0$ (depending continuously on the mean and covariance functions of \mathbf{f}) so that for all $\epsilon > 0$ we have*

$$\mathbf{P}(A_\epsilon) \leq c\epsilon^{3-2\delta}.$$

(ii) $\mathbf{P}(A_\epsilon \mid a_1, a_2, \dots) \leq C\epsilon^3$, where C may depend on (a_1, a_2, \dots) but is finite almost surely.

PROOF. The first statement is precisely Lemma 2.4.2. To prove the second we refer the reader back to the notations used in the proof of that lemma.

The argument used to bound $\mathbf{P}(F)$ in Lemma 2.4.2 also yields that

$$\mathbf{P}\left(\min_{z \in \partial B_\epsilon} |\mathbf{h}(z)| < 2|a_2|\epsilon^2 \mid \{a_j\}_{j \geq 1}\right) \leq c_7\epsilon^3.$$

An application of Rouché's Theorem concludes the proof. □

LEMMA 5.3.5. Denote by Ω_ϵ the event that the power series \mathbf{f}_L defined in (5.1.1) has a zero in $B_\epsilon(0)$. As $\epsilon \rightarrow 0$, the conditional distribution of the coefficients a_1, a_2, a_3, \dots given Ω_ϵ , converges to a product law where a_1 is rotationally symmetric, $|a_1|$ has density $2r^3 e^{-r^2}$, and a_2, a_3, \dots are standard complex Gaussian.

PROOF. Let a_0, a_1 be i.i.d. standard complex normal random variables, and $L > 0$. Consider first the limiting distribution, as $\epsilon \rightarrow 0$, of a_1 given that the equation $a_0 + a_1 \sqrt{L}z = 0$ has a root Z in $B_\epsilon(0)$. The limiting distribution must be rotationally symmetric, so it suffices to compute its radial part. If $S = |a_0|^2$ and $T = |a_1|^2$, set $U = L|Z|^2 = S/T$. The joint density of (S, T) is e^{-s-t} , so the joint density of (U, T) is e^{-ut-t} . Thus as $\epsilon \rightarrow 0$, the conditional density of T given $U < L\epsilon^2$ converges to the conditional density given $U = 0$, that is te^{-t} . This means that the conditional distribution of a_1 is not normal, rather, its radial part has density $2r^3 e^{-r^2}$.

We can now prove the lemma. The conditional density of the coefficients a_1, a_2, \dots given Ω_ϵ , with respect to their original product law, is given by the ratio $\mathbf{P}(\Omega_\epsilon | a_1, a_2, \dots) / \mathbf{P}(\Omega_\epsilon)$. By Lemma 5.3.4, the limit of this ratio is not affected if we replace \mathbf{f}_L by its linearization $a_0 + a_1 \sqrt{L}z$. This yields the statement of the lemma. \square

Kakutani's absolute continuity criterion (see Williams (87), Theorem 14.17) applied to the coefficients gives the following

LEMMA 5.3.6. The distributions of the random functions $\mathbf{f}_L(z)$ and $(\mathbf{f}_L(z) - a_0)/z$ are mutually absolutely continuous.

By Lemma 5.3.5, conditioning on $0 \in Z_L$ amounts to setting $a_0 = 0$ and changing the distribution of a_1 in an absolutely continuous manner. Thus, by Lemma 5.3.6, given $0 \in Z_L$ the distribution of the random function $\mathbf{g}(z) = \mathbf{f}_L(z)/z$ is absolutely continuous with respect to the distribution of the unconditioned $\mathbf{f}_L(z)$. Hence we may apply Theorem 5.3.1 to $\mathbf{g}(z)$ and get that given $0 \in Z_L$, if we order the *other* zeros of \mathbf{f}_L in increasing absolute value as $\{z_k\}_{k=1}^\infty$, then

$$(5.3.10) \quad |\mathbf{f}'_{\mathbb{D},L}(0)| = |\mathbf{g}(0)| = c_L \prod_{k=1}^{\infty} e^{L/(2k)} |z_k| \quad \text{a.s.}$$

5.4. Notes

5.4.1. Extensions of the determinantal formula. It is natural to ask if the results in this chapter can be extended to random functions on more general domains. The answer is affirmative. We begin by explaining how the Szegő and Bergman kernels are defined for general domains and then describe the the random analytic function which replaces the i.i.d. power series of (5.1.1). Let D be a bounded planar domain with a C^∞ smooth boundary (the regularity assumption can be weakened). Consider the set of complex analytic functions in D which extend continuously to the boundary ∂D . The classical Hardy space $H^2(D)$ is given by the L^2 -closure of this set with respect to length measure on ∂D . Every element of $H^2(D)$ can be identified with a unique analytic function in D via the Cauchy integral (see Bell (4), Section 6).

Consider an orthonormal basis $\{\psi_n\}_{n \geq 0}$ for $H^2(D)$; e.g. in the unit disk, take $\psi_n(z) = \frac{z^n}{\sqrt{2\pi}}$ for $n \geq 0$. The Szegő kernel S_D is given by the expression

$$(5.4.1) \quad S_D(z, w) = \sum_{n=0}^{\infty} \psi_n(z) \overline{\psi_n(w)}$$

is the Szegő kernel in D . It does not depend on the choice of orthonormal basis and is positive definite (i.e. for points $z_j \in D$ the matrix $(S_D(z_j, z_k))_{j,k}$ is positive definite). Now let $T: \Lambda \rightarrow D$

be a conformal homeomorphism between two bounded domains with C^∞ smooth boundary. The derivative T' of the conformal map has a well-defined square root, see (4) p. 43. If $\{\psi_n\}_{n \geq 0}$ is an orthonormal basis for $H^2(D)$, then $\{\sqrt{T'} \cdot (\psi_n \circ T)\}_{n \geq 0}$ forms an orthonormal basis for $H^2(\Lambda)$. Hence, the Szegő kernels satisfy the transformation rule

$$(5.4.2) \quad S_\Lambda(z, w) = T'(z)^{1/2} \overline{T'(w)^{1/2}} S_D(T(z), T(w)).$$

When D is a simply connected domain, it follows from (5.4.2) that S_D does not vanish in the interior of D , so for arbitrary $\alpha > 0$ powers S_D^α are defined.

To define the Bergman kernel, let $\{\eta_n\}_{n \geq 0}$ be an orthonormal basis of the subspace of complex analytic functions in $L^2(D)$ with respect to Lebesgue area measure. The Bergman kernel is defined to be

$$\mathbb{K}_D(z, w) = \sum_{n=0}^{\infty} \eta_n(z) \overline{\eta_n(w)}$$

and is independent of the basis chosen, see Nehari (60), formula (132).

Now use i.i.d. complex Gaussians $\{a_n\}_{n \geq 0}$ to define the random analytic function

$$(5.4.3) \quad \mathbf{f}_{D,1}(z) = \sqrt{2\pi} \sum_{n=0}^{\infty} a_n \psi_n(z).$$

(cf. (6) in Shiffman and Zelditch (74)). The factor of $\sqrt{2\pi}$ is included just to simplify formulas in the case where D is the unit disk. The covariance function of $\mathbf{f}_{D,1}$ is given by $2\pi S_D(z, w)$, and one can prove the following corollary to Theorem 5.1.1

COROLLARY 5.4.1. *Let D be a simply connected bounded planar domain, with a C^∞ smooth boundary. The joint intensity of zeros for the Gaussian analytic function \mathbf{f}_D is given by the determinant of the Bergman kernel*

$$\rho_n(z_1, \dots, z_n) = \det[\mathbb{K}_D(z_i, z_j)]_{i,j}.$$

Note that for simply connected domains as in the corollary, the Bergman and Szegő kernels satisfy $\mathbb{K}_D(z, w) = 4\pi S_D(z, w)^2$, see Bell (4), Theorem 23.1.

5.4.2. The Szegő random functions. Recall the one-parameter family of Gaussian analytic functions \mathbf{f}_L defined in (5.1.1), whose zero sets are invariant in distribution under conformal maps preserving the unit disk (Möbius transformations). Using the binomial expansion we compute the covariance structure

$$(5.4.4) \quad \begin{aligned} \mathbf{E}(\mathbf{f}_L(z) \overline{\mathbf{f}_L(w)}) &= \sum_{n=0}^{\infty} \binom{-L}{n} z^n \overline{w}^n \\ &= \sum_{n=0}^{\infty} \binom{-L}{n} (-z\overline{w})^n = (1 - z\overline{w})^{-L} = [2\pi S_{\mathbb{D}}(z, w)]^L. \end{aligned}$$

The random function $\mathbf{f}_{D,1}$ defined in 5.4.3 provides a generalization of $\mathbf{f}_{\mathbb{D},1}$ to more general domains. The following proposition explains that appropriate generalizations for other values of L also exist.

PROPOSITION 5.4.2. *Let D be a bounded planar domain with a C^∞ boundary and let $L > 0$. Suppose that either (i) D is simply connected or (ii) L is an integer. Then there is a mean zero Gaussian analytic function $\mathbf{f}_{D,L}$ in D with covariance structure*

$$\mathbf{E}(\mathbf{f}_{D,L}(z) \overline{\mathbf{f}_{D,L}(w)}) = [2\pi S_D(z, w)]^L \quad \text{for } z, w \in D.$$

The zero set $Z_{D,L}$ of $\mathbf{f}_{D,L}$ has a conformally invariant distribution: if Λ is another bounded domain with a smooth boundary, and $T: \Lambda \rightarrow D$ is a conformal homeomorphism, then $T(Z_{\Lambda,L})$ has the same distribution as $Z_{D,L}$. Moreover, the following two random functions have the same distribution:

$$(5.4.5) \quad \mathbf{f}_{\Lambda,L}(z) \stackrel{d}{=} T'(z)^{L/2} \cdot (\mathbf{f}_{D,L} \circ T)(z).$$

We call the Gaussian analytic function $\mathbf{f}_{D,L}$ described in the proposition the **Szegő random function with parameter L** in D .

PROOF. Case (i): D is simply connected. Let $\Psi : D \rightarrow \mathbb{D}$ be a conformal map onto \mathbb{D} , and let $\{a_n\}$ be i.i.d. standard complex Gaussians. We claim that

$$(5.4.6) \quad \mathbf{f}(z) = \Psi'(z)^{L/2} \sum_{n=0}^{\infty} \binom{-L}{n}^{\frac{1}{2}} a_n \Psi(z)^n$$

is a suitable candidate for $\mathbf{f}_{D,L}$. Indeed, repeating the calculation in (5.4.4), we find that

$$\begin{aligned} \mathbf{E}\left(\mathbf{f}(z)\overline{\mathbf{f}(w)}\right) &= [\Psi'(z)\overline{\Psi'(w)}]^{L/2} (1 - \Psi(z)\overline{\Psi(w)})^{-L} \\ &= [\Psi'(z)\overline{\Psi'(w)}]^{L/2} \cdot [2\pi S_{\mathbb{D}}(\Psi(z), \Psi(w))]^L. \end{aligned}$$

The last expression equals $[2\pi S_D(z, w)]^L$ by the transformation formula (5.4.2). Thus we may define $\mathbf{f}_{D,L}$ by the right hand side of (5.4.6). If $T : \Lambda \rightarrow D$ is a conformal homeomorphism, then $\Psi \circ T$ is a conformal map from Λ to \mathbb{D} , so (5.4.6) and the chain rule give the equality in law (5.4.5). Since T' does not have zeros in Λ , multiplying $\mathbf{f}_{D,L} \circ T$ by a power of T' does not change its zero set in Λ , and it follows that $T(Z_{\Lambda,L})$ and $Z_{D,L}$ have the same distribution.

Case (ii): L is an integer. Let $\{\psi_n\}_{n \geq 0}$ be an orthonormal basis for $H^2(D)$. Use i.i.d. complex Gaussians $\{a_{n_1, \dots, n_L} : n_1, \dots, n_L \geq 0\}$ to define the random analytic function

$$(5.4.7) \quad \mathbf{f}_{D,L}(z) = (2\pi)^{L/2} \sum_{n_1, \dots, n_L \geq 0} a_{n_1, \dots, n_L} \psi_{n_1}(z) \cdots \psi_{n_L}(z);$$

see Sodin (78) for convergence. A direct calculation shows that $\mathbf{f}_{D,L}$, thus defined, satisfies

$$\mathbf{E}\left(\mathbf{f}_{D,L}(z)\overline{\mathbf{f}_{D,L}(w)}\right) = (2\pi)^L \sum_{n_1, \dots, n_L \geq 0} \psi_{n_1}(z)\overline{\psi_{n_1}(w)} \cdots \psi_{n_L}(z)\overline{\psi_{n_L}(w)} = [2\pi S_D(z, w)]^L.$$

The transformation formula (5.4.2) implies that the two sides of (5.4.5) have the same covariance structure, $[2\pi S_{\Lambda}(z, w)]^L$. This establishes (5.4.5) and completes the proof of the proposition. \square

5.4.3. The analytic extension of white noise. Here we show that up to the constant term, the power series \mathbf{f}_1 has the same distribution as the analytic extension of white noise on the unit circle. Let $B(\cdot)$ be a standard real Brownian motion, and let

$$u(z) = \int_0^{2\pi} \text{Poi}(z, e^{it}) dB(t).$$

The integral with respect to B can be interpreted either as a stochastic integral, or as a Riemann-Stieltjes integral, using integration by parts and the smoothness of the Poisson kernel. Recall that the Poisson kernel

$$\text{Poi}(z, w) = \frac{1}{2\pi} \text{Re} \left(\frac{1+z\bar{w}}{1-z\bar{w}} \right) = \frac{1}{2\pi} \text{Re} \left(\frac{2}{1-z\bar{w}} - 1 \right) = 2 \text{Re} S_{\mathbb{D}}(z, w) - \frac{1}{2\pi}$$

has the kernel property

$$\text{Poi}(z, w) = \int_0^{2\pi} \text{Poi}(z, e^{it}) \text{Poi}(e^{it}, w) dt.$$

(This follows from the Poisson formula for harmonic functions, see Ahlfors (1), Section 6.3). The white noise dB has the property that if f_1, f_2 are smooth functions on an interval and $\mathbf{f}_i^{\#} = \int f_i(t) dB(t)$ then $\mathbf{E}[\mathbf{f}_1^{\#} \mathbf{f}_2^{\#}] = \int f_1(t) f_2(t) dt$. By this and the kernel property we get $\mathbf{E}\left(u(z)u(w)\right) = \text{Poi}(z, w)$. Therefore if b is a standard real Gaussian independent of $B(\cdot)$, then

$$(5.4.8) \quad \tilde{u}(z) = \sqrt{\frac{\pi}{2}} u(z) + \frac{b}{2}$$

has covariance structure $\mathbf{E}[\tilde{u}(z)\tilde{u}(w)] = \pi \operatorname{Re} S_{\mathbb{D}}(z, w)$. Now if v, v' are mean 0 complex Gaussians, then $\operatorname{Re} \mathbf{E} v \bar{v}' = 2\mathbf{E}(\operatorname{Re} v \operatorname{Re} v')$; thus

$$(5.4.9) \quad \mathbf{E}(\mathbf{f}_{\mathbb{D}}(z)\overline{\mathbf{f}_{\mathbb{D}}(w)}) = \sum_{n=0}^{\infty} (z\bar{w})^n = (1 - z\bar{w})^{-1}.$$

implies that \tilde{u} has the same distribution as Ref_1 .

Remark. Similarly, since $\mathbf{f}_{\mathbb{D},2}$ is the derivative of $\sum_{m=1}^{\infty} a_m z^m / \sqrt{m}$, the zero set $Z_{\mathbb{D},2}$ can be interpreted as the set of saddle points of the random harmonic function

$$u(z) = \sum_{m=1}^{\infty} \operatorname{Re}(a_m z^m) / \sqrt{m}$$

in \mathbb{D} . More generally, in any domain D , the zero set $Z_{D,2}$ can be interpreted as the set of saddle points of the Gaussian free field (with free boundary conditions) restricted to harmonic functions.

5.5. Hints and solutions

Exercise 5.1.2 Computing

$$\begin{aligned} \mathbf{E}f_L(z)\overline{\mathbf{f}_{\mathbb{D},L}(w)} &= \frac{1}{(1 - z\bar{w})^L} \\ \mathbf{E}f'_{\mathbb{D},L}(z)\overline{\mathbf{f}_{\mathbb{D},L}(w)} &= \frac{L\bar{w}}{(1 - z\bar{w})^{L+1}} \\ \mathbf{E}f'_{\mathbb{D},L}(z)\overline{\mathbf{f}'_{\mathbb{D},L}(w)} &= \frac{L^2 z\bar{w} + L}{(1 - z\bar{w})^{L+2}} \end{aligned}$$

and applying (3.4.2) we see that

$$(5.5.1) \quad \rho_2(0, r) = \frac{\operatorname{per}(C - BA^{-1}B^*)}{\det(\pi A)}$$

where

$$(5.5.2) \quad A = \begin{pmatrix} 1 & 1 \\ 1 & s^{-L} \end{pmatrix}$$

$$(5.5.3) \quad B = \begin{pmatrix} 0 & Lr \\ 0 & Lrs^{-(L+1)} \end{pmatrix}$$

$$(5.5.4) \quad C = \begin{pmatrix} L & L \\ L & (L^2 r^2 + L)s^{-(L+2)} \end{pmatrix}.$$

Also, by (2.4.8) we have that

$$\rho_1(z) = \frac{L}{\pi} \frac{1}{(1 - z\bar{z})^2}$$

so $\rho_1(0) = L\pi^{-1}$ and $\rho_1(r) = L(\pi s^2)^{-1}$.

A Determinantal Zoo

In chapter 4 we saw the general theory of determinantal point processes and in chapter 5 we saw one prime example of a determinantal process that was also the zero set of a Gaussian analytic function. In this chapter we delve more deeply into examples. Of particular interest to us is the example of matrix-analytic functions, introduced in section 4.3.11, to be proved in section 6.7. This example lies at the intersection of determinantal processes and zeros of random analytic functions and is a natural generalization of the i.i.d. power series. However the proof we give is quite different from the one in chapter 5 and makes use of random matrix ensembles of the earlier sections of this chapter. In particular, it gives a new proof of Theorem 5.1.1.

How does one check if a given point process is determinantal or not? If it happens that $\rho_2(x, y) > \rho_1(x)\rho_1(y)$ for even a single pair of points $x, y \in \Lambda$, then the process is definitely not determinantal (caution: this applies only if we restrict ourselves to Hermitian kernels, as we do). One can often calculate the first two joint intensities, at least numerically, and hence, this is a valuable check that can rule out false guesses. In chapter 5, this criterion showed us that zero sets of many Gaussian analytic functions are not determinantal (see Figure 1). But when it comes to checking that a point process *is* indeed determinantal, there is no single method, nor is it a trivial exercise (usually). All the examples considered in this chapter were stated in section 4.3, but not all examples listed there will be given proofs. In each section of this chapter, we use the notations of the corresponding subsection in chapter 4, section 4.3 without further comment.

6.1. Uniform spanning trees

We outline the proof of Burton-Pemantle theorem as given in BLPS (5).

Sketch of proof: In proving (4.3.8), we assume that $\{e_1, \dots, e_k\}$ does not contain any cycle. For, if it did, the left hand side is obviously zero, by definition of tree, and the right hand side vanishes because the matrix under consideration is a Gram matrix with entries (I^{e_i}, I^{e_j}) and because for any cycle e_1, \dots, e_r , the sum $\epsilon_1 I^{e_1} + \dots + \epsilon_r I^{e_r}$ is zero where the $\epsilon_i = \pm 1$ are orientations chosen so that $\epsilon_1 e_1, \dots, \epsilon_r e_r$ is a directed cycle.

Again, because the right hand side of (4.3.8) is a Gram determinant, its value is the squared volume of the parallelepiped spanned by its determining vectors. Thus

$$(6.1.1) \quad \det(\mathbb{K}(e_i, e_j))_{1 \leq i, j \leq k} = \prod_{i=1}^k \left\| P_{Z_i}^\perp I^{e_i} \right\|^2,$$

where Z_i is the linear span of $I^{e_1}, \dots, I^{e_{i-1}}$ and $P_{Z_i}^\perp$ is the projection onto Z_i^\perp . The left hand side of (4.3.8) can also be written as a product

$$\begin{aligned} \mathbf{P}[e_1, \dots, e_k \in T] &= \prod_{i=1}^k \mathbf{P}[e_i \in T \mid e_j \in T \text{ for } j < i] \\ &= \prod_{i=1}^k \mathbf{P}[e_i \in T_i] \end{aligned}$$

where T_i is the uniform spanning tree on a new graph got by identifying every pair of vertices connected by e_1, \dots, e_{i-1} and denoted $G/\{e_1, \dots, e_{i-1}\}$. Comparison with (6.1.1) shows that to establish (4.3.8), it suffices to prove

$$(6.1.2) \quad \mathbf{P}[e_i \in T_i] = \left\| P_{Z_i}^\perp I^{e_i} \right\|^2.$$

This leads us to examine the effect of contracting edges in G , in terms of the inner product space \mathbf{H} . Fix a finite set F of edges, and let $\mathbf{h}\star$ denote the subspace of \mathbf{H} spanned by the stars of G/F , and let $\mathbf{h}\diamond$ denote the space of cycles (including loops) of G/F . It is easy to see that $\mathbf{h}\diamond = \diamond + \langle \chi^F \rangle$, where $\langle \chi^F \rangle$ is the linear span of $\{\chi^f : f \in F\}$. Consequently, $\mathbf{h}\diamond \supset \diamond$ and $\mathbf{h}\star \subset \star$. Let $Z := P_\star \langle \chi^F \rangle$, which is the linear span of $\{I^f : f \in F\}$. Since $\mathbf{h}\star \subset \star$ and $\mathbf{h}\star$ is the orthogonal complement of $\mathbf{h}\diamond$, we have $P_\star \mathbf{h}\diamond = \star \cap \mathbf{h}\diamond$. Consequently,

$$\star \cap \mathbf{h}\diamond = P_\star \mathbf{h}\diamond = P_\star \diamond + P_\star \langle \chi^F \rangle = Z,$$

and we obtain the orthogonal decomposition

$$\mathbf{H} = \mathbf{h}\star \oplus Z \oplus \diamond,$$

where $\star = \mathbf{h}\star \oplus Z$ and $\mathbf{h}\diamond = \diamond \oplus Z$.

Let e be an edge that does not form a cycle together with edges in F . Set $\mathbf{h}I^e := P_{\mathbf{h}\star} \chi^e$; this is the analogue of I^e in the network G/F . The above decomposition tells us that

$$\mathbf{h}I^e = P_{\mathbf{h}\star} \chi^e = P_Z^\perp P_\star \chi^e = P_Z^\perp I^e.$$

From (6.1.2), all that is left to prove is that for any graph G ,

$$\mathbf{P}[e \in T] = \left\| I^e \right\|^2.$$

(Then we apply it to $G/\{e_1, \dots, e_{i-1}\}$ for each i). This is exactly (4.3.8) with $k = 1$ and was proved by Kirchoff (48) in 1847. We omit the proof and direct the interested reader to Thomassen (85) for a short combinatorial argument (see the notes).

6.2. Circular unitary ensemble

We give the proof of Theorem 4.3.9 in three steps. In the first, we write the Haar measure on $\mathcal{U}(n)$ in a workable explicit form. In the second step, we represent a unitary matrix in terms of its eigenvalues and auxiliary variables. Finally, in the third step, we compute the Jacobian determinant of this change of variables and integrate out the auxiliary variables to get the distribution of eigenvalues.

Haar measure on $\mathcal{U}(n)$: The Haar measure on $\mathcal{U}(n)$ is the unique Borel probability measure on $\mathcal{U}(n)$ that is invariant under left and right multiplication by unitary matrices. Our first task is to write this measure more explicitly. On $\mathcal{U}(n)$, we have the following n^2 smooth functions

$$u_{i,j}(U) = U_{i,j},$$

where $U_{i,j}$ is the (i,j) entry of the matrix U . Define the matrix-valued one form $\Omega(U) = U^* dU$. This just means that we define n^2 one-forms on $\mathcal{U}(n)$, by

$$\Omega_{i,j}(U) = \sum_{k=1}^n \bar{u}_{k,i}(U) du_{k,j}(U),$$

and put them together in a matrix. The matrix notation is for convenience. One property of Ω is that it is skew-Hermitian, that is $\Omega_{i,j} = -\bar{\Omega}_{j,i}$. Another property is its invariance, in the following sense.

For a fixed $W \in \mathcal{U}(n)$, consider the left-translation map $L_W : \mathcal{U}(n) \rightarrow \mathcal{U}(n)$ defined as $L_W(U) = WU$. The pullback of Ω under L_W is

$$\begin{aligned} L_W^* \Omega(U) &= \Omega(WU) \\ &= (WU)^* d(WU) \\ &= U^* W^* W dU \\ &= \Omega(U). \end{aligned}$$

Thus Ω is a left-invariant, Hermitian matrix-valued one-form on $\mathcal{U}(n)$ (called the “left Maurer-Cartan” form of $\mathcal{U}(n)$). Analogously, the form $U dU^*$ is right-invariant. Now we define the n^2 -form on $\mathcal{U}(n)$

$$\omega := \left(\bigwedge_i \Omega_{i,i} \right) \wedge \left(\bigwedge_{i < j} (\Omega_{i,j} \wedge \bar{\Omega}_{i,j}) \right).$$

To prevent ambiguity, let us fix the order in the first wedge product as $i = 1, 2, \dots, n$ and in the second as $(i, j) = (1, 2), (1, 3), \dots, (1, n), (2, 3), \dots, (n-1, n)$. This is not important, as a change in order may only change the overall sign. Now, ω is left-invariant, i.e., $L_W^* \omega = \omega$, since Ω has the same property. Also, the dimension of $\mathcal{U}(n)$ is n^2 and ω is clearly not zero. Thus for each U , up to scalar multiplication, $\omega(U)$ is the unique n^2 -form in the tangent space to $\mathcal{U}(n)$ at U . Therefore integration against ω is the unique (up to constant) left-invariant bounded linear functional on the space of continuous functions on $\mathcal{U}(n)$. It is important to note that ω is not zero! See remark 6.2.1 below. That is, for any continuous function $f : \mathcal{U}(n) \rightarrow \mathbb{R}$ and $W \in \mathcal{U}(n)$, we have $\omega(f \circ L_W) = \omega(f)$, where $L_W U = WU$. We may scale ω by a constant κ so that it is positive (in other words, if $f \geq 0$, then $\kappa \int f \omega \geq 0$) and so that $\kappa \int \omega = 1$. To see that it can be made positive, note that for any $S \subset \mathcal{U}(n)$, and any $W \in \mathcal{U}(n)$, we have $\int_S \omega = \int_{L_W^{-1}(S)} \omega$, whence ω is everywhere positive or everywhere negative.

Then we can define the left-Haar measure of $\mathcal{U}(n)$ as the measure μ such that for any continuous function $f : \mathcal{U}(n) \rightarrow \mathbb{R}$,

$$\int_{\mathcal{U}(n)} f(U) d\mu(U) = \kappa \int_{\mathcal{U}(n)} f(U) \omega(U).$$

It is a fact that the left-Haar measure is also right-invariant for any compact group (see the first paragraph of 4.3.6 and the reference therein). Hence, μ is a bi-invariant probability measure and ω is bi-invariant. In effect, we have constructed the Haar measure on $\mathcal{U}(n)$.

REMARK 6.2.1. Naturally, one must check that ω is not zero. By invariance, it suffices to check this at the identity, that is, $\omega(I) \neq 0$. Indeed, the exponential map $X \rightarrow e^X$, from the Lie algebra of skew Hermitian matrices $su(n)$ to the unitary group $\mathcal{U}(n)$, is a diffeomorphism of some neighbourhood of 0 in $su(n)$ onto some neighbourhood of the identity in $\mathcal{U}(n)$. On the Lie algebra side, $X_{i,i}, i \leq n$ and the

real and imaginary parts of $X_{i,j}$, $i < j$, form a co-ordinate system and hence $\tilde{\omega} = \wedge_i dX_{i,i} \wedge_{i < j} (dX_{i,j} \wedge dX_{j,i})$ is not zero. And $\omega(U)$ is easily seen to be nothing but the push forward of $\tilde{\omega}$ under the exponential map.

Choosing eigenvectors and eigenvalues: Now let U be a unitary matrix. By the spectral theorem for normal matrices, we may write

$$U = V\Delta V^*$$

where $\Delta = \text{diagonal}(\lambda_1, \dots, \lambda_n)$ is the diagonal matrix of eigenvalues of U , and V is a unitary matrix whose j^{th} column is an eigenvector of U with eigenvalue λ_j . To have a unique representation of U in terms of its eigenvalues and eigenvectors, we must impose extra constraints.

Eigenvalues are uniquely defined only as a set. To define Δ uniquely, we order the eigenvalues so that $\lambda_j = e^{i\alpha_j}$, with $0 < \alpha_1 < \alpha_2 \dots < \alpha_n < 2\pi$. (We may omit the lower-dimensional sub-manifold of matrices with two or more equal eigenvalues or having eigenvalue equal to 1). Once Δ is fixed, V is determined up to right multiplication by a diagonal unitary matrix $\Theta = \text{diagonal}(e^{i\theta_1}, \dots, e^{i\theta_n})$, where $\theta_j \in \mathbb{R}$. We impose the conditions $V_{i,i} \geq 0$, which then determine V uniquely. Then Δ and V are smooth functions of U , outside of the submanifold of matrices that we omitted.

Eigenvalue density: Write $U = V\Delta V^*$, where $\Delta = \Delta(U)$ and $V = V(U)$ are chosen as above. Then

$$\begin{aligned} dU &= V(d\Delta)V^* + (dV)\Delta V^* + V\Delta d(V^*) \\ &= V(d\Delta)V^* + (dV)\Delta V^* - V\Delta V^* dV V^*, \end{aligned}$$

where we used the fact that $dV^* = -V^*(dV)V^*$ (because $VV^* = I$). Thus

$$(6.2.1) \quad V^*U^*(dU)V = \Delta^* d\Delta + \Delta^* V^* dV \Delta - V^*(dV).$$

From the alternating property $dx \wedge dy = -dy \wedge dx$, we recall that if $dy_j = \sum_{k=1}^n a_{j,k} dx_k$, for $1 \leq j \leq n$, then

$$(6.2.2) \quad dy_1 \wedge dy_2 \dots \wedge dy_n = \det(a_{j,k})_{j,k \leq n} dx_1 \wedge dx_2 \dots \wedge dx_n.$$

We apply this to both sides of (6.2.1). For brevity, call the matrix-valued one-forms on the left and right of (6.2.1) as L and M , respectively. Then, by (6.2.2),

$$(6.2.3) \quad \left(\bigwedge_i L_{i,i} \right) \wedge \left(\bigwedge_{i < j} (L_{i,j} \wedge L_{j,i}) \right) = \omega(U)$$

because, for $V \in \mathcal{U}(n)$, the linear transformation $X \rightarrow V^*XV$ on the space of matrices is also unitary. Next, rewrite the right hand side of (6.2.1) as

$$M_{j,k} = \begin{cases} id\alpha_j & \text{if } j = k. \\ (e^{-i\alpha_j} e^{i\alpha_k} - 1)(V^*dV)_{j,k} & \text{if } j \neq k. \end{cases}$$

Equality (6.2.1) asserts that $L = M$, and hence by (6.2.3) it follows that

$$(6.2.4) \quad \omega(U) = i^n \left(\bigwedge_j d\alpha_j \right) \wedge \left(\bigwedge_{j < k} |e^{-i\alpha_j} e^{i\alpha_k} - 1|^2 (V^*dV)_{j,k} \wedge (V^*dV)_{k,j} \right).$$

Recalling that $\kappa\omega$ is what defines the Haar measure, we see that we have decomposed the Haar measure into a product of two measures, one on Δ and the other on V . Integrating out the V part gives the eigenvalue density as proportional to

$$(6.2.5) \quad \prod_{j < k} |e^{i\alpha_j} - e^{i\alpha_k}|^2 \bigwedge_{j=1}^n d\alpha_j.$$

Since $e^{i\theta_k}$ are orthogonal in $L^2(S^1)$, by writing the density as the determinant of BB^* where, $B = (e^{ira_s})_{r,s}$ and expanding the determinants as usual, we get the normalizing factor. The kernel is also read off from BB^* .

REMARK 6.2.2. From 6.2.4, we see that the measure on V is given by the $n(n-1)$ -form

$$(6.2.6) \quad \bigwedge_{i < j} ((V^* dV)_{i,j} \wedge (V^* dV)_{j,i}).$$

Had there been an extra factor of $\wedge_j (V^* dV)_{j,j}$, this would have been the Haar measure on $\mathcal{U}(n)$. But constraints such as $V_{j,j} > 0$, that we imposed to define V uniquely, prevent this. We may avoid this irksomeness by stating Theorem 4.3.9 in the reverse direction: If V is sampled from Haar distribution on $\mathcal{U}(n)$ and Δ is sampled according to density (6.2.5) independently of V , then the matrix $U = V\Delta V^*$ has Haar distribution on $\mathcal{U}(n)$.

6.3. Non-normal matrices, Schur decomposition and a change of measure

For unitary and Hermitian matrix models, to find the law of eigenvalues, we always take auxiliary variables to be the eigenvectors of the matrix. This is because the eigenvectors may be normalized to form an orthonormal basis, or what is the same, the matrix can be diagonalized by a unitary matrix. The GUE and CUE are examples of this.

However, the case of non-normal matrix models (means A and A^* do not commute) is completely different. This applies to the examples of sections 4.3.7, 4.3.8 and 4.3.9. The eigenvectors do not form an orthonormal basis, but only a linear basis (almost surely, in all our examples). This complicates the relationship between the entries of the matrix and the eigenvalues. In fact it is remarkable that the eigenvalue density for these three models can be found explicitly. We are not aware of any other non-normal random matrix models that have been solved exactly.

A non-normal matrix is not unitarily equivalent to a diagonal matrix, but can be diagonalized by a non-unitary matrix (Ginibre's approach) or triangularized by a unitary matrix (Dyson's approach). We take the latter route, which is considerably simpler than the former. In this section we deduce a fundamental Jacobian determinant formula for the change of variables from a matrix to its triangular form. In the next three sections to follow, we shall apply this formula to three non-normal matrix models. The deduction of the Jacobian determinant is due to Dyson and appears in the appendices of Mehta's book (56). However, there seems to be a slight problem with the proof given there, which we have corrected below (see the notes at the end of the chapter for a discussion of this point).

Schur decomposition: Any matrix $M \in gl(n, \mathbb{C})$ can be written as

$$(6.3.1) \quad M = V(Z + T)V^*,$$

where V is unitary, T is strictly upper triangular and Z is diagonal. The decomposition is not unique for the following reasons.

Firstly, $Z = \text{diagonal}(z_1, \dots, z_n)$ has the eigenvalues of M along its diagonal, and hence is determined only up to a permutation. Use the lexicographic order on complex numbers ($u + iv \leq u' + iv'$ if $u < u'$ or if $u = u'$ and $v \leq v'$) to arrange the eigenvalues in increasing order. Thus, $z_1 \leq z_2 \leq \dots \leq z_n$. But we shall omit all matrices with two or more equal eigenvalues (a lower dimensional set and hence also of zero Lebesgue measure), and then strict inequalities hold.

Once Z is fixed, V, T may be replaced by $V\Theta, \Theta^*T\Theta$ where $\Theta = \text{diagonal}(e^{i\theta_1}, \dots, e^{i\theta_n})$ is any diagonal unitary matrix. If the eigenvalues are distinct, this is the only source of non-uniqueness. We restore uniqueness of the decomposition by requiring that $V_{i,i} \geq 0$.

From (6.3.1) we get

$$\begin{aligned} dM &= (dV)(Z+T)V^* + V(dZ+dT)V^* + V(Z+T)d(V^*) \\ &= (dV)(Z+T)V^* + V(dZ+dT)V^* - V(Z+T)V^*(dV)V^* \\ &= V[(V^*dV)(Z+T) - (Z+T)(V^*dV) + dZ+dT]V^*. \end{aligned}$$

It will be convenient to introduce the notations $\Lambda := V^*(dM)V$, $\Omega := V^*dV$ and $S = Z+T$ so that $dS = dZ+dT$. Thus $\Lambda = (\lambda_{i,j})$ and $\Omega = (\omega_{i,j})$ are $n \times n$ matrices of one-forms. Moreover, Ω is skew-Hermitian as we saw in section 6.2. Then the above equation may be written succinctly as

$$(6.3.2) \quad \Lambda = \Omega S - S\Omega + dS.$$

Integration of a function of M with respect to Lebesgue measure is the same as integrating against the $2n^2$ -form

$$\bigwedge_{i,j} (dM_{i,j} \wedge d\bar{M}_{i,j}).$$

Actually, there should be a factor of $2^{n^2}i^{n^2}$, but to make life less painful for ourselves and our readers, we shall omit constants at will in all Jacobian determinant computations to follow. Where probability measures are involved, these constants can be reclaimed at the end by finding normalization constants.

We want to write the Lebesgue measure on M in terms of Z, V, T . For this we must find the Jacobian determinant for the change of variables from $\{dM_{i,j}, d\bar{M}_{i,j}\}$ to $dz_i, 1 \leq i \leq n$, $dT_{i,j}, i < j$, and Ω . Since for any fixed unitary matrix W , the transformation $M \rightarrow WMW^*$ is unitary on $g\ell(n, \mathbb{C})$, we have

$$(6.3.3) \quad \bigwedge_{i,j} (dM_{i,j} \wedge d\bar{M}_{i,j}) = \bigwedge_{i,j} (\lambda_{i,j} \wedge \bar{\lambda}_{i,j}).$$

Thus we only need to find the Jacobian determinant for change of variables from Λ to Ω, dS (and their conjugates). We write equation (6.3.2) in the following manner.

$$\begin{aligned} \lambda_{i,j} &= \sum_{k=1}^j S_{k,j} \omega_{i,k} - \sum_{k=i}^n S_{i,k} \omega_{k,j} + dS_{i,j} \\ &= \begin{cases} (S_{j,j} - S_{i,i})\omega_{i,j} + \left[\sum_{k=1}^{j-1} S_{k,j} \omega_{i,k} - \sum_{k=i+1}^n S_{i,k} \omega_{k,j} \right] & \text{if } i > j. \\ dS_{i,j} + S_{i,j}(\omega_{i,i} - \omega_{j,j}) + \left[\sum_{\substack{k=1 \\ k \neq i}}^j S_{k,j} \omega_{i,k} - \sum_{\substack{k=i+1 \\ k \neq j}}^n S_{i,k} \omega_{k,j} \right] & \text{if } i \leq j. \end{cases} \end{aligned}$$

Now arrange $\{\lambda_{i,j}, \bar{\lambda}_{i,j}\}$ in the order

$$\lambda_{n,1}, \bar{\lambda}_{n,1}, \dots, \lambda_{n,n}, \bar{\lambda}_{n,n}, \lambda_{n-1,1}, \bar{\lambda}_{n-1,1}, \dots, \lambda_{n-1,n}, \bar{\lambda}_{n-1,n}, \dots, \lambda_{1,1}, \bar{\lambda}_{1,1}, \dots, \lambda_{1,n}, \bar{\lambda}_{1,n}.$$

Return to the transformation rules given above for $\lambda_{i,j}$ in terms of Ω, dS . The expressions inside square brackets involve only one-forms that have already appeared before (in the given ordering of one-forms). Here it is necessary to recall

that $\omega_{i,j} = -\bar{\omega}_{j,i}$. Therefore, upon taking wedge products, we arrive at (for brevity we write $|\omega|^2$ for $\omega \wedge \bar{\omega}$)

$$(6.3.4) \quad \bigwedge_{i,j} |\lambda_{i,j}|^2 = \left(\prod_{i>j} |z_i - z_j|^2 \right) \bigwedge_{i>j} |\omega_{i,j}|^2 \bigwedge_i |dz_i|^2 \bigwedge_{i<j} |dt_{i,j} + t_{i,j}(\omega_{i,i} - \omega_{j,j})|^2.$$

Here we have reverted to Z and T in place of S . We make the following claim.

CLAIM 6.3.1. *For any $k \leq n$, we have $\omega_{k,k} \wedge_{i>j} |\omega_{i,j}|^2 = 0$.*

PROOF. Let $\mathcal{M} = \{V : V^*V = I, V_{i,i} \geq 0\}$ be the set of unitary matrices with non-negative diagonal entries. If we omit all V that have at least one zero entry on the diagonal, \mathcal{M} is a smooth manifold of dimension $n^2 - n$. This is because $\mathcal{U}(n)$ is a manifold of dimension n^2 and \mathcal{M} is a sub-manifold thereof, got by imposing n constraints. (Alternately, just observe that $\{V_{j,k} : j > k\}$ parameterize \mathcal{M} . This is because given $\{V_{j,k} : j > k\}$, using orthonormality of the columns of V , we may inductively solve for $V_{1,1}, \{V_{1,2}, V_{2,2}\}, \dots, \{V_{1,n}, \dots, V_{n,n}\}$ in that order).

Since the dimension of \mathcal{M} is only $n^2 - n$ whereas $\omega_{k,k} \wedge_{i>j} |\omega_{i,j}|^2$ is an $(n^2 - n + 1)$ -form, it must be zero. \square

From the claim, thanks to (6.3.4) and (6.3.3), we arrive at the following Jacobian determinant formula, which we shall also refer to as Ginibre's measure decomposition.

$$(6.3.5) \quad \bigwedge_{i,j} (dM_{i,j} \wedge d\bar{M}_{i,j}) = \left(\prod_{i>j} |z_i - z_j|^2 \right) \bigwedge_{i>j} |\omega_{i,j}|^2 \bigwedge_i |dz_i|^2 \bigwedge_{i<j} |dt_{i,j}|^2.$$

This Jacobian determinant formula will be of vital importance to us in the three examples to follow.

REMARK 6.3.2. In place of $V_{i,i} \geq 0$, we may impose constraints of the form $\arg V_{i,i} = \alpha_i$, for some fixed α_i s and arrive at the same formula (6.3.5).

6.4. Ginibre ensemble

Proof of Theorem 4.3.10. If the entries of M are i.i.d. standard complex Gaussian, the joint density is proportional to

$$e^{-\text{tr}(M^*M)} \bigwedge_{i,j} (dM_{i,j} \wedge d\bar{M}_{i,j}).$$

Now use the Schur decomposition of M as $V(Z+T)V^*$, with $V_{i,i} \geq 0$ and observe that

$$\begin{aligned} \text{tr}(M^*M) &= \text{tr}[(Z+T)^*(Z+T)] \\ &= \text{tr}(Z^*Z) + \text{tr}(T^*T) \end{aligned}$$

because Z^*T and T^*Z have zeros on the diagonal. From the Jacobian determinant formula (6.3.5), we see that the probability density of Z, V, T is proportional to

$$\left(e^{-\text{tr}(Z^*Z)} \prod_{i<j} |z_i - z_j|^2 \bigwedge_i (dz_i \wedge d\bar{z}_i) \right) \left(e^{-\text{tr}(T^*T)} \bigwedge_{i<j} (dT_{i,j} \wedge d\bar{T}_{i,j}) \right) \left(\bigwedge_{i>j} \omega_{i,j} \wedge \bar{\omega}_{i,j} \right)$$

where $\omega_{i,j} := (V^*dV)_{i,j}$. Thus Z, T, V are independent, and integrating over V, T , we conclude that the density of Z with respect to Lebesgue measure on \mathbb{C}^n is proportional to

$$\prod_{k=1}^n e^{-|z_k|^2} \prod_{i<j} |z_i - z_j|^2.$$

The normalization constant is found in the usual manner, by orthogonalizing $\{z^k\}$ with respect to $e^{-|z|^2} dm(z)$. \square

6.5. Spherical ensemble

Proof of Theorem 4.3.11. (A, B) has density

$$\frac{1}{\pi^{2n^2}} \exp\{-\operatorname{tr}AA^* - \operatorname{tr}BB^*\} \bigwedge_{i,j} |dA_{i,j}|^2 \bigwedge_{i,j} |dB_{i,j}|^2.$$

Hence with $M = A^{-1}B$, the joint density of (M, A) is

$$\frac{1}{\pi^{2n^2}} |\det(A)|^{2n} \exp\{-\operatorname{tr}A^*(I + MM^*)A\} \bigwedge_{i,j} |dA_{i,j}|^2 \bigwedge_{i,j} |dM_{i,j}|^2.$$

Now, for any fixed positive definite matrix Σ , a standard Gaussian integral computation shows that

$$\int_{\mathfrak{gl}(n, \mathbb{C})} |\det(A)|^{2n} \exp\{-\operatorname{tr}A^*\Sigma A\} \bigwedge_{i,j} |dA_{i,j}|^2 = \pi^{n^2} C_n |\det(\Sigma)|^{-2n}.$$

In fact one may show that $C_n = \prod_{k=1}^n \frac{(k+n-1)!}{(k-1)!}$, but we shall not need this. Thus, the density of M is found to be

$$\frac{C_n}{\pi^{n^2}} \frac{1}{\det(I + MM^*)^{2n}} \bigwedge_{i,j} |dM_{i,j}|^2.$$

Let $M = V(Z + T)V^*$, with $V_{i,i} \geq 0$, be the Schur decomposition (6.3.1) of M . By Ginibre's measure decomposition (6.3.5), we get the measure of Z, V, T to be

$$(6.5.1) \quad \frac{C_n}{\pi^{n^2}} \prod_{i < j} |z_i - z_j|^2 \frac{1}{\det(I + (Z + T)(Z + T)^*)^{2n}} \bigwedge_{i < j} |dT_{i,j}|^2 \bigwedge_{i > j} |\omega_{i,j}^V|^2 \bigwedge_i |dz_i|^2.$$

As usual $\omega^V = V^* dV$. The density does not depend on V , and hence, to get the density of eigenvalues of M , all we need is to compute the integral over T . Unlike in the Ginibre ensemble, here T and Z not independent. Define

$$I(n, p) = \int_{\mathcal{T}_n} \frac{1}{\det(I + (Z + T)(Z + T)^*)^p} \bigwedge_{i < j} |dT_{i,j}|^2.$$

where \mathcal{T}_n is the space of $n \times n$ strictly upper triangular matrices. We compute $I(n, p)$ recursively. Write $S_n = Z + T$ so that

$$I + S_n S_n^* = \begin{bmatrix} I + S_{n-1} S_{n-1}^* + \mathbf{u} \mathbf{u}^* & \bar{z}_n \mathbf{u} \\ z_n \mathbf{u}^* & 1 + |z_n|^2 \end{bmatrix}$$

where $\mathbf{u} = [T_{1,n} \dots T_{n-1,n}]^t$. We want to integrate over \mathbf{u} first. For this, observe that

$$\begin{aligned} \det(I + S_n S_n^*) &= (1 + |z_n|^2) \det\left(I + S_{n-1} S_{n-1}^* + \mathbf{u} \mathbf{u}^* - \frac{|z_n|^2}{1 + |z_n|^2} \mathbf{u} \mathbf{u}^*\right) \\ &= (1 + |z_n|^2) \det\left(I + S_{n-1} S_{n-1}^* - \frac{1}{1 + |z_n|^2} \mathbf{u} \mathbf{u}^*\right) \\ &= (1 + |z_n|^2) \det(I + S_{n-1} S_{n-1}^*) \left(1 - \frac{\mathbf{u}^* (I + S_{n-1} S_{n-1}^*)^{-1} \mathbf{u}}{1 + |z_n|^2}\right) \end{aligned}$$

where in the last line we made the following observation. For any $A_{n \times n}$ and $\mathbf{u} \in \mathbb{C}^n$, the matrix $A^{-1}\mathbf{u}\mathbf{u}^*$ has rank one, and hence its only eigenvalues is its trace, which is $\mathbf{u}^*A^{-1}\mathbf{u}$. Therefore,

$$\det(A + \mathbf{u}\mathbf{u}^*) = \det(A)\det(I + A^{-1}\mathbf{u}\mathbf{u}^*) = \det(A)(1 + \mathbf{u}^*A^{-1}\mathbf{u}).$$

Hence

$$\begin{aligned} \int_{\mathbb{C}^{n-1}} \frac{\wedge_{i < n} |dT_{i,n}|^2}{\det(I + S_n S_n^*)^p} &= \frac{1}{(1 + |z_n|^2)^p \det(I + S_{n-1} S_{n-1}^*)^p} \int_{\mathbb{C}^{n-1}} \frac{\wedge |du_i|^2}{\left(1 - \frac{1}{1 + |z_n|^2} \mathbf{u}^*(I + S_{n-1} S_{n-1}^*)^{-1}\mathbf{u}\right)^p} \\ &= \frac{C(n, p)}{(1 + |z_n|^2)^{p-n+1} \det(I + S_{n-1} S_{n-1}^*)^{p-1}} \end{aligned}$$

where the last line results from making the change of variables $\mathbf{v} = \frac{1}{\sqrt{1 + |z_n|^2}}(I + S_{n-1} S_{n-1}^*)^{-\frac{1}{2}}\mathbf{u}$. Again, one may compute that

$$C(n, p) = \int_{\mathbb{C}^{n-1}} \frac{\wedge_i |dv_i|^2}{(1 + \mathbf{v}^*\mathbf{v})^p} = \frac{\pi}{2} \text{Beta}(p - n + 1, n - 1)$$

but we shall not need it. Thus we get the recursion

$$I(n, p) = \frac{C(n, p)}{(1 + |z_n|^2)^{p-n+1}} I(n - 1, p - 1).$$

What we need is $I(n, 2n)$, which by the recursion gives

$$I(n, 2n) = C'_n \prod_{k=1}^n \frac{1}{(1 + |z_k|^2)^{n+1}}.$$

Using this result back in (6.5.1), we see that the density of eigenvalues of M is

$$C''_n \prod_{k=1}^n \frac{1}{(1 + |z_k|^2)^{n+1}} \prod_{i < j} |z_i - z_j|^2.$$

To compute the constant, note that

$$\left\{ \sqrt{\frac{n(n-1)}{\pi}} \frac{z^k}{(1 + |z|^2)^{\frac{n+1}{2}}} \right\}_{0 \leq k \leq n-1}$$

is an orthonormal set in $L^2(\mathbb{C})$. The projection operator on the Hilbert space generated by these functions defines a determinantal process whose kernel is as given in (4.3.12). Writing out the density of this determinantal process shows that it has the same form as the eigenvalue density that we have determined. Hence the constants must match and we obtain C'_n . \square

6.6. Truncated unitary matrices

We give a proof of Theorem 4.3.13 for the case $m = 1$. The general case follows the same ideas but the notations are somewhat more complicated (see notes).

Consider an $(n + 1) \times (n + 1)$ complex matrix

$$M = \begin{bmatrix} X & \mathbf{c} \\ \mathbf{b}^* & a \end{bmatrix}$$

and assume that M and X are non-singular and that the eigenvalues of X are all distinct. Our first step will be to transform Lebesgue measure on the entires of

M into co-ordinates involving eigenvalues of X and some auxiliary variables. The situation in Theorem 4.3.13 is that we want to find the measure on eigenvalues of X , but when M is chosen from the submanifold $\mathcal{U}(n+1)$ of $g\ell(n+1, \mathbb{C})$. Therefore, some further work will be required to use the Jacobian determinant from Lebesgue measure on M to the latter case when M has Haar measure on $\mathcal{U}(n+1)$.

We shall need the following decompositions of M .

- (1) **Polar decomposition:** $M = UP^{1/2}$, where U is unitary and $P^{1/2}$ is the positive definite square root of a positive definite matrix P . The decomposition is unique, the only choice being $P = M^*M$ and $U = P^{-1/2}M$.
- (2) **Schur decomposition of X :** Write $M = WYW^*$ where

$$W = \begin{bmatrix} V & 0 \\ 0 & 1 \end{bmatrix}, \quad Y = \begin{bmatrix} Z+T & \mathbf{v} \\ \mathbf{u}^* & a \end{bmatrix},$$

where V is unitary with $V_{i,i} \geq 0$, T is strictly upper triangular, $Z = \text{diag}(z_1, \dots, z_n)$ is the diagonal matrix of eigenvalues of X , and $\mathbf{u} = V^*\mathbf{b}$, $\mathbf{v} = V^*\mathbf{c}$. Since z_i are distinct, if we fix their order in some manner, then this decomposition is unique (see 6.3.1).

- (3) **Modified Schur decomposition:** Use the notations in the previous two decompositions. As our final goal is to take M to be unitary, we want to find a new set of co-ordinates for M with the property that the submanifold $\mathcal{U}(n+1)$ is represented in a simple way in these co-ordinates. An obvious choice is to use P , since $\mathcal{U}(n+1)$ is the same as $\{P = I\}$. Obviously we want Z to be part of our co-ordinates. Thus we have $(n+1)^2$ degrees of freedom in P and $2n$ degrees of freedom in Z and need co-ordinates for $n^2 + 1$ further degrees of freedom (the total being $2(n+1)^2$ for M). The matrix V will furnish $n^2 - n$ of them and the angular parts of \mathbf{u} and a will provide the remaining $n+1$. We now delve into the details.

Write $u_k = r_k e^{i\alpha_k}$, $1 \leq k \leq n$ and $a = r e^{i\theta}$. Set $Q = W^*PW$, so that $Y^*Y = Q$. Let \mathcal{A}_k and \mathcal{Q}_k be the submatrices consisting of the first k rows and columns of Y and Q respectively. Let $\mathbf{u}_k = [u_1 \dots u_k]^t$ and $\mathbf{v}_k = [v_1 \dots v_k]^t$ denote the vectors consisting of the first k co-ordinates of \mathbf{u} and \mathbf{v} respectively. In particular $\mathbf{u}_n = \mathbf{u}$ and $\mathbf{v}_n = \mathbf{v}$. Also let $\mathbf{t}_k = [T_{1,k}, T_{2,k}, \dots, T_{k-1,k}]^t$ and $\mathbf{q}_k = [Q_{1,k}, Q_{2,k}, \dots, Q_{k-1,k}]^t$ for $k \geq 2$.

Then from the off-diagonal equations of $Y^*Y = Q$, we get

$$\mathcal{A}_k^* \mathbf{t}_{k+1} + \bar{u}_{k+1} \mathbf{u}_k = \mathbf{q}_{k+1} \quad \text{for } 1 \leq k \leq n-1, \quad \mathcal{A}_n^* \mathbf{v} + a \mathbf{u} = \mathbf{q}_{n+1}.$$

The matrices \mathcal{A}_k are upper triangular their diagonal entries are z_i s which are all assumed non-zero. Therefore, we can inductively solve for $\mathbf{t}_2, \dots, \mathbf{t}_n$ and \mathbf{v} in terms of Q, Z, \mathbf{u} and a . Thus we get

$$(6.6.1) \quad \mathbf{t}_{k+1} = \mathcal{A}_k^{*-1}(\mathbf{q}_{k+1} - \bar{u}_{k+1} \mathbf{u}_k),$$

$$(6.6.2) \quad \mathbf{v} = \mathcal{A}_n^{*-1}(\mathbf{q}_{n+1} - a \mathbf{u}).$$

From the diagonal equations of $Y^*Y = Q$, we get

$$r_1^2 = Q_{1,1} - |z_1|^2, \quad r_k^2 + \|\mathbf{t}_k\|^2 = Q_{k,k} - |z_k|^2, \quad \text{for } 2 \leq k \leq n, \quad r^2 + \|\mathbf{v}\|^2 = Q_{n+1, n+1}.$$

As equations (6.6.1) show, \mathbf{t}_{k+1} depends only on z_j , $j \leq k$, and u_j , $j \leq k+1$, and Q , it is possible to successively solve for r_1, \dots, r_n and r in terms of Q, Z, θ and α_k , $1 \leq k \leq n$. This is done as follows.

The first equation $r_1^2 = \mathbf{Q}_{1,1} - |z_1|^2$ can be solved uniquely for $r_1 > 0$, provided $\mathbf{Q}_{1,1} \geq |z_1|^2$. Substitute from (6.6.1) for \mathbf{t}_{k+1} in the equation for r_{k+1}^2 to get

$$\begin{aligned} \mathbf{Q}_{k+1,k+1} - |z_{k+1}|^2 &= r_{k+1}^2 + (\mathbf{q}_{k+1} - \bar{u}_{k+1} \mathbf{u}_k)^* (\mathcal{A}_k^* \mathcal{A}_k)^{-1} (\mathbf{q}_{k+1} - \bar{u}_{k+1} \mathbf{u}_k) \\ &= r_{k+1}^2 \{1 + \mathbf{u}_k^* (\mathcal{A}_k^* \mathcal{A}_k)^{-1} \mathbf{u}_k\} - 2r_{k+1} \operatorname{Re}\{e^{-i\alpha_{k+1}} \mathbf{u}_k^* (\mathcal{A}_k^* \mathcal{A}_k)^{-1} \mathbf{q}_{k+1}\} \\ &\quad + \mathbf{q}_{k+1}^* (\mathcal{A}_k^* \mathcal{A}_k)^{-1} \mathbf{q}_{k+1}. \end{aligned} \quad (6.6.3)$$

An identical consideration applies to the equation for r and we get

$$r^2 \{1 + \mathbf{u}^* (\mathcal{A}_n^* \mathcal{A}_n)^{-1} \mathbf{u}\} - 2r \operatorname{Re}\{e^{-i\theta} \mathbf{u}^* (\mathcal{A}_n^* \mathcal{A}_n)^{-1} \mathbf{q}_{n+1}\} = \mathbf{Q}_{n+1,n+1} - \mathbf{q}_{n+1}^* (\mathcal{A}_n^* \mathcal{A}_n)^{-1} \mathbf{q}_{n+1}. \quad (6.6.4)$$

A quadratic $ax^2 + bx + c$ with $a > 0$ and b, c real, has a unique positive solution if and only if $c < 0$. Thus, the constraints under which we can solve for positive numbers r_k and r , uniquely in terms of \mathbf{Q}, \mathbf{Z} and α_k , $1 \leq k \leq n$, are (interpret $\mathbf{q}_1 = 0, \mathcal{A}_0 = 0$)

$$|z_k|^2 < \mathbf{Q}_{k,k} - \mathbf{q}_k^* (\mathcal{A}_{k-1}^* \mathcal{A}_{k-1})^{-1} \mathbf{q}_k, \quad \mathbf{q}_{n+1}^* (\mathcal{A}_n^* \mathcal{A}_n)^{-1} \mathbf{q}_{n+1} < \mathbf{Q}_{n+1,n+1}. \quad (6.6.5)$$

Thus we may take our independent variables to be Z, V, P, θ and α_k , $k \leq n$, subject to the constraints (6.6.5). Then we decompose M as WYW^* , where we now regard T, \mathbf{v}, r and r_k , $k \leq n$ as functions of Z, V, P, θ and α_k s, got from equations (6.6.1)-(6.6.4). Clearly this decomposition is also unique, because Schur decomposition is.

The following lemmas express the Lebesgue measure in terms of the variables in polar decomposition and modified Schur decompositions, respectively.

LEMMA 6.6.1. *Let $UP^{1/2}$ be the polar decomposition of M . Then*

$$\bigwedge_{i,j} |dM_{i,j}|^2 = f(P) \bigwedge_{i,j} dP_{i,j} \bigwedge_{i,j} \omega_{i,j}^U$$

where f is some smooth function of P while $dP = (dP_{i,j})$ and $\omega^U = U^* dU$ are Hermitian and skew Hermitian, respectively.

LEMMA 6.6.2. *Let WYW^* , with T, \mathbf{v}, r and r_k , $k \leq n$ being functions of Z, V, P, θ and α_k , $k \leq n$, be the modified Schur decomposition of M . Then*

$$\bigwedge_{i,j} |dM_{i,j}|^2 = \frac{\left(\prod_{i < j} |z_i - z_j|^2 \right) \mathbf{1}_{(6.6.5)} \wedge_i |dz_i|^2 \wedge_{i,j} dP_{i,j} \wedge_{i \neq j} \omega_{i,j}^V \wedge_k d\alpha_k \wedge d\theta}{\prod_{k=1}^n |\det(\mathcal{A}_k)|^2 \left(1 + \mathbf{u}_k^* (\mathcal{A}_k^* \mathcal{A}_k)^{-1} \mathbf{u}_k - \frac{1}{r_k} \operatorname{Re}\{e^{-i\alpha_{k+1}} \mathbf{u}_k^* (\mathcal{A}_k^* \mathcal{A}_k)^{-1} \mathbf{q}_{k+1}\} \right)}. \quad (6.6.6)$$

where the notations are as defined earlier, and $\omega^V = V^* dV$. Here $\mathbf{1}_{(6.6.5)}$ denotes the indicator function of the constraints stated in the display (6.6.5) on Z and \mathbf{Q} , where \mathbf{Q} is related to P by $\mathbf{Q} = W^* P W$.

Assuming the validity of these lemmas, we now deduce Theorem 4.3.13. First we state an elementary fact that we leave for the reader to verify.

FACT 6.6.3. Let M be a manifold and suppose that $\{x_i : i \leq k\} \cup \{y_j : j \leq \ell\}$ and $\{x_i : i \leq k\} \cup \{z_j : j \leq \ell\}$ are two sets of co-ordinates on M . Let $\mathbf{x} = (x_1, \dots, x_k)$ and similarly define \mathbf{y} and \mathbf{z} . If the volume form on M is given in the two co-ordinate systems by $f(\mathbf{x}, \mathbf{y}) \wedge_i dx_i \wedge_j dy_j$ and by $g(\mathbf{x}, \mathbf{z}) \wedge_i dx_i \wedge_j dz_j$ respectively, then, on the submanifold $\mathbf{x} = 0$, the two ℓ -forms $f(\mathbf{0}, \mathbf{y}) \wedge_j dy_j$ and $g(\mathbf{0}, \mathbf{z}) \wedge_j dz_j$ are equal.

Proof of Theorem 4.3.13. The unitary group is the submanifold $P = I$ of $g\ell(n, \mathbb{C})$. Therefore, by Lemma 6.6.1, Lemma 6.6.2 and Fact 6.6.3, we may conclude that

$$f(I) \bigwedge_{i,j} \omega_{i,j}^U = \frac{\prod_{i<j} |z_i - z_j|^2}{\prod_{k=1}^n |\det(\mathcal{A}_k)|^2 (1 + \mathbf{u}_k^* (\mathcal{A}_k^* \mathcal{A}_k)^{-1} \mathbf{u}_k)} \mathbf{1}_{(6.6.5)} \bigwedge_i |dz_i|^2 \bigwedge_{i \neq j} \omega_{i,j}^V \bigwedge_k d\alpha_k \bigwedge d\theta.$$

The denominator is much simpler than in (6.6.6) because, when P is the identity, so is Q , and hence $\mathbf{q}_{k+1} = 0$ for each $1 \leq k \leq n$. For the same reason, and because $Q_{k,k} = 1$, the constraints (6.6.5) simplify to $|z_k|^2 < 1$, $1 \leq k \leq n$.

The denominator can be further simplified. Using $Y^*Y = Q = I$ which gives

$$\mathcal{A}_k^* \mathcal{A}_k + \mathbf{u}_k \mathbf{u}_k^* = I_k, \text{ for } k \leq n.$$

From this we see that

$$\begin{aligned} 1 &= \det(\mathcal{A}_k^* \mathcal{A}_k + \mathbf{u}_k \mathbf{u}_k^*) \\ &= |\det(\mathcal{A}_k)|^2 \det(I + [(\mathcal{A}_k^*)^{-1} \mathbf{u}_k][(\mathcal{A}_k^*)^{-1} \mathbf{u}_k]^*) \\ &= |\det(\mathcal{A}_k)|^2 (1 + \mathbf{u}_k^* (\mathcal{A}_k^* \mathcal{A}_k)^{-1} \mathbf{u}_k) \end{aligned}$$

where the last line employs the identity $\det(I + \mathbf{w}\mathbf{w}^*) = 1 + \mathbf{w}^*\mathbf{w}$ for any vector \mathbf{w} . This identity holds because \mathbf{w} is an eigenvector of $I + \mathbf{w}\mathbf{w}^*$ with eigenvalue $1 + \mathbf{w}^*\mathbf{w}$, while vectors orthogonal to \mathbf{w} are eigenvectors with eigenvalue 1. Thus we arrive at

$$\bigwedge_{i,j} \omega_{i,j}^U = C \prod_{i<j} |z_i - z_j|^2 \bigwedge_i |dz_i|^2 \bigwedge_{i \neq j} \omega_{i,j}^V \bigwedge_k d\alpha_k \bigwedge d\theta$$

for some constant C . This gives the density of Z as proportional to $\prod_{i<j} |z_i - z_j|^2$, for $|z_j| < 1$, $j \leq n$. This is exactly what we wanted to prove. \square

It remains to prove Lemma 6.6.1 and Lemma 6.6.2.

Proof of Lemma 6.6.1. The bijection $M \rightarrow (U, P)$ from $GL(n, \mathbb{C})$ onto the space $\{\text{p.d. matrices}\} \times \mathcal{U}(n)$ is clearly smooth. Thus we must have

$$\bigwedge_{i,j} |dM_{i,j}|^2 = f(P, U) \bigwedge_{i,j} dP_{i,j} \bigwedge_{i,j} \omega_{i,j}^U$$

because $dP_{i,j}, \omega_{i,j}^U$ are $2n^2$ independent one-forms on the $2n^2$ -dimensional space $\{\text{p.d. matrices}\} \times \mathcal{U}(n)$.

For any fixed unitary matrix U_0 , the transformation $M \rightarrow U_0 M$ preserves the Lebesgue measure while it transforms (U, P) to $(U_0 U, P)$. From the invariance of ω^U , it follows that $f(P, U_0 U) = f(P, U)$ which in turn just means that f is a function of P alone. \square

Proof of Lemma 6.6.2. First consider the (unmodified) Schur decomposition $M = WYV^*$, where the effect is to just change from X to V, Z, T , while \mathbf{b}, \mathbf{c} undergo unitary transformations to \mathbf{u}, \mathbf{v} respectively. Using Ginibre's measure decomposition (6.3.5) to make the change from X to V, Z, T , we get

$$(6.6.7) \quad \bigwedge_{i,j} |dM_{i,j}|^2 = \prod_{i<j} |z_i - z_j|^2 \bigwedge_i |dz_i|^2 \bigwedge_{i \neq j} \omega_{i,j}^V \bigwedge_{i<j} |dT_{i,j}|^2 \bigwedge_k |dv_k|^2 \bigwedge_k (r_k dr_k \wedge d\alpha_k) \bigwedge (r dr \wedge d\theta).$$

Here we have expressed $|dv_k|^2$ and $|da|^2$ in polar co-ordinates. Recall equations (6.6.1)-(6.6.4) that express T, \mathbf{v}, r and $r_k, k \leq n$ as functions of Z, V and P . From

(6.6.1) and (6.6.2), we get

$$\bigwedge_{i=1}^k dT_{i,k+1} = \frac{1}{\det(\mathcal{A}_k)} \bigwedge_{i=1}^k dQ_{i,k+1} + [\dots], \quad \bigwedge_{i=1}^n dv_i = \frac{1}{\det(\mathcal{A}_n)} \bigwedge_{i=1}^n dQ_{i,n+1} + [\dots]$$

where [...] consists of many terms involving du_i, dz_i , as well as $dT_{i,j}$ for $j \leq k$. Therefore, when we take wedge product of these expressions and their conjugates over k , all terms inside [...] containing any $dT_{i,j}$ or $d\bar{T}_{i,j}$ factors vanish, and we get

$$\bigwedge_{i<j} |dT_{i,j}|^2 \bigwedge_i |dv_i|^2 = \frac{1}{\prod_{k=1}^n |\det(\mathcal{A}_k)|^2} \bigwedge_{i<j} |dQ_{i,j}|^2 + [\dots]$$

where [...] consists of many terms involving du_i, dz_i , their conjugates. Substitute this into the right hand side of (6.6.7), and observe that all terms coming from [...] give zero because du_i, dz_i and their conjugates already appear in (6.6.7). Thus

(6.6.8)

$$\bigwedge_{i,j} |dM_{i,j}|^2 = \frac{\prod_{i<j} |z_i - z_j|^2}{\prod_{k=1}^n |\det(\mathcal{A}_k)|^2} \bigwedge_i |dz_i|^2 \bigwedge_{i<j} |dQ_{i,j}|^2 \bigwedge_{i \neq j} \omega_{i,j}^V \bigwedge_k (r_k dr_k \wedge d\alpha_k) \bigwedge (r dr \wedge d\theta).$$

Since Q is Hermitian, we have written $|dQ_{i,j}|^2$ as $dQ_{i,j} \wedge dQ_{j,i}$. We are being cavalier about the signs that come from interchanging order of wedge products, but that can be fixed at the end as we know that we are dealing with positive measures.

Next, apply (6.6.3) and (6.6.4) to write

$$\bigwedge_k (2r_k dr_k) \bigwedge r dr = \frac{dQ_{1,1} \wedge \dots \wedge dQ_{n+1,n+1}}{\prod_{k=1}^n \left(1 + \mathbf{u}_k^*(\mathcal{A}_k^* \mathcal{A}_k)^{-1} \mathbf{u}_k - \frac{1}{r_k} \operatorname{Re}\{e^{-i\alpha_{k+1}} \mathbf{u}_k^*(\mathcal{A}_k^* \mathcal{A}_k)^{-1} \mathbf{q}_{k+1}\}\right)} + [\dots].$$

Again the terms included in [...] yield zero when “wedged” with the other terms in (6.6.8). Thus,

(6.6.9)

$$\bigwedge_{i,j} |dM_{i,j}|^2 = \frac{\left(\prod_{i<j} |z_i - z_j|^2\right) \bigwedge_i |dz_i|^2 \bigwedge_{i,j} dQ_{i,j} \bigwedge_{i \neq j} \omega_{i,j}^V \bigwedge_k d\alpha_k \bigwedge d\theta}{\prod_{k=1}^n |\det(\mathcal{A}_k)|^2 \left(1 + \mathbf{u}_k^*(\mathcal{A}_k^* \mathcal{A}_k)^{-1} \mathbf{u}_k - \frac{1}{r_k} \operatorname{Re}\{e^{-i\alpha_{k+1}} \mathbf{u}_k^*(\mathcal{A}_k^* \mathcal{A}_k)^{-1} \mathbf{q}_{k+1}\}\right)}.$$

This is almost the same as the statement of the lemma, except that we have dQ in place of dP . However from $P = WQW^*$, and the definition of W we get

$$dP = W \left(dQ + \begin{bmatrix} \omega^V & 0 \\ 0 & 1 \end{bmatrix} P - P \begin{bmatrix} \omega^V & 0 \\ 0 & 1 \end{bmatrix} \right) W^*.$$

As we have seen before, the map $M \rightarrow W^* M W$ is unitary, which implies that $\bigwedge dP_{i,j} = \bigwedge (W^* dP W)_{i,j}$, which by the above equation shows that $\bigwedge_{i,j} Q_{i,j} = \bigwedge_{i,j} dP_{i,j} + [\dots]$, where again the terms brushed under [...] are those that yield zero when substituted into (6.6.9). Therefore

$$\bigwedge_{i,j} |dM_{i,j}|^2 = \frac{\left(\prod_{i<j} |z_i - z_j|^2\right) \bigwedge_i |dz_i|^2 \bigwedge_{i,j} dP_{i,j} \bigwedge_{i \neq j} \omega_{i,j}^V \bigwedge_k d\alpha_k \bigwedge d\theta}{\prod_{k=1}^n |\det(\mathcal{A}_k)|^2 \left(1 + \mathbf{u}_k^*(\mathcal{A}_k^* \mathcal{A}_k)^{-1} \mathbf{u}_k - \frac{1}{r_k} \operatorname{Re}\{e^{-i\alpha_{k+1}} \mathbf{u}_k^*(\mathcal{A}_k^* \mathcal{A}_k)^{-1} \mathbf{q}_{k+1}\}\right)}.$$

□

6.7. Singular points of matrix-valued GAFs

Now we use Theorem 4.3.13 to prove Theorem 4.3.15 . This gives an alternate proof to Theorem 5.1.1, different from the one that was given in the chapter 5. The proof given here is due to appear in the paper of Katsnelson, Kirstein and Krishnapur (45) and is simpler than the original one in (?).

We split the proof into two lemmas, the first of which establishes the link between submatrices of Haar unitary matrices and Gaussian matrices and the second which uses Theorem 4.3.13 and in which a central idea is a link between (deterministic) unitary matrices and analytic functions on the unit disk.

LEMMA 6.7.1. *Let U be an $N \times N$ random unitary matrix sampled from the Haar measure. Fix $n \geq 1$. After multiplication by \sqrt{N} , the first principal $n \times n$ sub-matrices of U^p , $p \geq 1$, converge in distribution to independent matrices with i.i.d. standard complex Gaussian entries. In symbols,*

$$\sqrt{N}([U]_{i,j \leq n}, [U^2]_{i,j \leq n}, \dots) \xrightarrow{d} (G_1, G_2, \dots)$$

where G_i are independent $n \times n$ matrices with i.i.d. standard complex Gaussian entries. That is, any finite number of random variables $\sqrt{N}[U^p]_{i,j}$, $p \geq 1$, $i, j \leq n$, converge in distribution to independent standard complex Gaussians.

LEMMA 6.7.2. *Let U be any unitary matrix of size $N + m$. Write it in the block form*

$$U = \begin{bmatrix} A_{m \times m} & B \\ C & V_{N \times N} \end{bmatrix}.$$

Then,

$$\frac{\det(zI - V^*)}{\det(I - zV)} = (-1)^N \det(U^*) \det(A + zB(I - zV)^{-1}C).$$

Assuming the lemmas, we deduce Theorem 4.3.15.

Proof of Theorem 4.3.15. Let U be sampled from Haar measure on $\mathcal{U}(N + m)$ and write it in block form as in Lemma 6.7.2. Define

$$\mathbf{f}_N(z) = (-1)^N \det(U) \frac{\det(zI - V^*)}{\det(I - zV)}.$$

Since V^* has the same law as V , by Theorem 4.3.13, the zeros of \mathbf{f}_N are determinantal with kernel

$$\mathbb{K}_N^{(m)}(z, w) = \sum_{k=0}^{N-1} \frac{(m+1) \dots (m+k)}{k!} (z\bar{w})^k$$

Hence, to prove Theorem 4.3.15, it suffices to show that

$$(6.7.1) \quad N^{m/2} \mathbf{f}_N(z) \xrightarrow{d} \det(G_0 + zG_1 + z^2G_2 + \dots),$$

where the distributional convergence is not for a fixed z but in the space of functions analytic in the unit disk, with respect to the topology of uniform convergence on compact subsets. By Lemma 6.7.2, we see that

$$\begin{aligned} N^{m/2} \mathbf{f}_N(z) &= \det\left(\sqrt{N}(A + zB(I - zV)^{-1}C)\right) \\ &= \det\left(\sqrt{N}(A + zBC + z^2BVC + z^3BV^2C + \dots)\right). \end{aligned}$$

Now observe that $A = [U]_{i,j \leq m}$. Hence, by Lemma 6.7.1, it follows that

$$(6.7.2) \quad \sqrt{N}A \xrightarrow{d} G_0.$$

Further,

$$\sqrt{N}[U^2]_{i,j \leq m} = \sqrt{N}A^2 + \sqrt{N}BC.$$

By (6.7.2), we see that $\sqrt{N}A^2 \xrightarrow{P} 0$, and thus, an application of Lemma 6.7.1 implies that

$$\left(\sqrt{N}A, \sqrt{N}BC \right) \xrightarrow{d} (G_0, G_1).$$

Inductively, we see that $BV^k C = [U^{k+2}]_{i,j \leq m} + O_P(1/N)$. Here, by $O_P(1/N)$ we mean a quantity which upon dividing by N^{-1} remains tight. Thus Lemma 6.7.1 implies that

$$\sqrt{N}(A, BC, BVC, BV^2 C, \dots) \xrightarrow{d} (G_0, G_1, G_2, \dots).$$

This convergence is meant in the sense that any finite set of the random variables on the left converge in distribution to the corresponding ones on the right. Surely, this implies that the coefficients in the power series expansion of $N^{m/2} \mathbf{f}_N$ converge in distribution to those of $\det(G_0 + zG_1 + z^2G_2 + \dots)$. However, to say that the zeros of $N^{m/2} \mathbf{f}_N$ converge (in distribution) to those of $\det(\sum G_k z^k)$, we need to show weak convergence in the space of analytic functions on the unit disk with respect to the topology of uniform convergence on compact sets. Since we already have convergence of coefficients, this can be done by proving that $\sup_{z \in K} |N^{m/2} \mathbf{f}_N(z)|$ is tight, for any compact $K \subset \mathbb{D}$. We skip this boring issue and refer the reader to Lemma 14 in (?). This completes the proof. \square

A word of explanation on the question of tightness in the last part of the proof. To see that there is an issue here, consider the sequence of analytic functions $g_n(z) = c_n z^n$. All the coefficients of g_n converge to 0 rapidly, but g_n may converge uniformly on compact sets in the whole plane ($c_n = 2^{-n^2}$) or only in a disk ($c_n = 1$) or merely at one point ($c_n = 2^{n^2}$). Which of these happens can be decided by the asking on what sets is the sequence g_n uniformly bounded.

It remains to prove the two lemmas. In proving Lemma 6.7.1, we shall make use of the following ‘‘Wick formula’’ for joint moments of entries of a unitary matrix (compare with the Gaussian Wick formula of Lemma 2.1.7). We state a weaker form that is sufficient for our purpose. In Nica and Speicher (62), page 381, one may find a stronger result, as well as a proof.

RESULT 6.7.3. Let $U = ((u_{i,j}))_{i,j \leq N}$ be chosen from Haar measure on $\mathcal{U}(N)$. Let $k \leq N$ and fix $i(\ell), j(\ell), i'(\ell), j'(\ell)$ for $1 \leq \ell \leq k$. Then

$$(6.7.3) \quad \mathbf{E} \left[\prod_{\ell=1}^k u_{i(\ell), j(\ell)} \prod_{\ell=1}^k \bar{u}_{i'(\ell), j'(\ell)} \right] = \sum_{\pi, \sigma \in \mathcal{S}_k} \text{Wg}(N, \pi \sigma^{-1}) \prod_{\ell=1}^k \mathbf{1}_{i(\ell)=i'(\pi \ell)} \mathbf{1}_{j(\ell)=j'(\sigma \ell)}$$

where Wg (called ‘‘Weingarten function’’) has the property that as $N \rightarrow \infty$,

$$(6.7.4) \quad \text{Wg}(N, \tau) = \begin{cases} N^{-k} + O(N^{-k-1}) & \text{if } \tau = e \text{ (‘‘identity’’)}. \\ O(N^{-k-1}) & \text{if } \tau \neq e. \end{cases}$$

Proof of Lemma 6.7.1. We want to show that $\sqrt{N}(U^k)_{\alpha, \beta}$, $k \geq 1$, $1 \leq \alpha, \beta \leq n$ converge (jointly) in distribution to independent standard complex Gaussians. To use the method of moments consider two finite products of these random variables

$$(6.7.5) \quad S = \prod_{i=1}^m [(U^{k_i})_{\alpha_i, \beta_i}]^{p_i} \quad \text{and} \quad T = \prod_{i=1}^{m'} [(U^{k'_i})_{\alpha'_i, \beta'_i}]^{p'_i}.$$

where $m, m', p_i, p'_i, k_i, k'_i \geq 1$ and $1 \leq \alpha_i, \beta_i, \alpha'_i, \beta'_i \leq n$ are fixed. We want to find $\mathbf{E}[ST]$ asymptotically as $N \rightarrow \infty$.

The idea is simple-minded. We expand each $(U^k)_{\alpha,\beta}$ as a sum of products of entries of U . Then we get a huge sum of products and we evaluate the expectation of each product using Result 6.7.3. Among the summands that do not vanish, most have the same contribution and the rest are negligible. We now delve into the details.

Let $\mathcal{P}_k(\alpha, \beta)$ denote all “paths” γ of length k connecting α to β . This just means that $\gamma \in [N]^{k+1}$, $\gamma(1) = \alpha$ and $\gamma(k+1) = \beta$. Then we write

$$(6.7.6) \quad (U^k)_{\alpha,\beta} = \sum_{\gamma \in \mathcal{P}_k(\alpha,\beta)} \prod_{j=1}^k u_{\gamma(j),\gamma(j+1)}.$$

Expanding each factor in the definition of S like this, we get

$$(6.7.7) \quad S = \sum_{\substack{\gamma_i^\ell \in \mathcal{P}_{k_i}(\alpha_i, \beta_i) \\ i \leq m; \ell \leq p_i}} \prod_{i=1}^m \prod_{\ell=1}^{p_i} \prod_{j=1}^{k_i} u_{\gamma_i^\ell(j), \gamma_i^\ell(j+1)}.$$

In words, we are summing over a packet of p_1 paths of length k_1 from α_1 to β_1 , a packet of p_2 paths of length k_2 from α_2 to β_2 , etc. T may similarly be expanded as

$$(6.7.8) \quad T = \sum_{\substack{\Gamma_i^\ell \in \mathcal{P}_{k'_i}(\alpha'_i, \beta'_i) \\ i \leq m'; \ell \leq p'_i}} \prod_{i=1}^{m'} \prod_{\ell=1}^{p'_i} \prod_{j=1}^{k'_i} u_{\Gamma_i^\ell(j), \Gamma_i^\ell(j+1)}.$$

To evaluate $\mathbf{E}[S\bar{T}]$, for each pair of collections $\gamma = \{\gamma_i^\ell\}$ and $\Gamma = \{\Gamma_i^\ell\}$, we must find

$$(6.7.9) \quad \mathbf{E} \left[\left(\prod_{i=1}^m \prod_{\ell=1}^{p_i} \prod_{j=1}^{k_i} u_{\gamma_i^\ell(j), \gamma_i^\ell(j+1)} \right) \left(\prod_{i=1}^{m'} \prod_{\ell=1}^{p'_i} \prod_{j=1}^{k'_i} \bar{u}_{\Gamma_i^\ell(j), \Gamma_i^\ell(j+1)} \right) \right].$$

Fix a collection of packets $\gamma_i^\ell \in \mathcal{P}_{k_i}(\alpha_i, \beta_i)$. For which collections $\Gamma_i^\ell \in \mathcal{P}_{k'_i}(\alpha'_i, \beta'_i)$ does (6.7.9) give a nonzero answer? For that to happen, the number of $u_{i,j}$ s and the number of $\bar{u}_{i,j}$ s inside the expectation must be the same (because $e^{i\theta} U \stackrel{d}{=} U$ for any $\theta \in \mathbb{R}$). Assume that this is the case.

It will be convenient to write $\gamma(i, \ell, j)$ in place of $\gamma_i^\ell(j)$. From Result 6.7.3, to get a nonzero answer in (6.7.9) we must have bijections

$$\begin{aligned} \{(i, \ell, j) : i \leq m, \ell \leq p_i, 1 \leq j \leq k_i\} &\xrightarrow{\pi} \{(i, \ell, j) : i \leq m', \ell \leq p'_i, 1 \leq j \leq k'_i\} \\ \{(i, \ell, j) : i \leq m, \ell \leq p_i, 2 \leq j \leq k_i + 1\} &\xrightarrow{\sigma} \{(i, \ell, j) : i \leq m', \ell \leq p'_i, 2 \leq j \leq k'_i + 1\} \end{aligned}$$

such that

$$\begin{aligned} (\gamma(i, \ell, j))_{i \leq m, \ell \leq p_i, 1 \leq j \leq k_i} &= (\Gamma(\pi(i, \ell, j)))_{i \leq m', \ell \leq p'_i, 1 \leq j \leq k'_i} \\ (\gamma(i, \ell, j))_{i \leq m, \ell \leq p_i, 2 \leq j \leq k_i + 1} &= (\Gamma(\sigma(i, \ell, j)))_{i \leq m', \ell \leq p'_i, 2 \leq j \leq k'_i + 1}. \end{aligned}$$

And for each such pair of bijections π, σ , we get a contribution of $\text{Wg}(N, \pi\sigma^{-1})$.

Let us call the collection of packets γ **typical**, if all the paths γ_i^ℓ are pairwise disjoint (except possibly at the initial and final points) and also non self-intersecting (again, if $\alpha_i = \beta_i$, the paths in packet i intersect themselves, but only at the end points).

If γ is typical, then it is clear that for Γ to yield a nonzero contribution, Γ must consist of exactly the same paths as γ . This forces $k_i = k'_i$ and $p_i = p'_i$ and $\alpha_i = \alpha'_i, \beta_i = \beta'_i$ for every i . If this is so, then the only pairs of bijections (π, σ) that yield a non zero contribution are those for which

- $\pi = \sigma$ (From the disjointness of the paths).
- π permutes each packet of paths among itself. In particular there are $\prod_{i=1}^m p_i!$ such permutations.

This shows that for a typical γ , the expectation in (6.7.9) is equal to

$$(6.7.10) \quad \mathbf{1}_{\Gamma=\gamma} \left(\prod_{i=1}^m p_i! \right) \text{Wg}(N, e).$$

Here $\gamma = \Gamma$ means that the two sets of paths are the same. Now suppose γ is atypical. For any fixed γ , typical or atypical, the number of Γ for which (6.7.9) is nonzero is clearly bounded uniformly by m and $p_i, k_i, i \leq m$. In particular it is independent of N . Therefore the expected value in (6.7.9) is bounded in absolute value by

$$(6.7.11) \quad C \sup_{\tau} \text{Wg}(N, \tau).$$

Now for an atypical γ , at least two of $\gamma_i^\ell(j)$, $1 \leq i \leq m$, $1 \leq \ell \leq p_i$, $2 \leq j \leq k_i$, must be equal (our definition of ‘‘typical’’ did not impose any condition on the initial and final points of the paths, which are anyway fixed throughout). Thus, if we set $r = p_1(k_1 - 1) + \dots + p_m(k_m - 1)$, then it follows that the total number of atypical γ is less than $r^2 N^{r-1}$. Since the total number of γ is precisely N^r , this also tells us that there are at least $N^r - r^2 N^{r-1}$ typical γ . Put these counts together with the contributions of each typical and atypical path, as given in (6.7.10) and (6.7.11), respectively. Note that we get nonzero contribution from typical paths only if $S = T$. Also, the total number of factors in S is $r + \sum p_i$ (this is the ‘‘ k ’’ in Result 6.7.3). Hence

$$\begin{aligned} \mathbf{E}[S\bar{T}] &= \mathbf{1}_{S=T} N^r (1 - O(1/N)) \text{Wg}(N, e) \prod_{i=1}^m p_i! + O(N^{r-1}) \sup_{\tau \in \mathcal{S}_{r+\sum p_i}} \text{Wg}(N, \tau) \\ &= \mathbf{1}_{S=T} N^{-\sum p_i} \left(\prod_{i=1}^m p_i! \right) \left(1 + O\left(\frac{1}{N}\right) \right) \end{aligned}$$

by virtue of the asymptotics of the Weingarten function, as given in Result 6.7.3.

The factor $N^{\sum p_i}$ is precisely compensated for, once we scale $(U^k)_{\alpha, \beta}$ by \sqrt{N} , as in the statement of the lemma. Since the moments of standard complex Gaussian are easily seen to be $\mathbf{E}[g^p \bar{g}^q] = p! \mathbf{1}_{p=q}$, we have shown that $\sqrt{N}(U^k)_{\alpha, \beta}$, $k \geq 1$, $\alpha, \beta \leq n$, converge to independent standard complex Gaussians. \square

Proof of Lemma 6.7.2. Consider the matrix

$$X = \begin{bmatrix} z^{-1}A & B \\ C & -(I - zV) \end{bmatrix}.$$

Then

$$\det(z^{-1}A) \det(-(I - zV) - C(z^{-1}A)^{-1}B) = \det(-(I - zV)) \det(z^{-1}A + B(I - zV)^{-1}C)$$

because both sides are equal to $\det(X)$. Factor out z to get

$$(6.7.12) \quad \det(A) \det(-I + z(V - CA^{-1}B)) = (-1)^N \det(I - zV) \det(A + zB(I - zV)^{-1}C).$$

The left hand side may be written as

$$[\det(A) \det(V - CA^{-1}B)] [\det(zI - \tilde{V})]$$

where $\tilde{V} = (V - CA^{-1}B)^{-1}$ is the lower right $N \times N$ block of U^{-1} (by well known formulas for the inverse of a block matrix). Since U is unitary, it follows that $\tilde{V} = V^*$.

Further, the quantity inside the first bracket is just $\det(U)$. Using these inferences in equation (6.7.12), we obtain

$$\frac{\det(zI - V^*)}{\det(I - zV)} = (-1)^N \det(U^*) \det(A + zB(I - zV)^{-1}C)$$

as claimed. \square

6.8. Notes

- We omitted the proof of $\mathbf{P}[e \in T] = \|I^e\|^2$. This was originally proved by Kirchoff (48) in 1847. Thomassen proves it by showing that if $e = \vec{x}\vec{y}$, then

$$I^e = \frac{1}{\text{no. spanning trees of } G} \sum_T (\chi^{f_1} + \dots + \chi^{f_k})$$

where the sum is over all spanning trees of G and f_1, \dots, f_k is the unique path in T from x to y . The Burton Pemantle theorem was proved for two edges by Brooks, Smith, Stone and Tutte (10) and in general by Burton and Pemantle (11). The joint distribution of the set of edges in T in terms of independent Bernoullis (as in Theorem 4.5.3) was found by Bapat (3), but the determinantal nature was not realized. The proof presented here is due to Benjamini, Lyons, Peres and Schramm (5), where one may also find much more about uniform spanning trees and forests, including the situation of an infinite underlying graph.

- The derivation of random matrix eigenvalue densities presented here using various decompositions and wedge products may be found in Forrester (25) and Mehta (56). We have cleaned up some proofs in places where they seemed insufficiently rigorous (see below). The eigenvalue density of a Haar unitary matrix is well known in representation theory and is called the Weyl integration formula for $\mathcal{U}(n)$. It was introduced in the random matrix context by Dyson (21) in his "three-fold classification".
- Ginibre's original proof of Theorem 4.3.10 used the diagonalization of M as $X\Delta X^{-1}$. The proof given here based on Schur decomposition follows appendix 35 of Mehta (56) who attributes the proof to Dyson. The mistake referred to in our proof is as follows. In that proof they "impose the constraints, $(V^*dV)_{j,j} = 0$ ", justifying it by saying that there are n more degrees of freedom in (V, Z, T) as compared to M , and hence, n constraints may be imposed. The freedom of choice we have in Schur decomposition is that (V, T) may be replaced by $(V\Theta, \Theta^*V\Theta)$, where $\Theta_{j,j} = e^{i\theta_j}$. This changes $(V^*dV)_{j,j}$ to $(V^*dV)_{j,j} + i d\theta_j$. It is not clear to us that this can be made to vanish. However, as our proof shows, it suffices to have $(V^*dV)_{j,j}$ be a linear combination of $(V^*dV)_{k,\ell}$, $k \neq \ell$, since upon taking wedge products, that will render the $(V^*dV)_{j,j}$ terms irrelevant.
- Theorem 4.3.13 was proved in (88) and the same proof with more details is presented in Forrester's book (25). We have supplied the same proof with additional arguments, such as the use of polar decomposition in Lemma 6.6.1, to render the proof rigorous. As explained in the text, we start with Lebesgue measure on $g^\ell(n, \mathbb{C})$, so as to make ready use of Ginibre's measure decomposition (6.3.5). A possible alternate approach would be to start with the Haar measure $\wedge_{i,j} (U^*dU)_{i,j}$ on $\mathcal{U}(n+1)$, and use Schur decomposition of X . Then we write U^*dU in terms of V^*dV , dZ $d\alpha_k$ etc., in effect carrying out the proof of (6.3.5) in the current context where we have only $(n+1)^2$ degrees of freedom, instead of $2(n+1)^2$.

When we go to the general case of $m \geq 2$, the bulk of the proof remains unchanged. However, \mathbf{b} is now an $m \times n$ matrix, and if we write the j^{th} column of \mathbf{b} as $r_j \omega_j$ in spherical co-ordinates, then the measure on U does not quite come out as a product form as it did for $m = 1$. There will be an extra step of integration over $\omega_n, \dots, \omega_1$, in that order, before we get the density of eigenvalues.

- Theorem 4.3.15 was first proved in (?). The original proof did use Lemma 6.7.1 and then constructed the functions \mathbf{f}_N . In taking the limit of $N^{m/2}\mathbf{f}_N$ however, the proof was much more complicated, working with the coefficients in the power series expansion of \mathbf{f}_N . The idea of using Lemma 6.7.2 in its place was pointed out by Katsnelson and Kirstein (personal communication, but see (44)) and simplifies the proof a great deal. It is due to appear in (45). The idea of associating to a unitary matrix U (with blocks A, B, C, V), the rational function $\det(A + zB(I - zV)^{-1}C)$, called the characteristic function, was due to Livshits (see (44) and references therein) and is extensively used in system theory.
- Lemma 6.7.1 may be read as follows. Let $e_i, i \leq m$ be any orthonormal set in \mathbb{C}^N . Then the spectral measures $\mu_{i,j}$ defined by $\langle U^k e_j, e_i \rangle = \int e^{ik\theta} d\mu_{i,j}$ converge to independent complex white noises on the unit circle. The lemma was proved in (?). It is similar in spirit to well-known results on Gaussian approximation to traces of powers of U (due to Diaconis and Shahshahani, and Evans) and to entries of U itself, in which the best known results are due to Tiefeng Jiang, who showed that the top $\sqrt{n} \times \sqrt{n}$ submatrix of $\sqrt{N}U$ is approximately Gaussian. There is a long history going back to Maxwell (see (17) and references therein).

Large Deviations for Zeros

7.1. An Offord type estimate

In this chapter we study probabilities of various unlikely events of random zeros. We begin with a large deviation estimate which is valid for arbitrary Gaussian analytic functions and then describe more specialized results. This estimate bounds the probability for a linear statistic of the zero set of a GAF to deviate from its mean. This result is taken from Sodin (78) who extended the ideas of Offord (64). Offord proved the same theorem for Nevanlinna's integrated counting function, $N(f, w, r) := \int_0^r \frac{N_f(u)}{u} du$, where $N_f(u)$ is the number of zeros of f in the disk of $\mathbb{D}(w, u)$. The Nevanlinna counting function is easily seen to be equal to $\int \log_+(r/|z-w|) dn_f(z)$, where n_f is the counting measure of zeros of f , and hence for w, r fixed, $N(f, w, r)$ is a linear statistic of the zeros of f . In this form, Sodin's extension amounts to replacing "log₊" by arbitrary smooth functions. We present this result in the case when \mathbf{f} is Gaussian.

THEOREM 7.1.1. *Let \mathbf{f} be a Gaussian analytic function on a domain $\Lambda \subset \mathbb{C}$. Let $n_{\mathbf{f}}$ denote the counting measure of the zero set of \mathbf{f} and let μ be the expectation of $n_{\mathbf{f}}$, i.e., $\mu(A) = \mathbf{E}[n_{\mathbf{f}}(A)]$. Let $\varphi \in C_c^2(\Lambda)$ be a test function with compact support in Λ . Then, for every $\lambda > 0$,*

$$(7.1.1) \quad \mathbf{P} \left[\left| \int_{\Lambda} \varphi(dn_{\mathbf{f}} - d\mu) \right| \geq \lambda \right] \leq 3e^{-\pi\lambda/\|\Delta\varphi\|_{L^1}}.$$

The following lemma is the key ingredient in Offord's approach.

LEMMA 7.1.2. *Let a be a complex Gaussian random variable with zero mean and variance σ^2 . Then, for any event E in the probability space, we have*

$$(7.1.2) \quad \left| \mathbf{E}[\mathbf{1}_E \log |a|] - \mathbf{P}(E) \log \sigma \right| \leq \mathbf{P}(E) \left[2 \log \frac{1}{\mathbf{P}(E)} + \frac{\mathbf{P}(E)}{2} \right].$$

PROOF. Upper bound: Replace a by a/σ to assume without losing generality that $\sigma = 1$. From Jensen's inequality $\mathbf{E}[\log |a|^2 | E] \leq \log \mathbf{E}[|a|^2 | E]$. Rewrite this as

$$\frac{1}{\mathbf{P}(E)} \mathbf{E}[\mathbf{1}_E \log |a|^2] \leq \log \left(\frac{1}{\mathbf{P}(E)} \mathbf{E}[|a|^2 \mathbf{1}_E] \right) \leq \log \left(\frac{1}{\mathbf{P}(E)} \right),$$

the last inequality being valid because $\mathbf{E}[|a|^2 \mathbf{1}_E] \leq 1$. Thus $\mathbf{E}[\mathbf{1}_E \log |a|] \leq -\frac{1}{2} \mathbf{P}(E) \log \mathbf{P}(E)$ which is more than we need.

Lower bound: Next we prove the lower bound in (7.1.2). Let $\log^- x = -\min\{0, \log x\}$. Then,

$$\begin{aligned} \mathbf{E}[\log |a| \mathbf{1}_E] &\geq -\mathbf{E}[\log^- |a| \mathbf{1}_E] \\ &= -\mathbf{E}[\log^- |a| \mathbf{1}_{E \cap \{|a| \leq \mathbf{P}(E)\}}] - \mathbf{E}[\log^- |a| \mathbf{1}_{E \cap \{|a| > \mathbf{P}(E)\}}]. \end{aligned}$$

The second term may be bounded below by

$$(7.1.3) \quad -\mathbf{E}[\log^- |a| \mathbf{1}_{E \cap \{|a| > \mathbf{P}(E)\}}] \geq -\mathbf{P}(E) \log \left(\frac{1}{\mathbf{P}(E)} \right).$$

while for the first term we have

$$\begin{aligned} -\mathbf{E}[\log^- |a| \mathbf{1}_{E \cap \{|a| \leq \mathbf{P}(E)\}}] &\geq -\mathbf{E}[\log^- |a| \mathbf{1}_{|a| \leq \mathbf{P}(E)}] \\ &= -\mathbf{E} \left[\mathbf{1}_{|a| \leq \mathbf{P}(E)} \int_0^1 \mathbf{1}_{s > |a|} \frac{ds}{s} \right] \\ &= -\int_0^1 \mathbf{P}[|a| < \min\{\mathbf{P}(E), s\}] \frac{ds}{s} \\ &= -\log(s) \mathbf{P}[|a| < \min\{\mathbf{P}(E), s\}] \Big|_0^1 \\ &\quad + \int_0^1 \log(s) \frac{d}{ds} \mathbf{P}[|a| < \min\{\mathbf{P}(E), s\}] ds. \end{aligned}$$

The first summand is zero. For the second, observe that $\mathbf{P}[|a| < s] = 1 - e^{-s^2}$, whence

$$\begin{aligned} -\mathbf{E}[\log^- |a| \mathbf{1}_{E \cap \{|a| \leq \mathbf{P}(E)\}}] &\geq \int_0^{\mathbf{P}(E)} \log(s) e^{-s^2} 2s ds \\ &= \frac{1}{2} \int_0^{\mathbf{P}(E)^2} \log(t) e^{-t} dt \\ &\geq \frac{1}{2} \int_0^{\mathbf{P}(E)^2} \log(t) dt \\ &= -\mathbf{P}(E)^2 \log \left(\frac{1}{\mathbf{P}(E)} \right) - \frac{1}{2} \mathbf{P}(E)^2. \end{aligned}$$

Adding this with (7.1.3) gives

$$\begin{aligned} \mathbf{E}[\log |a| \mathbf{1}_E] &\geq -\mathbf{P}(E)^2 \log \left(\frac{1}{\mathbf{P}(E)} \right) - \frac{1}{2} \mathbf{P}(E)^2 - \mathbf{P}(E) \log \left(\frac{1}{\mathbf{P}(E)} \right) \\ &\geq -2\mathbf{P}(E) \log \left(\frac{1}{\mathbf{P}(E)} \right) - \frac{1}{2} \mathbf{P}(E)^2 \end{aligned}$$

as claimed. \square

We are ready to prove Theorem 7.1.1.

PROOF. [Proof of Theorem 7.1.1] Fix $\lambda > 0$, and define the events

$$A_+ = \left\{ \int_{\Lambda} \varphi(dn_{\mathbf{f}} - d\mu) \geq \lambda \right\}, \quad A_- = \left\{ \int_{\Lambda} \varphi(dn_{\mathbf{f}} - d\mu) \leq -\lambda \right\}.$$

Using Lemma 7.1.2, we obtain upper bounds for the probabilities of A_+ and A_- and thus the deviation inequality we are after. First, consider A_+ .

Since deterministic zeros of \mathbf{f} do not contribute to the deviation at all, we exclude all of them from the domain Λ without losing generality. Then the expected measure of the zeros, μ , is absolutely continuous with respect to Lebesgue measure

and the first intensity (the Radon-Nikodym derivative of μ with respect to Lebesgue measure) is given by (2.4.8). Recall (2.4.4) to deduce that

$$\int_{\Lambda} \varphi(z)(dn_{\mathbf{f}}(z) - d\mu(z)) = \frac{1}{2\pi} \int_{\Lambda} (\Delta\varphi)(z) \left\{ \log |\mathbf{f}(z)| - \log \sqrt{K(z, z)} \right\} dm(z).$$

As φ is compactly supported and twice differentiable,

$$(7.1.4) \quad \int_{\Lambda} \mathbf{E} \left[|\Delta\varphi(z)| \left(|\log |\mathbf{f}(z)|| + |\log \sqrt{K(z, z)}| \right) \right] dm(z) < \infty.$$

This would be valid even if \mathbf{f} had some deterministic zeros, because the integrand has only logarithmic singularities. But anyway, we have assumed that there are no deterministic zeros, so the integrand is bounded almost surely. This justifies the interchange of integral and expectation below.

$$\begin{aligned} \lambda \mathbf{P}[A_+] &\leq \mathbf{E} \left[\mathbf{1}_{A_+} \cdot \frac{1}{2\pi} \int_{\Lambda} \Delta\varphi(z) \left\{ \log |\mathbf{f}(z)| - \log \sqrt{K(z, z)} \right\} dm(z) \right] \\ &= \frac{1}{2\pi} \int_{\Lambda} \Delta\varphi(z) \mathbf{E} \left[\mathbf{1}_{A_+} \left\{ \log |\mathbf{f}(z)| - \log \sqrt{K(z, z)} \right\} \right] dm(z) \\ &= \frac{1}{2\pi} \int_{\Lambda} \Delta\varphi(z) \left(\mathbf{E} [\mathbf{1}_{A_+} \log |\mathbf{f}(z)|] - \mathbf{P}[A_+] \log \sqrt{K(z, z)} \right) dm(z) \end{aligned}$$

Applying Lemma 7.1.2 to estimate the quantity inside the brackets, we get

$$\begin{aligned} \lambda \mathbf{P}[A_+] &\leq \frac{1}{2\pi} \int_{\Lambda} |\Delta\varphi(z)| \mathbf{P}[A_+] \left(2|\log \mathbf{P}[A_+]| + \frac{\mathbf{P}(A_+)}{2} \right) dm(z) \\ &\leq \frac{1}{2\pi} \mathbf{P}[A_+] \left(2|\log \mathbf{P}[A_+]| + \frac{1}{2} \right) \|\Delta\varphi\|_{L^1}. \end{aligned}$$

This gives,

$$\mathbf{P}[A_+] \leq e^{-\frac{\pi\lambda}{\|\Delta\varphi\|_{L^1}} + \frac{1}{4}}.$$

The same estimate holds for A_- and the theorem follows because $2e^{\frac{1}{4}} < 3$. \square

7.2. Hole probabilities

One quantity of interest that is informative about the “rigidity” of a point process in the plane is the decay of **hole probability**, i.e., the probability that a disk of radius r contains no points, as $r \rightarrow \infty$. Before posing this question for zeros of Gaussian analytic functions, we compute the hole probabilities for several other point processes. Here below, $n(r)$ will denote the number of points in the disk of radius r centered at the origin. The center does not matter if the point process is translation invariant.

- **Poisson process in the plane with constant intensity λ :** By definition, the number of points $n(r)$ in a disk of radius r has Poisson distribution with mean $\lambda\pi r^2$. Therefore

$$(7.2.1) \quad \mathbf{P}[n(r) = 0] = e^{-\lambda\pi r^2}.$$

- **Perturbed lattice:** Let $a_{k,\ell}$ be i.i.d. $N_{\mathbb{C}}(0,1)$ random variables, for $(k,\ell) \in \mathbb{Z}^2$. Then let \mathcal{X} be the point process $\{(k,\ell) + a_{k,\ell}\}$. In this case, for the event $\{n(r) = 0\}$ to occur, we must have $|(k,\ell) + a_{k,\ell}| > r$ for every k,ℓ .

For $k^2 + \ell^2 < \frac{r^2}{4}$, this implies that $|a_{k,\ell}|^2 > \frac{r^2}{4}$. Since there are more than $\frac{\pi(1-\epsilon)}{4}r^2$ such pairs (k,ℓ) , (for any $\epsilon > 0$, this is true for large enough r), we see that for some C_1

$$\mathbf{P}[n(r) = 0] \leq e^{-C_1 r^4}.$$

On the other hand, it is easy to see that that the event $\mathcal{A} = \{|(k,\ell) + a_{k,\ell}| > r \text{ for every } k^2 + \ell^2 > 2r^2\}$ has a positive probability at least $C' > 0$, where C' is independent of r (in fact we can let C' go to 1 as $r \rightarrow \infty$). Moreover, if $k^2 + \ell^2 \leq 2r^2$, then $\mathbf{P}[|(k,\ell) + a_{k,\ell}| > r] \geq \mathbf{P}[|a_{k,\ell}| > 4r] = e^{-16r^2}$. Therefore

$$\begin{aligned} \mathbf{P}[n(r) = 0] &\geq \mathbf{P}[\mathcal{A}] \prod_{k^2 + \ell^2 \leq 2r^2} \mathbf{P}[|a_{k,\ell}| > 4r] \\ &\geq C' e^{-C_2 r^4}. \end{aligned}$$

The interested reader may try to find the sharp constant in the exponent.

- **Ginibre ensemble:** For the infinite Ginibre ensemble, we saw the result of Kostlan in Theorem 4.7.3 that the set of absolute values of the points has the same distribution as $\{R_1, R_2, \dots\}$, where R_k^2 has distribution $\text{Gamma}(k, 1)$ and all the R_k s are independent. Therefore

$$\mathbf{P}[n(r) = 0] = \prod_{k=1}^{\infty} \mathbf{P}[R_k^2 > r^2].$$

The moment generating function of R_k^2 exists for $\theta < 1$ and yields

$$\begin{aligned} \mathbf{P}[R_k^2 > r^2] &\leq e^{-\theta r^2} \mathbf{E}[e^{\theta R_k^2}] \\ &= e^{-\theta r^2} (1 - \theta)^{-k}. \end{aligned}$$

For $k < r^2$, the bound is optimized for $\theta = 1 - \frac{k}{r^2}$. This gives (we write as if r^2 is an integer. This is hardly essential).

$$\begin{aligned} \mathbf{P}[n(r) = 0] &\leq \prod_{k=1}^{r^2} \mathbf{P}[R_k^2 > r^2] \\ &\leq \prod_{k=1}^{r^2} e^{-(1 - \frac{k}{r^2})r^2 - k \log(\frac{k}{r^2})} \\ &= e^{-\frac{1}{2}r^2(r^2 - 1) - r^4 \int_0^1 x \log(x) dx + O(r^2 \log r)} \\ &= e^{-\frac{1}{4}r^4(1 + o(1))}. \end{aligned}$$

Next we want to get a lower bound for $\prod_{k=1}^{\infty} \mathbf{P}[R_k^2 > r^2]$. Recall that

$$\mathbf{P}[\text{Gamma}(n, 1) > \lambda] = \mathbf{P}[\text{Poisson}(\lambda) < n].$$

Therefore,

$$\begin{aligned} \mathbf{P}[R_k^2 > r^2] &= \mathbf{P}[\text{Poisson}(r^2) \leq k - 1] \\ &\geq e^{-r^2} \frac{r^{2(k-1)}}{(k-1)!}. \end{aligned}$$

Use this inequality for $k \leq r^2$ to obtain,

$$\begin{aligned}
\prod_{k=1}^{r^2} \mathbf{P}[R_k^2 > r^2] &\geq \prod_{k=1}^{r^2} e^{-r^2} \frac{r^{2(k-1)}}{(k-1)!} \\
&= \exp\{-r^4 + \sum_{k < r^2} k \log(r^2) - \log(k!)\} \\
&= \exp\{-r^4 + \sum_{k < r^2} k \log(r^2) - \sum_{k < r^2} (r^2 - k) \log(k)\} \\
&= \exp\{-r^4 + \sum_{k < r^2} (r^2 - k) \log(r^2) - \sum_{k < r^2} (r^2 - k) \log(k)\} \\
&= \exp\{-r^4 - \sum_{k < r^2} (r^2 - k) \log\left(\frac{k}{r^2}\right)\}.
\end{aligned}$$

As before,

$$\begin{aligned}
\sum_{k < r^2} (r^2 - k) \log\left(\frac{k}{r^2}\right) &= r^4 \int_0^1 (1-x) \log(x) dx + O(r^2 \log r) \\
&= -\frac{3}{4} r^4 + O(r^2 \log r).
\end{aligned}$$

This yields

$$(7.2.2) \quad \prod_{k=1}^{r^2} \mathbf{P}[R_k^2 > r^2] \geq e^{-\frac{r^4}{4} + O(r^2 \log r)}.$$

Since $\mathbf{P}[\text{Poisson}(\lambda) > \lambda] \rightarrow \frac{1}{2}$ as $\lambda \rightarrow \infty$, it follows that for large enough r , for any $k > r^2$, we have $\mathbf{P}[R_k^2 > r^2] \geq \frac{1}{4}$. Therefore, for large enough r , we have

$$(7.2.3) \quad \prod_{k=r^2+1}^{2r^2} \mathbf{P}[R_k^2 > r^2] \geq e^{-r^2 \log(4)}.$$

For large enough r , with probability at least $\frac{1}{2}$, the event $\{R_k^2 > r^2, \text{ for every } k > 2r^2\}$ occurs. To see this, recall that the large deviation principle (Cramer's bound) for exponential random variables with mean 1 gives

$$\mathbf{P}[R_k^2 < \frac{k}{2}] \leq e^{-ck},$$

for a constant c independent of k . Therefore, for large r

$$\sum_{k > 2r^2} \mathbf{P}[R_k^2 < r^2] < \frac{1}{2}.$$

Then,

$$(7.2.4) \quad \prod_{k=2r^2+1}^{\infty} \mathbf{P}[R_k^2 > r^2] \geq \frac{1}{2}.$$

From (7.2.2), (7.2.3) and (7.2.4) we get

$$\prod_{k=1}^{\infty} \mathbf{P}[R_k^2 > r^2] \geq e^{-\frac{1}{4}r^4 + O(r^2 \log r)}.$$

Thus we have proved

PROPOSITION 7.2.1. *For the Ginibre ensemble, $\frac{1}{r^4} \log \mathbf{P}[n(r) = 0] \rightarrow -\frac{1}{4}$, as $r \rightarrow \infty$.*

7.2.1. Hole probability for the planar Gaussian analytic function. Coming back to zeros of Gaussian analytic functions, Theorem 7.1.1 provides as an easy corollary, an upper bound for the hole probability for any Gaussian analytic function \mathbf{f} on a domain Λ . As we shall see, this estimate is far from optimal in general.

Firstly apply Theorem 7.1.1 with $\lambda = \int \varphi d\mu$, where μ is the first intensity measure, to get

$$(7.2.5) \quad \mathbf{P}\left(\int_{\Lambda} \varphi dn_{\mathbf{f}} = 0\right) \leq 3 \exp\left\{-\frac{\pi}{\|\Delta\varphi\|_{L^1}} \int \varphi d\mu\right\}.$$

Now let $D_R \subset \Lambda$ be a disk of radius R , and let D_r , $r < R$, be a concentric disk of a smaller radius r . Without loss of generality, let the common center be 0.

Fix a smooth function $h : \mathbb{R} \rightarrow [0, 1]$ that equals 1 on $(-\infty, 0]$ and equals 0 on $[1, \infty)$ and $0 < h(x) < 1$ for $x \in (0, 1)$. Then define a test-function $\varphi : \Lambda \rightarrow \mathbb{R}$ by $\varphi(z) = h\left(\frac{|z|-r}{R-r}\right)$. Clearly, φ vanishes outside D_R and equals 1 on D_r . Furthermore, with $|z| = t$, we have

$$(7.2.6) \quad \frac{\partial^p \varphi(z)}{\partial t^p} = (R-r)^{-p} h^{(p)}\left(\frac{t-r}{R-r}\right).$$

For a radial function, it is easy to see that $\Delta\varphi(z) = \left(\frac{\partial^2}{\partial t^2} + \frac{1}{t} \frac{\partial}{\partial t}\right)\varphi(t)$. Thus,

$$\begin{aligned} \|\Delta\varphi\|_{L^1} &= 2\pi \int_r^R \left| t \frac{\partial^2 \varphi(t)}{\partial t^2} + \frac{\partial \varphi(t)}{\partial t} \right| dt \\ &\leq 2\pi \int_0^1 |h'(t)| dt + \frac{2\pi R}{R-r} \int_0^1 |h''(t)| dt \\ &\leq C \frac{R+r}{R-r}, \end{aligned}$$

for a constant C that depends only on h . Then it follows from (7.2.5) that

COROLLARY 7.2.2.

$$\mathbf{P}(n_{\mathbf{f}}(R) = 0) \leq 3 \exp\left[-c\mu(D_r) \frac{R-r}{R+r}\right], \quad \text{for any } 0 < r < R.$$

We now focus our attention on the planar GAF,

$$(7.2.7) \quad \mathbf{f}(z) = \sum_{k=0}^{\infty} a_k \frac{z^k}{\sqrt{k!}},$$

where a_k are i.i.d. $\sim N_{\mathbb{C}}(0, 1)$, and consider the hole probability $\mathbf{P}(n_{\mathbf{f}}(r) = 0)$. As a consequence of Corollary 7.2.2 we get $\mathbf{P}(n_{\mathbf{f}}(r) = 0) \leq \exp(-cr^2)$. However, this is the same asymptotic rate of decay that we obtained for the Poisson process in (7.2.1). As a glance at Figure 1 suggests, the zeros should at least exhibit some local repulsion. In fact, the local repulsion for the zeros is more like that of the Ginibre ensemble. Hence we might expect the hole probability of the zeros to decay like $\exp\{-cr^4\}$, as it does for the Ginibre case. The next result, due to Sodin and Tsirelson (81), shows that this is indeed the case.

THEOREM 7.2.3 (Sodin and Tsirelson). *There exist positive constants c and C such that for all $r \geq 1$, we have*

$$\exp(-Cr^4) \leq \mathbf{P}(n_{\mathbf{f}}(r) = 0) \leq \exp(-cr^4).$$

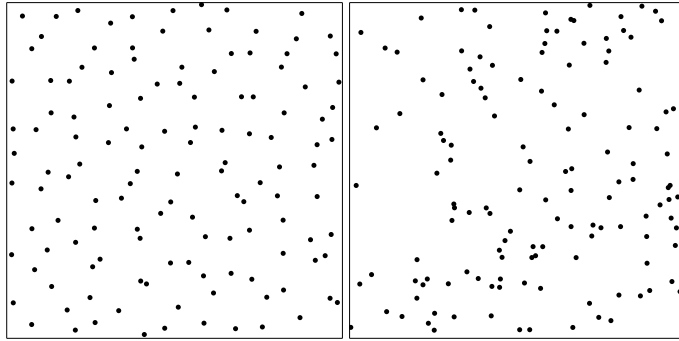


FIGURE 1. The zero set of \mathbf{f} (left) and a Poisson point process with the same intensity.

In this section, by c and C we denote various positive numerical constants whose values can be different at each occurrence.

REMARK 7.2.4. Theorem 7.2.3 above shows that the hole probability for the zeros of the planar GAF \mathbf{f} decays exponentially in the square of the area of the hole, just as for the perturbed lattice. This motivates a question as to whether the zeros of \mathbf{f} can in fact be thought of as a perturbed lattice? Obviously we do not expect the zeros to be exactly distributed as the lattice with i.i.d. perturbations. One way to make the question precise is whether there is a *matching* (this term will be precisely defined in chapter 8) between the zeros of \mathbf{f} and the lattice in such a manner that the distance between matched pairs has small tails. Sodin and Tsirelson showed that there is indeed a matching with sub-Gaussian tails that is also invariant under translations by \mathbb{Z}^2 . In chapter 8 we shall discuss this and the closely related question of translation invariant transportation between Lebesgue measure and the counting measure on zeros.

In addition to hole probability, one may ask for a large deviation estimate for $n(r)$ as $r \rightarrow \infty$. Sodin and Tsirelson proved such an estimate (without sharp constants). In fact this deviation inequality is used in proving the upper bound on hole probability, but it is also of independent interest.

THEOREM 7.2.5. *For any $\delta > 0$, there exists $c(\delta) > 0$, $r(\delta) > 0$ such that for any $r \geq r(\delta)$,*

$$(7.2.8) \quad \mathbf{P} \left(\left| \frac{n_{\mathbf{f}}(r)}{r^2} - 1 \right| \geq \delta \right) \leq \exp\{-c(\delta)r^4\}.$$

In what follows, by $c(\delta)$ we denote various positive constants which depend on δ only and which may change from one occurrence to the next. A natural and very interesting question here is that of finding sharp constants in the exponents in Theorem 7.2.3 and Theorem 7.2.5. See the notes at the end of the chapter for a discussion of some recent developments in this direction.

PROOF. [Theorem 7.2.3] The lower bound is considerably easier than the upper bound. This because one can easily find conditions on the coefficients that are sufficient to force the event under question (a hole of radius r) to occur but much harder to find a necessary one.

Lower bound There will be no zeros in $D(0, r)$ if the constant coefficient a_0 dominates the rest of the series for \mathbf{f} on the disk of radius r , that is, if

$$|a_0| > \left| \sum_{k=1}^{\infty} a_k \frac{z^k}{\sqrt{k!}} \right| \quad \forall |z| \leq r.$$

For the series on the right hand side, namely $\mathbf{f}(z) - a_0$, to be small all over the disk $D(0, r)$, we shall impose some stringent conditions on the first few coefficients. The later ones are easily taken care of by the rapidly decreasing factor $z^k/\sqrt{k!}$. For, if $|z| \leq r$, then

$$\begin{aligned} \left| \sum_{k=m+1}^{\infty} a_k \frac{z^k}{\sqrt{k!}} \right| &\leq \sum_{k=m+1}^{\infty} |a_k| \frac{r^k}{\sqrt{k!}} \\ &\leq \sum_{k=m+1}^{\infty} |a_k| \left(\frac{er^2}{k} \right)^{\frac{k}{2}} \end{aligned}$$

by the elementary inequality $k! \geq k^k e^{-k}$. Choose $m = e(1+\delta)^2 r^2$ where $\delta > 0$. Then the factors in the series above are bounded by $(1+\delta)^{-k}$. Define the event

$$A := \{|a_k| < k \quad \forall k > m\}.$$

If the event A occurs then for sufficiently large r we have

$$(7.2.9) \quad \left| \sum_{k=m+1}^{\infty} a_k \frac{z^k}{\sqrt{k!}} \right| \leq \sum_{k=m+1}^{\infty} \frac{k}{(1+\delta)^k} \leq \frac{1}{2}.$$

Now consider

$$\begin{aligned} \left| \sum_{k=1}^m a_k \frac{z^k}{\sqrt{k!}} \right|^2 &\leq \left(\sum_{k=1}^m |a_k|^2 \right) \left(\sum_{k=1}^m \frac{r^{2k}}{k!} \right) \\ &\leq e^{r^2} \sum_{k=1}^m |a_k|^2. \end{aligned}$$

Define the event

$$B := \left\{ |a_k|^2 < e^{-r^2} \frac{1}{4m} \quad \forall 1 \leq k \leq m \right\}.$$

If B occurs, then it follows that

$$(7.2.10) \quad \left| \sum_{k=1}^m a_k \frac{z^k}{\sqrt{k!}} \right| \leq \frac{1}{2}.$$

We also define a third event $C := \{|a_0| > 1\}$. If A, B, C all occur, then by (7.2.9) and (7.2.10) we see that $n_{\mathbf{f}}(r) = 0$. Recall that $|a_k|^2$ are independent exponentials to deduce that for r sufficiently large, we have (with $m = e(1+\delta)^2 r^2$),

$$\begin{aligned} \mathbf{P}(A) &\geq 1 - \sum_{k=m+1}^{\infty} e^{-k^2} \geq \frac{1}{2}, \\ \mathbf{P}(B) &= (1 - \exp\{-e^{-r^2} (4m)^{-1}\})^m \geq e^{-mr^2} (8m)^{-m}, \\ \mathbf{P}(C) &= e^{-1}. \end{aligned}$$

In estimating $\mathbf{P}(B)$, we used the simple fact that $1 - e^{-x} \geq \frac{x}{2}$ for $x \in [0, 1]$. Thus

$$\begin{aligned} \mathbf{P}(n_{\mathbf{f}}(r) = 0) &\geq \mathbf{P}(A) \cdot \mathbf{P}(B) \cdot \mathbf{P}(C) \\ &\geq \frac{1}{2} e^{-1} e^{-mr^2} (8m)^{-m} \\ &= e^{-\alpha r^4(1+o(1))} \end{aligned}$$

for any $\alpha > e$. This is the desired lower bound.

Upper bound The upper bound is much harder but is a direct corollary of Theorem 7.2.5 which is proved next. Unlike in the lower bound we do not have a good numerical value of the exponent here. \square

7.2.2. Proof of Theorem 7.2.5. Recall Jensen's formula (see (1), chapter 5, section 3.2 or (71), section 15.16)

$$(7.2.11) \quad \log |\mathbf{f}(0)| + \sum_{\substack{\alpha \in \mathbf{f}^{-1}(0) \\ |\alpha| < r}} \log \left(\frac{r}{|\alpha|} \right) = \int_0^{2\pi} \log |\mathbf{f}(r e^{i\theta})| \frac{d\theta}{2\pi}.$$

Observe that the summation on the left hand side may also be written as $\int_0^r \frac{n(t)}{t} dt$.

Fix $\kappa = 1 + \delta$ and observe that

$$\int_{\frac{r}{\kappa}}^r \frac{n(t)}{t} dt \leq n(r) \log \kappa \leq \int_r^{\kappa r} \frac{n(t)}{t} dt.$$

Thus (7.2.11) leads to the following upper and lower bounds for $n(r)$ in terms of the logarithmic integral of \mathbf{f} .

$$(7.2.12) \quad n(r) \log \kappa \leq \int_0^{2\pi} (\log |\mathbf{f}(\kappa r e^{i\theta})| - \log |\mathbf{f}(r e^{i\theta})|) \frac{d\theta}{2\pi}.$$

$$(7.2.13) \quad n(r) \log \kappa \geq \int_0^{2\pi} (\log |\mathbf{f}(r e^{i\theta})| - \log |\mathbf{f}(\kappa^{-1} r e^{i\theta})|) \frac{d\theta}{2\pi}.$$

Therefore the theorem immediately follows from Lemma 7.2.6 below. Indeed, to deduce Theorem 7.2.5, apply this lemma to say that with probability at least $1 - e^{c(\delta^2)r^4}$, we have

$$\left(\frac{1}{2} - \delta^2 \right) \kappa^2 r^2 \leq \int_0^{2\pi} \log |\mathbf{f}(\kappa r e^{i\theta})| \leq \left(\frac{1}{2} + \delta^2 \right) \kappa^2 r^2, \quad \left(\frac{1}{2} - \delta^2 \right) r^2 \leq \int_0^{2\pi} \log |\mathbf{f}(r e^{i\theta})| \leq \left(\frac{1}{2} + \delta^2 \right) r^2.$$

Without losing generality we assume that $\delta < 1$ so that $\delta - \frac{\delta^2}{2} \leq \log \kappa \leq \delta$. Then, under the above events, apply the upper bound (7.2.12) on $n(r)$ to get

$$\frac{n(r)}{r^2} \leq \frac{1}{\log \kappa} \left\{ \left(\frac{1}{2} + \delta^2 \right) \kappa^2 - \left(\frac{1}{2} - \delta^2 \right) \right\} \leq 1 + C\delta.$$

Similarly from (7.2.13) we get $n(r) \geq (1 - C\delta)r^2$. Thus the theorem follows.

LEMMA 7.2.6. *For any $\delta > 0$, there exists $c(\delta) > 0, r(\delta) > 0$ such that for any $r \geq r(\delta)$,*

$$\mathbf{P} \left[\int_0^{2\pi} \log |\mathbf{f}(re^{i\theta})| \frac{d\theta}{2\pi} \geq \left(\frac{1}{2} + \delta \right) r^2 \right] \leq e^{-c(\delta)e^{\delta r^2}}.$$

$$\mathbf{P} \left[\int_0^{2\pi} \log |\mathbf{f}(re^{i\theta})| \frac{d\theta}{2\pi} \leq \left(\frac{1}{2} - \delta \right) r^2 \right] \leq e^{-c(\delta)r^4}.$$

7.2.3. Proof of Lemma 7.2.6. Easier than the bounds for the logarithmic integral in Lemma 7.2.6 is the following analogous lemma for the maximum of $\log |\mathbf{f}|$ in a large disk. The lower and upper bounds for the maximum will be used in proving the lower and upper bounds for the logarithmic integral, respectively.

LEMMA 7.2.7. *Let \mathbf{f} be the planar Gaussian analytic function and let $M(r, \mathbf{f}) = \max_{|z| \leq r} |\mathbf{f}(z)|$. Given any $\delta > 0$, there exists $c(\delta) > 0, r(\delta) > 0$ such that for any $r \geq r(\delta)$,*

$$\mathbf{P} \left[\log M(r, \mathbf{f}) \geq \left(\frac{1}{2} + \delta \right) r^2 \right] \leq e^{-c(\delta)e^{\delta r^2}}.$$

$$\mathbf{P} \left[\log M(r, \mathbf{f}) \leq \left(\frac{1}{2} - \delta \right) r^2 \right] \leq e^{-c(\delta)r^4}.$$

PROOF. **Upper bound:** For any z with $|z| = r$, we have for any m ,

$$\begin{aligned} |\mathbf{f}(z)| &\leq \sum_{k \leq m} |a_k| \frac{r^k}{\sqrt{k!}} + \sum_{k > m} |a_k| \frac{r^k}{\sqrt{k!}} \\ &\leq \left(\sum_{k \leq m} |a_k|^2 \right)^{\frac{1}{2}} e^{\frac{1}{2}r^2} + \sum_{k > m} |a_k| \frac{r^k}{\sqrt{k!}}. \end{aligned}$$

Now set $m = 4er^2$. Suppose the following events occur.

$$(7.2.14) \quad |a_k| \leq \begin{cases} e^{\frac{2}{3}\delta r^2} & \text{for } k \leq m \\ 2^{\frac{k}{2}} & \text{for } k > m. \end{cases}$$

Then it follows that (use the inequality $k! > k^k e^{-k}$ in the second summand)

$$\begin{aligned} \max\{|\mathbf{f}(z)| : |z| = r\} &\leq \sqrt{m} e^{\frac{2}{3}\delta r^2} e^{\frac{1}{2}r^2} + \sum_{k > m} 2^{-\frac{k}{2}} \\ &\leq \sqrt{2e} r \exp \left\{ \left(\frac{1}{2} + \frac{2\delta}{3} \right) r^2 \right\} + 1 \\ &\leq \exp \left\{ \left(\frac{1}{2} + \delta \right) r^2 \right\}. \end{aligned}$$

Thus if (7.2.14) occurs, then $\log M(r, \mathbf{f}) \leq \left(\frac{1}{2} + \delta \right) r^2$ as desired. Now the probability of the events in (7.2.14) is

$$\begin{aligned} \mathbf{P}[(7.2.14)] &= \left(1 - \exp\{-e^{\frac{4}{3}\delta r^2}\} \right)^{4er^2} \prod_{k > m} (1 - e^{-2^k}) \\ &\geq 1 - \exp\{-e^{\delta r^2}\} \end{aligned}$$

for sufficiently large r . This proves the upper bound.

Lower bound: Suppose now that

$$(7.2.15) \quad \log M(r, \mathbf{f}) \leq \left(\frac{1}{2} - \delta\right)r^2.$$

Recall the Cauchy integral formula

$$\mathbf{f}^{(k)}(0) = k! \int_0^{2\pi} \frac{\mathbf{f}(re^{i\theta})}{r^k e^{ik\theta}} \frac{d\theta}{2\pi}.$$

We use this and Stirling's formula to show that the coefficients a_k must be unusually small, which again happens with very low probability.

$$\begin{aligned} |a_k| &= \frac{|\mathbf{f}^{(k)}(0)|}{\sqrt{k!}} \\ &\leq \sqrt{k!} \frac{M(r, \mathbf{f})}{r^k} \\ &\leq Ck^{1/4} \exp\left(\frac{k}{2} \log k - \frac{k}{2} + \left(\frac{1}{2} - \delta\right)r^2 - k \log r\right). \end{aligned}$$

Observe that the exponent equals

$$\frac{k}{2} \left((1 - 2\delta) \frac{r^2}{k} - \log \frac{r^2}{k} - 1 \right).$$

We note that $(1 - 2\delta) \frac{r^2}{k} - \log \frac{r^2}{k} - 1 < -\delta$ when r^2/k is close enough to 1. Whence, for $(1 - \epsilon)r^2 \leq k \leq r^2$,

$$|a_k| \leq Ck^{1/4} \exp\left(-\frac{k\delta}{2}\right).$$

The probability of this event is $\leq \exp\{-c(\delta)k\}$. Since a_k are independent, multiplying these probabilities, we see that

$$\exp\left(-c(\delta) \sum_{(1-\epsilon)r^2 \leq k \leq r^2} k\right) = \exp(-c_1(\delta)r^4)$$

is an upper bound for the probability that event (7.2.15) occurs. \square

Now we return to the proof of Lemma 7.2.6 which is the last thing needed to complete the proof of Theorem 7.2.5 and hence of Theorem 7.2.3 also.

PROOF. [Proof of Lemma 7.2.6]

Upper bound: We use the trivial bound

$$(7.2.16) \quad \int_0^{2\pi} \log |\mathbf{f}(re^{i\theta})| \frac{d\theta}{2\pi} \leq \log M(r, \mathbf{f}).$$

From Lemma 7.2.7, we get

$$\mathbf{P} \left[\int_0^{2\pi} \log |\mathbf{f}(re^{i\theta})| \frac{d\theta}{2\pi} \geq \left(\frac{1}{2} + \delta\right)r^2 \right] \leq \exp\{-c(\delta)e^{\delta r^2}\}$$

which is what we aimed to prove.

Lower bound:

LEMMA 7.2.8. *Given $\delta > 0$ there exists $r(\delta) > 0$, $c(\delta) > 0$ such that if $r \geq r(\delta)$, then for any z_0 with $\frac{1}{2}r \leq |z_0| \leq r$,*

$$\mathbf{P} \left[\exists a \in z_0 + \delta r \mathbb{D} \text{ with } \log |\mathbf{f}(a)| > \left(\frac{1}{2} - 3\delta \right) |z_0|^2 \right] \geq 1 - e^{-c(\delta)r^4}.$$

PROOF. The random potential $\log |\mathbf{f}(z)| - \frac{1}{2}|z|^2$ is shift-invariant in distribution (a direct consequence of (2.3.10)). In proving the lower bound for the potential in Lemma 7.2.7, in fact we proved the following

$$\mathbf{P} \left(\max_{z \in r \mathbb{D}} \log |\mathbf{f}(z)| - \frac{1}{2}|z|^2 \leq -\delta r^2 \right) \leq \exp\{-c(\delta)r^4\}.$$

Apply the same to the function $z \mapsto \log |\mathbf{f}(z_0 + z)| - \frac{1}{2}|z_0 + z|^2$ on $\delta r \mathbb{D}$. We get

$$\mathbb{P} \left(\max_{z \in \delta r \mathbb{D}} \log |\mathbf{f}(z_0 + z)| - \frac{1}{2}|z_0 + z|^2 \leq -\delta(\delta r)^2 \right) \leq \exp\{-c(\delta)r^4\}$$

for a different $c(\delta)$. Since $|z_0| \geq r/2$, if $|z| \leq \delta r$, then we get $\frac{1}{2}|z_0 + z|^2 \geq \frac{1}{2}|z_0|^2(1 - 2\delta)^2$ whence, outside an exceptional set of probability at most $\exp\{-c(\delta)r^4\}$, there is some $a \in z_0 + \delta r \mathbb{D}$ such that $\log |\mathbf{f}(a)| \geq (\frac{1}{2} - 3\delta)|z_0|^2$. \square

Now, set $\kappa = 1 - \delta^{1/4}$, take $N = [2\pi\delta^{-1}]$, and consider N disks with centers at equally spaced points on the circle of radius κr . That is, we take the centers to be $z_j = \kappa r e^{2\pi i j/N}$ and the disks to be $z_j + \delta r_j \mathbb{D}$, for $j \leq N$. Lemma 7.2.8 implies that outside an exceptional set of probability $N \exp(-c(\delta)r^4) = \exp(-c_1(\delta)r^4)$, we can choose N points $a_j \in z_j + \delta r \mathbb{D}$ such that

$$\log |\mathbf{f}(a_j)| \geq \left(\frac{1}{2} - 3\delta \right) |z_j|^2 \geq \left(\frac{1}{2} - C\delta^{1/4} \right) r^2.$$

Let $P(z, a)$ be the Poisson kernel for the disk $r \mathbb{D}$, $|z| = r$, $|a| < r$. We set $P_j(z) = P(z, a_j)$. For any analytic function f , the function $\log |f|$ is subharmonic, and hence if $\overline{\mathbb{D}(0, r)}$ is inside the domain of analyticity, then $\log |f(a)| \leq \int_0^{2\pi} \log |f(re^{i\theta})| P(re^{i\theta}, a) \frac{d\theta}{2\pi}$ for any $a \in \mathbb{D}(0, r)$. Applying this to \mathbf{f} and each a_j we get

$$\begin{aligned} \left(\frac{1}{2} - C\delta^{1/4} \right) r^2 &\leq \frac{1}{N} \sum_{j=0}^{N-1} \log |\mathbf{f}(a_j)| \\ &\leq \int_0^{2\pi} \left(\frac{1}{N} \sum_{j=0}^{N-1} P_j(re^{i\theta}) \right) \log |\mathbf{f}(re^{i\theta})| \frac{d\theta}{2\pi} \\ &= \int_0^{2\pi} \log |\mathbf{f}(re^{i\theta})| \frac{d\theta}{2\pi} + \int_0^{2\pi} \left(\frac{1}{N} \sum_{j=0}^{N-1} P_j(re^{i\theta}) - 1 \right) \log |\mathbf{f}(re^{i\theta})| \frac{d\theta}{2\pi}. \end{aligned}$$

The two claims 7.2.9, 7.2.10 below, immediately imply that the second integral is bounded in absolute value by $10C_0\sqrt{\delta}r^2$, outside an exceptional set of probability $\exp(-cr^4)$. This in turn shows that outside the exceptional set, the first integral

$$\int_0^{2\pi} \log |\mathbf{f}(re^{i\theta})| \frac{d\theta}{2\pi} \geq \left(\frac{1}{2} - C\delta^{1/4} - 10C_0\sqrt{\delta} \right) r^2$$

which is exactly the lower bound we are trying to prove.

Let \mathbb{T} denote the unit circle $\{|z| = 1\}$.

CLAIM 7.2.9.

$$\max_{z \in r\mathbb{T}} \left| \frac{1}{N} \sum_{j=0}^{N-1} P_j(z) - 1 \right| \leq C_0 \delta^{1/2}.$$

CLAIM 7.2.10.

$$\int_0^{2\pi} |\log |\mathbf{f}(re^{i\theta})|| \frac{d\theta}{2\pi} \leq 10r^2$$

outside an exceptional set of probability $\exp(-cr^4)$.

Proof of Claim 7.2.9. We start by recalling that for $\int_0^{2\pi} P(re^{i\theta}, a) \frac{d\theta}{2\pi} = 1$ for any $a \in \mathbb{D}(0, r)$. Split the circle $\kappa r\mathbb{T}$ into a union of N disjoint arcs I_j of equal angular measure $\mu(I_j) = \frac{1}{N}$ centered at z_j . Then if $|z| = r$,

$$1 = \frac{1}{N} \sum_{j=0}^{N-1} P(z, a_j) + \sum_{j=0}^{N-1} \int_{I_j} (P(z, a) - P(z, a_j)) |da|$$

where the last integral is with respect to the normalized angular measure on I_j . Also, by elementary and well known estimates on the Poisson kernel (consult (1) or (71))

$$\begin{aligned} |P(z, a) - P(z, a_j)| &\leq \max_{a \in I_j} |a - a_j| \cdot \max_{z, a} |\nabla_a P(z, a)| \\ &\leq C_1 \delta r \cdot \frac{C_2 r}{(r - |a|)^2} = \frac{C_0 \delta}{\delta^{1/2}} = C_0 \delta^{1/2}, \end{aligned}$$

proving the claim. \square

Proof of Claim 7.2.10. By Lemma 7.2.8, we know that if r is large enough, then outside an exceptional set of probability $\exp(-cr^4)$, there is a point $a \in \frac{1}{2}r\mathbb{T}$ such that $\log |\mathbf{f}(a)| \geq 0$. Fix such a point a . Then

$$0 \leq \int_0^{2\pi} P(re^{i\theta}, a) \log |\mathbf{f}(re^{i\theta})| \frac{d\theta}{2\pi},$$

and hence

$$\int_0^{2\pi} P(re^{i\theta}, a) \log^- |\mathbf{f}(re^{i\theta})| \frac{d\theta}{2\pi} \leq \int_0^{2\pi} P(re^{i\theta}, a) \log^+ |\mathbf{f}(re^{i\theta})| \frac{d\theta}{2\pi}.$$

It remains to recall that for $|z| = r$ and $|a| = \frac{1}{2}r$,

$$\frac{1}{3} \leq P(z, a) \leq 3,$$

and that

$$\int_0^{2\pi} \log^+ |\mathbf{f}(re^{i\theta})| \frac{d\theta}{2\pi} \leq \log M(r, \mathbf{f}) \leq r^2$$

(provided we are outside the exceptional set). Hence

$$\int_0^{2\pi} \log^- |\mathbf{f}(re^{i\theta})| \frac{d\theta}{2\pi} \leq 9r^2$$

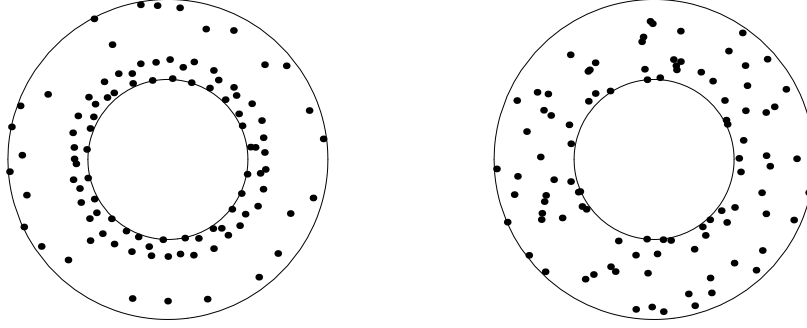


FIGURE 2. The zero set of $\mathbf{f}(\cdot, t)$ (left) and $Z_{pl}(t)$, conditioned to have a hole of radius five.

and

$$\int_0^{2\pi} \left| \log |\mathbf{f}(re^{i\theta})| \right| \frac{d\theta}{2\pi} \leq 10r^2,$$

proving the claim. □

□

7.3. Notes

- **Sharp constants:** Recently, Alon Nishry (63) has found a way to get sharp constants in the exponent for hole probability. In particular, for the planar GAF, he shows that $r^{-4} \log \mathbf{P}(n_{\mathbf{f}}(r) = 0) \rightarrow -\frac{3e^2}{4}$. In the same paper, he finds asymptotics for hole probabilities for zeros of a wide class of random entire function.
- **Time dependent processes:** We noted above (Remark 7.2.4) that the hole probability for the perturbed lattice

$$Z_{pl} = \{\sqrt{\pi}(k + i\ell) + ca_{k,\ell} : k, \ell \in \mathbb{Z}\}$$

has the same asymptotic decay as the hole probability for $Z_{\mathbf{f}}$, the zero set of the planar Gaussian analytic function. It turns out that natural time dependent versions of both these point processes exist, and that their large deviation behavior is strikingly different (see figure 2).

The perturbed lattice model can be made into a time homogeneous Markov process by allowing each lattice point to evolve as an independent Ornstein-Uhlenbeck process:

$$(7.3.1) \quad Z_{pl}(t) = \{\sqrt{\pi}(k + i\ell) + ca_{k,\ell}(t) : k, \ell \in \mathbb{Z}\}.$$

Specifically, $a_{k,\ell}(t) = e^{-t/2} B_{k,\ell}(e^t)$ where for each $n \in \mathbb{Z}^2$, we have a Brownian motion in \mathbb{C} that we write as $B_n(t) = \frac{1}{\sqrt{2}} (B_{n,1}(t) + iB_{n,2}(t))$.

One may construct a time dependent version of the planar GAF by defining

$$(7.3.2) \quad \mathbf{f}(z, t) = \sum_{n=0}^{\infty} a_n(t) \frac{z^n}{\sqrt{n!}}$$

where $a_n(t)$ are again i.i.d. complex valued Ornstein-Uhlenbeck processes. With probability one, this process defines an analytic function in the entire plane, and at any fixed time t the distribution of $Z_{\mathbf{f}}(t)$ is translation invariant. However, since some information is lost when one restricts attention from $\mathbf{f}(\cdot, t)$ to $Z_{\mathbf{f}}(t)$, it is not clear that $Z_{\mathbf{f}}(t)$ should even be Markovian. Fortunately, using an argument similar

to the one given for the hyperbolic GAF (Theorem 5.3.1), one may show that $|\mathbf{f}(z, t)|$ can be reconstructed from $Z_{\mathbf{f}}(t)$ and since the evolution of the coefficients is radially symmetric the zero set itself is a time homogeneous Markov process.

Whereas before we were interested in the hole probability that $r\mathbb{D}$ contains no points, it now makes sense to introduce the time dependent hole probability, $p(r, T)$ that $r\mathbb{D}$ contains no points of the process for all $t \in [0, T]$. Using straightforward estimates for Ornstein-Uhlenbeck processes, one can obtain the following (32)

PROPOSITION 7.3.1. *In the dynamical perturbed lattice model, let $H_k(T, R)$ denote the event that $R\mathbb{D}$ contains no points of the process for all $t \in [0, T]$. Then for any $R > R_* > 16$ and $T > T_*$, there exist positive constants c_1 and c_2 depending only on T_* and R_* so that*

$$(7.3.3) \quad \limsup_{T \rightarrow \infty} \frac{1}{T} \log(\mathbf{P}(H_k(T, R))) \leq -c_1 R^4$$

and

$$(7.3.4) \quad \liminf_{T \rightarrow \infty} \frac{1}{T} \log(\mathbf{P}(H_k(T, R))) \geq -c_2 R^4.$$

This result starkly contrasts with the time dependent hole probability for the planar GAF, as the following result shows (32).

THEOREM 7.3.2. *Let $H_{\mathbf{f}}(T, R)$ denote the event that the dynamical planar GAF does not have any zeros in $R\mathbb{D}$ for any $t \in [0, T]$. Then*

$$(7.3.5) \quad \limsup_{T \rightarrow \infty} \frac{1}{T} \log(\mathbf{P}(H_{\mathbf{f}}(T, R))) \leq -e^{(\frac{1}{3}-o(1))R^2}$$

and

$$(7.3.6) \quad \liminf_{T \rightarrow \infty} \frac{1}{T} \log(\mathbf{P}(H_{\mathbf{f}}(T, R))) \geq -e^{(\frac{1}{2}+o(1))R^2}.$$

- **Overcrowding** For the planar GAF, one can fix a disk of radius r and ask for the asymptotic behaviour of $\mathbf{P}[n(r) > m]$ as $m \rightarrow \infty$. Following a conjecture of Yuval Peres, it was proved in (52) that for any $r > 0$, $\log \mathbf{P}[n(r) > m] = -\frac{1}{2}m^2 \log(m)(1 + o(1))$. It is also shown there that for hyperbolic GAF with parameter ρ , there are upper and lower bounds of the form e^{-cm^2} for $\mathbf{P}[n(r) > m]$, for any fixed $r \in (0, 1)$.
- **Moderate and very large deviations** Inspired by the results obtained by Jancovici, Lebowitz and Manificat (36) for Coulomb gases in the plane (e.g., Ginibre ensemble), M.Sodin (79) conjectured the following.

Let $n(r)$ be the number of zeroes of the planar GAF in the disk $D(0, r)$. Then, as $r \rightarrow \infty$

$$(7.3.7) \quad \frac{\log \log \left(\frac{1}{\mathbf{P}[|n(r)-r^2| > r^\alpha]} \right)}{\log r} \rightarrow \begin{cases} 2\alpha - 1, & \frac{1}{2} \leq \alpha \leq 1; \\ 3\alpha - 2, & 1 \leq \alpha \leq 2; \\ 2\alpha, & 2 \leq \alpha. \end{cases}$$

The upper bound in the case $\alpha > 2$ follows by taking $\frac{1}{2}r^{\frac{\alpha}{2}}$ in place of r in Theorem 7.2.5 (In (52) it is shown that $\log \left(\frac{1}{\mathbf{P}[|n(r)-r^2| > r^\alpha]} \right)$ is asymptotic to $r^{2\alpha} \log(r)$, which is slightly stronger). A lower bound for the case $1 < \alpha < 2$ was proved in (52). All the remaining cases have been settled now by Sodin, Nazarov and Volberg (58).

Advanced Topics: Dynamics and Allocation to Random Zeros

8.1. Dynamics

8.1.1. Dynamics for the hyperbolic GAF. Recall the hyperbolic GAF

$$\mathbf{f}_L(z) = \sum_{n=0}^{\infty} a_n \frac{\sqrt{L(L+1)\dots(L+n-1)}}{\sqrt{n!}} z^n$$

which is defined for $L > 0$, and distinguished by the fact that its zero set is invariant in distribution under Möbius transformations preserving the unit disk

$$(8.1.1) \quad \varphi_{\alpha,\beta}(z) = \frac{\alpha z + \beta}{\beta z + \bar{\alpha}}, \quad z \in \mathbb{D}$$

with $|\alpha|^2 - |\beta|^2 = 1$. In order to understand the point process of zeros of \mathbf{f}_L it is useful to think of it as a stationary distribution of a time-homogeneous Markov process.

Define the complex Ornstein-Uhlenbeck process

$$a(t) := e^{-t/2} W(e^t), \quad W(t) := \frac{B_1(t) + iB_2(t)}{\sqrt{2}},$$

where B_1, B_2 are independent standard Brownian motions, and $W(t)$ is complex Brownian motion scaled so that $\mathbf{E}W(1)\overline{W(1)} = 1$. The process $\{a(t)\}$ is then stationary Markov with the standard complex Gaussian as its stationary distribution. Consider the process

$$\mathbf{f}_L(z, t) = \sum_{n=0}^{\infty} a_n(t) \frac{\sqrt{L(L+1)\dots(L+n-1)}}{\sqrt{n!}} z^n$$

where $a_n(t)$ are now i.i.d. Ornstein-Uhlenbeck processes. Then the entire process $\mathbf{f}_L(z, t)$ is conformally invariant in the sense that

$$\left\{ \left[\varphi'_{\alpha,\beta}(z) \right]^{L/2} \mathbf{f}_L(\varphi_{\alpha,\beta}(z), t) \right\}_{t>0}$$

has the same distribution as $\mathbf{f}_L(z, t)$, $t > 0$. For this, by continuity, it suffices to check that the covariances agree. Indeed, for $s \leq t$,

$$\mathbf{E} \mathbf{f}_L(z, s) \overline{\mathbf{f}_L(w, t)} = e^{(s-t)/2} \mathbf{E} \mathbf{f}_L(z, 0) \overline{\mathbf{f}_L(w, 0)}$$

so the problem is reduced to checking the equality of covariances for a fixed time, which has already been discussed in Proposition 2.3.4.

It follows automatically that the process $\{Z_{\mathbf{f}_L}(t)\}$ of zeros of $\mathbf{f}_L(\cdot, t)$ is conformally invariant. To check that it is a Markov process, recall from Section 5.4.1 that $\{Z_{\mathbf{f}_L}(t)\}$ determines $\mathbf{f}_L(\cdot, t)$ up to a multiplicative constant of modulus 1. Since the evolution of an Ornstein-Uhlenbeck process is radially symmetric it follows that $\mathbf{f}_L(\cdot, t)$ modulo such a constant is a Markov process; and hence $\mathbf{f}_L(\cdot, t)$ is a Markov process as well.

8.1.2. SDE for dynamics of one zero. Finally, we give an SDE description of the motion of zeros. Let $a_n(t) = e^{-t/2}W_n(e^t)$ be i.i.d. Ornstein-Uhlenbeck processes. Condition on starting at time 1 with a zero at the origin. This implies that $W_0(1) = 0$, and by the Markov property all the W_i are complex Brownian motions started from some initial distribution at time 1. For t in a small time interval $(1, 1 + \epsilon)$ and for z in the neighborhood of 0, we have

$$\varphi_t(z) = W_0(t) + W_1(t)z + W_2(t)z^2 + O(z^3).$$

If $W_1(1)W_2(1) \neq 0$, then the movement of the root z_t of φ_t where $z_1 = 0$ is described by the movement of the solution of the equation $W_0(t) + W_1(t)z_t + W_2(t)z_t^2 = O(z_t^3)$. Solving the quadratic gives

$$z_t = \frac{-W_1}{2W_2} \left(1 - \sqrt{1 - \frac{4W_0W_2}{W_1^2}} \right) + O(W_0^3).$$

Expanding the square root we get

$$z_t = -\frac{W_0}{W_1} - \frac{W_0^2W_2}{W_1^3} + O(W_0^3).$$

Since $W_0(t)$ is complex, $W_0^2(t)$ is a martingale, so there is no drift term. The noise term then has coefficient $-1/W_1$, so the movement of the zero at 0 is described by the SDE (at $t = 1$) $dz_t = -W_1(t)^{-1}dW_0(t)$ or, rescaling time for the time-homogeneous version, for any τ with $a_0(\tau) = 0$ we get

$$(8.1.2) \quad dz_\tau = -\frac{1}{a_1(\tau)} da_0(\tau).$$

The absence of drift in (8.1.2) can be understood as follows: in the neighborhood we are interested in, this solution z_t will be an analytic function of the $\{W_n\}$, and therefore has no drift.

For other values of L the same argument gives

$$dz_\tau = -\frac{1}{\sqrt{L}a_1(\tau)} da_0(\tau).$$

Of course, it is more informative to describe this movement in terms of the relationship to other zeros, as opposed to the coefficient a_1 . For this, we consider the reconstruction formula 5.3.10, which gives

$$|a_1| = |\mathbf{f}'_{\mathbb{D},L}(0)| = c_L \prod_{k=1}^{\infty} e^{L/(2k)} |z_k| \quad \text{a.s.}$$

This means that when there are many other zeros close to a zero, the noise term in its movement grows and it oscillates wildly. This produces a repulsion effect for zeros that we have already observed in the point process description. The equation (8.1.2) does not give a full description of the process as the noise terms for different zeros are correlated. We give a more complete description of the dynamics in subsection 8.3.2.

8.2. Allocation

8.2.1. Transportations of measures. Consider again the planar Gaussian analytic function defined by the random power series

$$(8.2.1) \quad \mathbf{f}(z) = \sum_{k \geq 0} a_k \frac{z^k}{\sqrt{k!}}$$

where a_k are independent standard complex Gaussian random variables (without loss of generality take $L = 1$ here). It is distinguished by the invariance of its distribution with respect to the rigid motions of the complex plane as described in Chapter 2. So far we have been concerned with computing various aspects of the distribution of zeros of random analytic functions. In this chapter we show that it is possible to tackle certain deep stochastic geometric questions regarding the zeros of \mathbf{f} . The stochastic geometric aspect that will be studied in this chapter is **transportation** or **matching** or **allocation**.

DEFINITION 8.2.1. Given two measures μ and ν on Λ , a transportation between μ and ν is a measure ρ on $\Lambda \times \Lambda$ whose first marginal is μ and the second marginal, ν . When μ and ν are both counting measures (i.e., atomic measures with atoms of size 1), and so is ρ , the transportation will be also called a matching. When μ is a counting measure and ν is the Lebesgue measure (or when μ is a point process and ν is a fixed deterministic measure), a transportation will be called an allocation.

Informally we think of ρ as taking a mass $d\mu(x)$ from the point x and spreading it over Λ by transporting a mass of $\rho(x, dy)$ to the point y . A matching is just what it says, a pairing of the support of μ with the support of ν (when both are counting measures). An allocation may be picturesquely described as a scheme for dividing up land (Lebesgue measure) among farmers (points of the point process) in a fair manner (each farmer gets unit area of land).

One use of transportation is to quantify how close the two measures μ and ν are. Indeed, the reader may be familiar with the fact that one can define a metric d (the Prohorov-metric) on the space of probability measures of a complete separable metric space by setting $d(\mu, \nu)$ equal to the smallest r (infimum, to be precise) for which one can find a transportation ρ of μ and ν that is supported in an r -neighbourhood of the diagonal, $\{(x, x) : x \in \Lambda\}$, of Λ^2 .

EXERCISE 8.2.2. Prove that d is indeed a metric.

Now consider a translation invariant simple point process \mathcal{X} in the plane, for example, the zeros of \mathbf{f} or a Poisson process with constant intensity. Then the expected measure $\mathbf{E}[\mathcal{X}(\cdot)]$ is a constant multiple of the Lebesgue measure on the plane. Now consider a transportation ρ between \mathcal{X} and c -Lebesgue measure (where c is the intensity of \mathcal{X}). Since \mathcal{X} is random, we would want ρ to be measurable (w.r.t. the natural sigma-field on the space of sigma-finite measures on Λ^2) and since \mathcal{X} is translation invariant, it is natural to ask for ρ to be diagonally invariant in the sense that

$$(8.2.2) \quad \rho(\cdot + w, \cdot + w) \stackrel{d}{=} \rho(\cdot, \cdot) \quad \text{for any } w \in \mathbb{R}^2.$$

Unlike in exercise 8.2.2 one cannot hope for a transportation that is supported within a finite distance of the diagonal. For, if \mathcal{X} has no points in $D(0, r)$, then for $|y| < \frac{r}{2}$, then $\rho(\cdot, y)$ is necessarily supported in $\{x : |x - y| > \frac{r}{2}\}$. For most point processes of interest, the event of $D(0, r)$ being a hole will have positive probability, no matter

how large r is, which implies that ρ cannot be supported within a finite distance of the diagonal of Λ^2 . Therefore we shall consider the *decay* of probability that mass is carried to a large distance r , as $r \rightarrow \infty$ as a measure of how localized a transportation is.

Let us make this notion precise. In this book we shall talk only of allocations, i.e., mass transportations from a point process to Lebesgue measure. Moreover, the point process being translation invariant, we shall always require (8.2.2) to hold. Therefore for every y , $\rho(\cdot, y)$ has the same law, and the quantity that we are interested in, is the asymptotic behaviour of $\mathbf{P}[\rho(D(0, r)^c, 0) > 0]$ as $r \rightarrow \infty$.

REMARK 8.2.3. The alert reader might wonder what we would do if we were dealing with matching or transportation between two independent copies $\mathcal{X}_1, \mathcal{X}_2$ of the point process. For, in that case we should consider $\mathbf{P}[\rho(D(y, r)^c, y) > 0]$ for a *typical point* $y \in \mathcal{X}_2$ and it is not obvious what that means. The notion of a typical point of a stationary point process can be given precise meaning, in terms of what is known as the **palm measure** of the point process (16). To get the palm version of \mathcal{X} , fix $r > 0$ and pick a point y uniformly at random from $\mathcal{X} \cap D(0, r)$ and translate the entire process by $-y$ so that the point at location y is brought to the origin. This defines a point process \mathcal{X}_r that has a point at 0, almost surely (If $\mathcal{X} \cap D(0, r) = \emptyset$, define $\mathcal{X}_r = \{0\}$). As $r \rightarrow \infty$, \mathcal{X}_r converges in distribution to a point process $\tilde{\mathcal{X}}$ that also has a point at the origin. This is the palm version of \mathcal{X} . When the matching scheme is applied to $\tilde{\mathcal{X}}$, the distance from 0 to its match can be justly interpreted as the typical distance to which a point of \mathcal{X}_2 is matched in the original setting. By limiting ourselves to allocations, we shall avoid the (minor) technicalities involved in dealing with palm measures.

In the next section, we describe a beautiful explicit allocation scheme due to Sodin and Tsirelson for the zeros of \mathbf{f} . We also give a brief sketch of the idea behind the proof of Nazarov, Sodin and Volberg (59) that the diameters of basins (allocated to a typical zero of \mathbf{f}) in this allocation have better than exponential tails.

8.2.2. The gravitational allocation scheme. Let f be an entire function with no multiple zeros. Set $u(z) = \log|f(z)| - \frac{1}{2}|z|^2$. Consider flow lines along the integral curves of the vector field $-\nabla u(z)$ (well defined off of the zero set of f). In other words, for each $z \in \mathbb{C} \setminus f^{-1}\{0\}$, consider the ODE

$$\frac{dZ(t)}{dt} = -\nabla u(Z(t))$$

with the initial condition $Z(0) = z$. We shall call these paths the “gradient” curves of u . Visualizing the potential as a height function, we may interpret these flow lines as the trajectories of particles without inertia in a gravitational field. Recall that $\frac{1}{2\pi} \Delta u(z) = dn_f(z) - \frac{1}{\pi} dm(z)$ in the distributional sense (see the explanation following (2.4.3)). Thus, outside of the zero set of f , the potential u is super harmonic, and therefore, u has no local minima other than the zeros of f . Therefore for a “typical” initial point z , the gradient curves will flow down to a zero of f (This cannot be true for all starting points, for instance if z is a saddle point of ∇u). For each $a \in f^{-1}\{0\}$, define its *basin*

$B(a) = \{z \in \mathbb{C} : \nabla u(z) \neq 0, \text{ and the gradient curve passing through } z \text{ terminates at } a\}$.

Clearly, each basin $B(a)$ is a connected open set, and $B(a) \cap B(a') = \emptyset$ if a and a' are two different zeros of f . The remarkable observation made by Sodin and Tsirelson (82)

is that, if a basin $B(a)$ is bounded and has a suitably nice boundary, then $B(a)$ has area exactly equal to π !

A heuristic argument: We give a heuristic argument that purports to show that the above scheme is in fact an allocation.

Fix $\epsilon > 0$ so small that $D(a, \epsilon) \subset B(a)$ and set $B_\epsilon = B(a) \setminus D(a, \epsilon)$. Then $\Delta u = -2$ on B_ϵ and by Green's theorem we find

$$\begin{aligned} -2|B_\epsilon| &= \int_{B_\epsilon} \Delta u(z) dm(z) \\ &= \int_{\partial B_\epsilon} \frac{\partial u}{\partial n}(z) |dz| \\ &= - \int_{\partial D(a, \epsilon)} \frac{\partial u}{\partial n}(z) |dz|, \end{aligned}$$

where in the last equality we used the intuitively obvious fact that $\frac{\partial u}{\partial n} = 0$ on $\partial B(a)$, since gradient lines must be flowing tangentially on the boundary of two basins. The negative sign is there because the outward facing normal on $\partial D(a, \epsilon)$ changes direction depending on whether we regard it as the boundary of $D(a, \epsilon)$ or the boundary of $B(a) \setminus D(a, \epsilon)$. This last integral can be written as

$$- \int_0^{2\pi} \operatorname{Re} \left\{ \left(\frac{f'(a + \epsilon e^{i\theta})}{f(a + \epsilon e^{i\theta})} - \frac{1}{2}(a + \epsilon e^{-i\theta}) \right) e^{i\theta} \right\} \epsilon d\theta = -2\pi + O(\epsilon),$$

because by Cauchy's theorem (the curve $\epsilon e^{i\theta}$ encloses a , a zero of f with unit multiplicity),

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{f'(a + \epsilon e^{i\theta})}{f(a + \epsilon e^{i\theta})} \epsilon d\theta = 1.$$

Thus by letting $\epsilon \rightarrow 0$, we deduce that $|B(a)| = \pi$ as we wanted to show.

The obvious gaps in this "back-of-the-envelope" calculation are that we have assumed a priori that the basins are bounded and have piecewise smooth boundaries. See Figure 1 for a picture of the potential and Figure 2 for a patch of the allocation defined by the gradient lines of the potential in the case when f is a sample of the planar Gaussian analytic function \mathbf{f} .

REMARK 8.2.4. Although this scheme gives a very explicit allocation of Lebesgue measure to the set of zeros, superficially it may seem as though the analytic function is essential to make it work. That is not quite correct, because at least when we have a finite set of points, it is possible to express everything in terms of the points of the point process alone, without recourse to the analytic function whose zeros they are. Given a finite collection of points $\{z_1, \dots, z_n\}$ in the complex plane, one may define $f(z) = \prod_{k=1}^n (z - z_k)$ and define $u(z)$ exactly as before. In this case

$$(8.2.3) \quad -\nabla u(z) = - \sum_{k=1}^n \frac{1}{z - z_k} + z,$$

so at the point z each zero z_k exerts a "gravitational force" of magnitude $\frac{1}{|z - z_k|}$ towards z_k . It is worth recalling here that the correct analogue of the gravitational

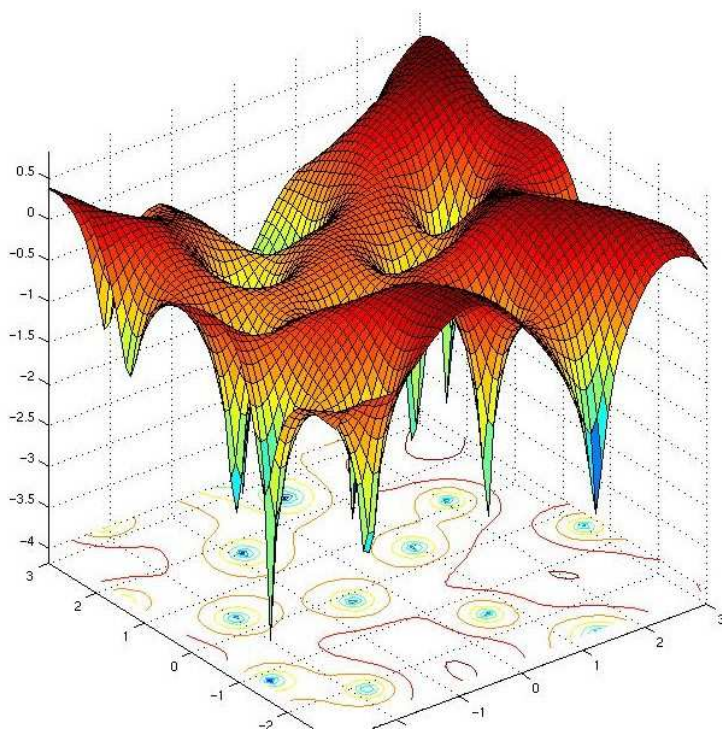


FIGURE 1. The potential function $u(z) = \log|\mathbf{f}(z)| - \frac{1}{2}|z|^2$.

potential (equivalently, the Green's function for the Laplacian) in two dimensions is $\log|z - w|$ while in \mathbb{R}^d for $d \geq 3$, it is $\|x - y\|^{-d+2}$. Henceforth we shall refer to this scheme as gravitational allocation. Figure 2 shows a piece of the allocation when applied to a finite number of points chosen uniformly from a square (a finite approximation to Poisson process on the plane), and visibly, the basins are more elongated compared to the case of the zeros. In \mathbb{R}^d with $d \geq 3$, the idea can be made to work for the Poisson process also. See the notes at the end of this chapter.

Here is a cute fact about the gravitational allocation scheme that has not found any application yet. This exercise is not essential to anything that comes later.

EXERCISE 8.2.5. The first part of Theorem 8.2.7 asserts that for the planar Gaussian analytic function \mathbf{f} , the allocation scheme described above does partition the whole plane into basins of equal area π . Assuming this, show that the *time* to flow from 0 into a zero of \mathbf{f} has exponential distribution with mean $\frac{1}{2}$. (Hint: Consider the time-derivative of the Jacobian of the reversed dynamics.)

In the following exercise, make appropriate assumptions that the relevant angles are well-defined, that the boundaries are smooth etc.

EXERCISE 8.2.6. Let f and g be two entire functions. Define the potential $v(z) = \log|f(z)| - \log|g(z)|$ and consider flow lines along the vector field ∇v . Since v is $+\infty$ ($-\infty$) at the zeros of g (respectively f), typical flow lines start at a zero of f and end

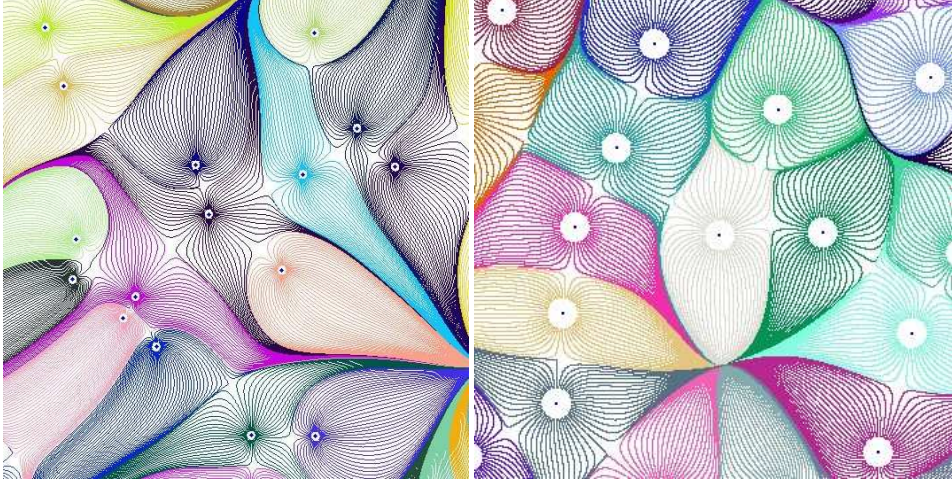


FIGURE 2. Gravitational allocation for a Poisson process (left) and for $Z_{\mathbf{f}}$.

at a zero of g .

Consider two gradient lines γ_1 and γ_2 that start at $a \in f^{-1}\{0\}$ and end at $b \in g^{-1}\{0\}$. Let θ_a be the angle between these two curves at a and θ_b the angle at b . Let Ω be the region bounded by these two curves and let $\Omega_\epsilon = \Omega \setminus [B(a, \epsilon) \cup B(b, \epsilon)]$. Assume that θ_a and θ_b exist and also that Ω contains no other zeros of f or g . Then apply Green's theorem to $\int_{\Omega_\epsilon} \Delta v$ and let $\epsilon \rightarrow 0$ to show that $\theta_a = \theta_b$. (For a picture

When f and g are independent samples of the planar Gaussian analytic function, see Figure 3). Having proved this, one can define the mass transportation between the zeros of f and g by setting

$$\rho(a, b) = \frac{1}{2\pi} \left\{ \begin{array}{l} \text{Angle of the sector of directions at } a \\ \text{along which the flow lines end up at } b. \end{array} \right.$$

8.2.3. Bounding the diameters of cells in the gravitational allocation.

The calculations in the previous section were somewhat formal, and in this section we state precise results on the gravitational allocation when applied to the planar Gaussian analytic function. The result that makes all the effort worthwhile is this.

THEOREM 8.2.7 (Nazarov, Sodin, Volberg). *Apply the gravitational allocation scheme to \mathbf{f} , the planar Gaussian analytic function.*

- (1) *Almost surely, each basin is bounded by finitely many smooth gradient curves (and, thereby, has area π), and $\mathbb{C} = \bigcup_{a \in Z_{\mathbf{f}}} B(a)$ up to a set of measure 0 (more, precisely, up to countably many smooth boundary curves).*
- (2) *For any point $z \in \mathbb{C}$, the probability of the event that the diameter of the basin containing z is greater than R is between $ce^{-CR(\log R)^{3/2}}$ and $Ce^{-cR\sqrt{\log R}}$.*

The proof of this theorem is quite intricate and is beyond the scope of this book. We shall merely whet the appetite of the reader by sketching an outline of the central

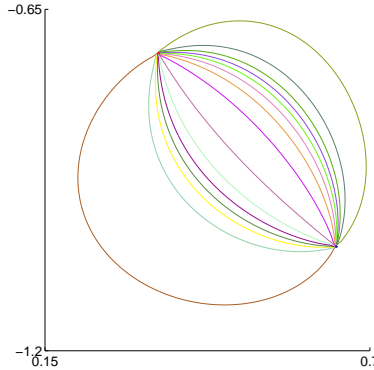


FIGURE 3. Gradient lines of $v(z) = \log|\mathbf{f}(z)| - \log|\mathbf{g}(z)|$.

part of the proof of Theorem 8.2.7 and direct those hungry for more to the original paper (59).

The diameter of a basin in the allocation can be large only if there is a long gradient line. Thus the following auxiliary result is of great relevance.

THEOREM 8.2.8 (Absence of long gradient curves). *Let $Q(w, s)$ be the square centered at w with side length s and let $\partial Q(w, s)$ be its boundary. Then there are constants c, C such that for any $R \geq 1$, the probability that there exists a gradient curve joining $\partial Q(0, R)$ with $\partial Q(0, 2R)$ does not exceed $Ce^{-cR\sqrt{\log R}}$.*

8.2.4. Proof sketch: absence of long gradient curves. First, notice that the potential u is shift invariant. Heuristically, we pretend that u is almost bounded. Thus, if a long gradient curve Γ exists, $|\nabla u|$ must be very small on Γ (about $\frac{1}{R}$). The second idea is to discretise the problem. Since it is hard to work with arbitrary curves (they are infinitely many), we want to replace each curve by a connected set of small squares covering it. Since the second derivatives of u are “morally bounded” and the smallness size of ∇u we need is $\frac{1}{R}$, it is natural to divide the square $Q(0, 2R)$ into squares of size $\frac{1}{R}$. Then, if $|\nabla u| < \frac{1}{R}$ at one point of the square $Q(w, \frac{1}{R})$, it is less than $\frac{C}{R}$ in the entire square, and, in particular at its center w . We shall call such a

square black. Now note that, since u is shift invariant and $\nabla u(z) = \overline{\left(\frac{\mathbf{f}(z)}{\mathbf{f}(z)}\right)} - z$, we have

$$\mathbf{P}\left\{|\nabla u(w)| < \frac{C}{R}\right\} = \mathbf{P}\left\{|\nabla u(0)| < \frac{C}{R}\right\} = \mathbf{P}\left\{\left|\frac{a_1}{a_0}\right| < \frac{C}{R}\right\} \leq CR^{-2}.$$

This means that the expected number of black squares in the entire square $Q(0, 2R)$ is bounded by $CR^4R^{-2} = CR^2$, which is barely enough to make a connected chain from $\partial Q(0, R)$ to $\partial Q(0, 2R)$. Moreover, if we take any smaller square $Q(w, 2r)$ with side length r , the expected number of black squares in it is about $(rR)^2R^{-2} = r^2$, which is much less than the number rR of squares needed to get a chain joining $\partial Q(w, r)$ to $\partial Q(w, 2r)$. This also gives an estimate r/R for the probability of existence of at least rR black squares in $Q(w, r)$ (just use the Chebyshev inequality). Hence, the probability of existence of a noticeable (i.e., comparable in length to the size of the square) piece of a black chain in $Q(w, 2r)$ also does not exceed r/R .

The next observation is that $u(w')$ and $u(w'')$ are almost independent if $|w' - w''|$ is large. More precisely, we have

$$\mathbf{E}\mathbf{f}(w')\overline{\mathbf{f}(w'')} = \sum_{k \geq 0} \frac{(w' \overline{w''})^k}{k!} = e^{w' \overline{w''}}.$$

This means that the absolute value of the covariance of the standard complex Gaussian random variables $\mathbf{f}(w')e^{-\frac{|w'|^2}{2}}$ and $\mathbf{f}(w'')e^{-\frac{|w''|^2}{2}}$ equals $e^{-\frac{|w' - w''|^2}{2}}$. Recall that two standard Gaussian random variables with covariance σ can be represented as two independent Gaussian random variables perturbed by something of size σ . This morally means that $\nabla u(w')$ and $\nabla u(w'')$ are independent up to an error of size $e^{-\frac{|w' - w''|^2}{2}}$. Since we want to estimate the probability that they are less than $\frac{1}{R}$, we can think of them as independent if $e^{-\frac{|w' - w''|^2}{2}} < \frac{1}{R}$, i.e., if $|w' - w''| > A\sqrt{\log R}$ where A is a large constant.

Thus, our situation can be approximately described by the following toy model. We have a big square $Q(0, 2R)$ partitioned into subsquares with side length $\frac{1}{R}$. Each small square is black with probability R^{-2} and the events that the small squares are black are independent if the distance between the centers of the squares is at least $\sqrt{\log R}$. Our aim is to estimate the probability of the event that there exists a chain of black squares connecting $\partial Q(0, R)$ and $\partial Q(0, 2R)$.

To solve this toy model problem, it is natural to switch to square blocks of squares of size $r = \sqrt{\log R}$ because then, roughly speaking, any two blocks are independent. Any chain of black squares with side length $\frac{1}{R}$ determines a chain of blocks of size r in which all blocks contain a noticeable piece of the chain of black squares. The probability that any particular chain of L blocks has this property is about $\left(\frac{r}{R}\right)^L < e^{-cL \log R}$ (due to independence). On the other hand, it is easy to estimate the number of connected chains of L blocks with side length r : there are $(R/r)^2$ blocks where we can start our chain and during each step we have a constant number of blocks to move to. This yields the estimate $(R/r)^2 e^{cL}$. Hence, the probability that there exists a chain of L blocks of side length r and each block, in turn, contains a noticeable piece of the chain of black squares of side length $1/R$, is bounded by $(R/r)^2 e^{L(C - c \log R)}$. Since our chains should connect $\partial Q(0, R)$ with $\partial Q(0, 2R)$, we need only the values $L \geq R/r$. For such L , we have $(R/r)^2 e^{L(C - c \log R)} \leq e^{-cL \log R}$. We conclude that the probability that there exists a chain of black squares of side length

$1/R$ connecting $\partial Q(0, R)$ and $\partial Q(0, 2R)$ is bounded by

$$\sum_{L \geq \frac{R}{\sqrt{\log R}}} e^{-cL \log R} \leq \exp \left\{ -c \frac{R}{\sqrt{\log R}} \log R \right\} = e^{-cR \sqrt{\log R}}.$$

There are several technical difficulties in the road to an honest proof. The first one is that it is hard to work with the random potential directly and everything has to be formulated in terms of \mathbf{f} . The second one is that the potential u is not exactly bounded: it can be both very large positive and very large negative. Large positive values are easy to control but large negative values are harder and we prefer to include the possibility that u is large negative into the definition of black squares. The last difficulty is that independence of the values of \mathbf{f} at distant points is not exact but only approximate and some work is needed to justify the product formula for the probability. All this makes the actual proof much more complicated and lengthy than the outline we just sketched.

8.3. Notes

8.3.1. Notes on Dynamics.

8.3.2. General formulation of dynamics. In this section we create a dynamical version of a GAF and hence of its zero set. We describe the motion of the zeros by a system of SDEs.

First consider a function $\mathbf{f}_t(z)$, where $t > 0$ and $z \in \Omega$, with the following properties.

- For each t , the function $z \rightarrow \mathbf{f}_t(z)$ is a (random) analytic function.
- For each $z \in \Omega$, the function $t \rightarrow \mathbf{f}_t(z)$ is a continuous semi-martingale.

Let $\{\zeta_k(t)\}_k$ be the set of zeros of \mathbf{f}_t . More precisely, index the zeros in an arbitrary way at $t = 0$. Then as t varies the function \mathbf{f}_t varies continuously and hence the zeros also trace continuous curves in Ω . There are two potential problems. Firstly, it may happen that the zeros collide and separate. More seriously zeros may escape to the boundary.

For now we assume that the above problems do not arise and work formally. Later in cases of interest to us, we shall see that these problems indeed do not arise.

Consider a zero curve $\zeta(t)$, and suppose that at time 0 we have $\zeta(0) = w$. By our assumption, the order to which \mathbf{f}_0 vanishes at w is 1. Hence by Rouché's theorem, we can fix a neighbourhood $D(w; r)$ of w and $\epsilon > 0$ (these depend on the sample path and hence are random), such that for any $t \in (0, \epsilon)$, $\zeta(t)$ is the unique zero of \mathbf{f}_t in $D(w; r)$. Fix such a t and expand \mathbf{f}_t around w . We obtain

$$(8.3.1) \quad \mathbf{f}_t(z) = \mathbf{f}_t(w) + \mathbf{f}'_t(w)(z-w) + \frac{\mathbf{f}''_t(w)}{2}(z-w)^2 + O((z-w)^3).$$

Therefore, one root of the equation

$$0 = \mathbf{f}_t(w) + \mathbf{f}'_t(w)(z-w) + \frac{\mathbf{f}''_t(w)}{2}(z-w)^2 + O((z-w)^3)$$

(the one closer to w) differs from $\zeta(t)$ by $O(\mathbf{f}_t(w)^3)$. The quadratic above can be solved explicitly and we get

$$\begin{aligned} \zeta(t) &= w - \frac{\mathbf{f}'_t(w)}{\mathbf{f}''_t(w)} \left(1 - \sqrt{1 - \frac{2\mathbf{f}_t(w)\mathbf{f}''_t(w)}{\mathbf{f}'_t(w)^2}} \right) + O(\mathbf{f}_t(w)^3) \\ &= w - \frac{\mathbf{f}_t(w)}{\mathbf{f}'_t(w)} + \frac{\mathbf{f}_t(w)^2 \mathbf{f}''_t(w)}{2\mathbf{f}'_t(w)^3} + O(\mathbf{f}_t(w)^3). \end{aligned}$$

Recall that $\mathbf{f}_t(w) = 0$ to get

$$d\zeta(0) = -\frac{d\mathbf{f}_t(w)}{\mathbf{f}'_t(w)}.$$

(Here 'd' denotes the Ito derivative.) The same calculations can be made for any t and all the zeros $\zeta_k(t)$ and we end up with

$$(8.3.2) \quad d\zeta_k(t) = -\frac{d\mathbf{f}_t(\zeta_k(t))}{\mathbf{f}'_t(\zeta_k(t))} \quad \text{for } k \geq 1.$$

In some cases the zeros of \mathbf{f}_t determine \mathbf{f}_t almost surely. Then obviously, the zero set will be a Markov process itself. In such cases the right hand side of the system of equations(8.3.2) can be expressed in terms of $\{\zeta_j(t)\}_j$ (the equation for $\zeta_k(t)$ will involve all the other ζ_j s, of course) and we have the equations for the diffusion of the zeros (possibly infinite dimensional).

Returning to Gaussian analytic functions, suppose we are given a GAF of the form $\mathbf{f}(z) = \sum_n a_n \psi_n(z)$ where a_n are i.i.d. complex normals and ψ_n are analytic functions. We now make a dynamical version of \mathbf{f} as follows. Let $a_n(t)$ be i.i.d. stationary complex Ornstein-Uhlenbeck processes defined as $a_n(t) = e^{-t/2} W_n(e^t)$, where W_n are i.i.d. standard complex Brownian motions. Here 'standard' means that $\mathbf{E}[|W_n(t)|^2] = 1$. It is well known and easy to see that they satisfy the SDEs

$$(8.3.3) \quad da_n(t) = -\frac{1}{2}a_n(t)dt + dW_n(t),$$

where W_n are i.i.d. standard complex Brownian motions.

Then set $\mathbf{f}_t(z) = \sum_n a_n(t)z^n$. Then the zero set of \mathbf{f}_t is isometry-invariant with the distribution of the zero set of \mathbf{f} . In this case, we can write equations (8.3.2) as

$$\begin{aligned} d\zeta_k(t) &= -\frac{d\mathbf{f}_t(\zeta_k(t))}{\mathbf{f}'_t(\zeta_k(t))} \\ &= -\frac{-\frac{1}{2}(\sum_n a_n(t)\psi_n(\zeta_k(t)))dt + \sum_n \psi_n(\zeta_k(t))dW_n(t)}{\mathbf{f}'_t(\zeta_k(t))} \\ &= -\frac{\sum_n \psi_n(\zeta_k(t))dW_n(t)}{\mathbf{f}'_t(\zeta_k(t))}, \end{aligned}$$

for every k . Here we used equations (8.3.3) to derive the second equality, and the fact that $\mathbf{f}_t(\zeta_k(t)) = 0$ to derive the third equality. In particular, we compute the covariances of the zeros to be

$$\begin{aligned} d\langle \zeta_k, \zeta_l \rangle(t) &= \frac{1}{\mathbf{f}'_t(\zeta_k(t))\overline{\mathbf{f}'_t(\zeta_l(t))}} \sum_n \psi_n(\zeta_k(t))\overline{\psi_n(\zeta_l(t))} \\ &= \frac{K(\zeta_k(t), \zeta_l(t))}{\mathbf{f}'_t(\zeta_k(t))\overline{\mathbf{f}'_t(\zeta_l(t))}}, \end{aligned}$$

where K is the covariance kernel of \mathbf{f} .

8.3.3. Notes on Allocation. It is natural to ask if there are other methods for allocating a discrete point set Ξ in the plane to regions of equal area. One such method, introduced by Hoffman, Holroyd and Peres in (31), produces matchings which are *stable* in the sense of the Gale-Shapley stable marriage problem (26). Intuitively, points in \mathbb{C} prefer to be matched with points of Ξ that are close to them in Euclidean distance and conversely, points of Ξ prefer regions of the plane close to themselves. An allocation is said to be *unstable* if there exist points $\xi \in \Xi$ and $z \in \mathbb{C}$ that are not allocated to each other but both prefer each other to their current allocations.

It is easy to see that a stable allocation of Ξ to \mathbb{C} will not in general allocate the points of Ξ to sets of equal area. To obtain an equal area allocation, one can impose the additional condition that each point in Ξ has appetite α , by which we mean that the Lebesgue measure of the set matched to each point $\xi \in \Xi$ cannot exceed α . Hoffman, Holroyd and Peres, show

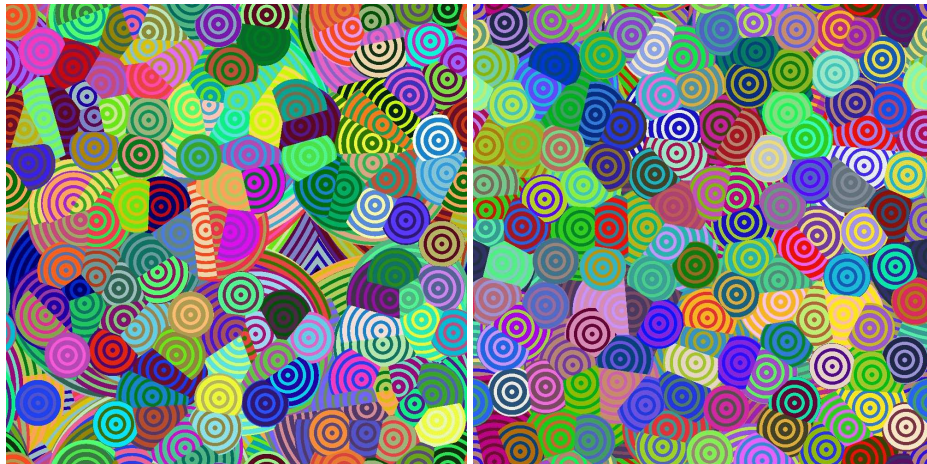


FIGURE 4. The stable marriage allocation for a Poisson process (left) and the zero set of the planar GAF.

that stable allocations with appetite α exist for any discrete point set Ξ . Moreover, they show that if the point process Ξ has intensity $\lambda \in (0, \infty)$ and is ergodic under translations, then with probability one there exists a Lebesgue-a.e. unique stable allocation with appetite $\frac{1}{\lambda}$ under which each point in Ξ is allocated a set of Lebesgue measure $\frac{1}{\lambda}$, and the set of unallocated points in \mathbb{C} has measure zero. Conceptually, this allocation is obtained by allowing each point $\xi \in \Xi$ to expand by growing a ball at a constant rate centered at ξ , and “capturing” all points in \mathbb{C} that it reaches first. Each point in Ξ “grows” according to this procedure until it has captured area equal to $\frac{1}{\lambda}$ at which point it stops growing. This description, of course, is non-rigorous and the interested reader is encouraged to consult (31) for precise statements and further details. Pictures of the resulting allocation obtained for the Poisson process and $Z_{\mathbf{f}}$ are given in Figure 4 (notice that the region allocated to a point $\xi \in \Xi$ need not be connected).

8.4. Hints and solutions

Exercise 8.2.5 Consider the reverse dynamics $\frac{dZ(t)}{dt} = \nabla u(Z(t))$. The forward- t map T_t , taking $Z(0)$ to $Z(t)$, is injective on $\mathbb{C} \setminus \{\mathbf{f}^{-1}(0)\}$. Moreover, for $z \notin \mathbf{f}^{-1}(0)$

$$\frac{d}{dt}DT_t(z) = D\left(\frac{dT_t(z)}{dt}\right) = \left(\left(\frac{\partial^2 u(T_t z)}{\partial x_i \partial x_j}\right)\right)_{i,j \leq 2}.$$

From this we get an expression for the derivative of the Jacobian determinant (this is called Liouville’s theorem)

$$\frac{d}{dt} \det(DT_t(z)) = \text{Trace} \left(\left(\frac{\partial^2 u(T_t z)}{\partial x_i \partial x_j} \right) \right)_{i,j \leq 2} = \Delta u(T_t z) = -2.$$

Let a be a zero of \mathbf{f} and let $B' = B(a) \setminus \{a\}$. Since T_0 is the identity map, from the derivative of the Jacobian determinant of T_t , we get $\frac{d|T_t(B')|}{dt} = -2|T_t(B')|$, which of course implies that $|T_t(B')| = e^{-2t}$. So far the argument was completely deterministic. But now observe that $T_t(B')$ is precisely the set of points in the basin of a which in the *forward dynamics* had not hit a by time t . By translation invariance, this shows that $\mathbf{P}[\text{time to fall into } \mathbf{f}^{-1}(0) \text{ starting from } 0 > t] = e^{-2t}$, as desired.

Exercise 8.2.6 $\Delta u = 0$ on Ω_ϵ . Further, the normal derivative $\frac{\partial u}{\partial n}$ w.r.t. Ω is zero on γ_1 and γ_2 . Hence by Green's theorem,

$$\int_{\Omega \cap \partial B(a, \epsilon)} \frac{\partial u}{\partial n} = \int_{\Omega \cap \partial B(b, \epsilon)} \frac{\partial u}{\partial n}.$$

Compute the normal derivatives of the potential by Taylor expansion of \mathbf{f}, \mathbf{g} at a and b to leading order in ϵ to obtain $\theta_a = \theta_b$.

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