Random Matrices and Related Problems

Lecture's notes for Beg Rohu Summer School 2008.

by

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Outline:

1) Introduction : universality and examples.
2) Gaussian Unitary / Orthogonal Ensembles (GUE / GOE):
   . An example to introduce the mathematical structure.
3) Determinantal point processes:
   . From point processes to Gap probability given by Fredholm det.
4) Edge scaling limit and Tracy-Widom distributions:
   . The universal limit distributions.
5) Extended determinantal point processes:
   . From Karlin-McGregor and LGV theorem to Airy_2 Process.

6.a) Application to the polynuclear growth in droplet geometry.

6.b) Application to the Totally Asymmetric Simple Exclusion Process.

Random matrices and related problems

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References used in the preparation of the lecture notes

1. Lecture notes on the same framework [19, 10]
2. My PhD thesis [6]
3. The standard book on Random Matrices [12]
4. Booklet on random matrices [11]
5. Universality in Mathematics and Physics [5]
6. Point processes [14] and determinantal class [17, 18, 10, 13, 1, 4]
7. Airy processes [15, 9, 2] (papers) and my review [7]
8. Tracy-Widom distributions [20, 8, 21]
9. Application to the PNG [15, 9, 3]
10. Application to the TASEP [2, 16]

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1) Introduction.

1.1) From micro to macro: universality.

. On a macroscopic scale, there are physical laws which are shared by different systems.

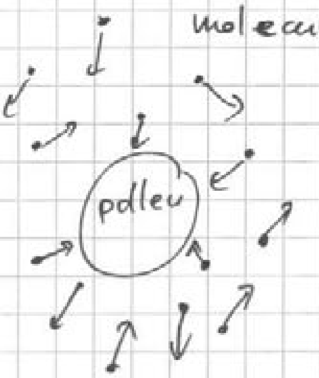
For example, consider the diffusion equation in the space-homogeneous case:

$$(1.1) \qquad \frac{\partial \phi}{\partial t} = D \cdot \nabla^2 \phi$$, for some function $\phi(x,t)$, $x \in \mathbb{R}^d$ the space, $t \in \mathbb{R}$ the time.

. "D" is called the diffusion coefficient.

. This equations appears in several situations, just two examples:

(a) ϕ represents the probability density of finding a grain of pollen in suspension in water, at position x and time t. The evolution of the grain of pollen being determined by the shocks with the water molecules: it looks random (→ Brownian Motion).



. Changing the dimension of the pollen or the water temperature, equation (1.1) still holds.
The only difference will be the diffusion coefficient D:

$$(1.2) \qquad D = \frac{k_B \cdot T}{m \cdot \gamma}$$, where: k_B = Boltzmann constant,
T = (absolute) temperature,
m = mass of pollen,
γ = friction coefficient of the liquid

(b) ϕ represents the temperature profile in a metal. Again, equation (1.1) holds and the only material dependence is in D.

. The key observation is that the same macroscopic laws emerges "no matters" of the details of the microscopic interactions.

- In the above example, the details of the atomic interactions emerge only in the diffusion constant D (which is material/system dependent), but <u>the diffusion equation is universal.</u>

<u>Remark:</u> It is the emergence of such universal behaviors for macroscopic systems, which allow the existence of physical law. If the form of (11) would change for every mass, temperature, ..., then one would not have the law of diffusion.

1.2) A simple mathematical example for universality.

- The simplest example is the Central Limit Theorem (CLT).
Let $\{X_n\}_{n\geq 1}$ be iid random variables. Assume that $\mu = \mathbb{E}(X_i)$, $\sigma^2 = \mathrm{Var}(X_i)$ are both finite (and $\mathbb{E}(|X_i|^3) < \infty$ too). Then,

$$\text{(13)} \quad \lim_{N\to\infty} \mathbb{P}\left(\frac{\sum_{i=1}^{N} x_i - \mu \cdot N}{\sigma\sqrt{N}} \leq S \right) = \int_{-\infty}^{S} du \, \frac{e^{-u^2/2}}{\sqrt{2\pi}} .$$

- <u>The r.h.s. of (13) is universal</u>: it does not depend on the details of the distribution of the x_i's. The details of it enters only via the centering (μ) and rescaling (σ).

  $S_N := \sum_{i=1}^{N} x_i$ is the macroscopic observable.

- <u>Two universal quantities:</u> $\begin{cases} \text{(a) Fluctuation exponent is } \frac{1}{2}: \; S_N - \mu N \simeq N^{1/2} \\ \text{(b) Limit law is Gaussian : r.h.s. of (3).} \end{cases}$

1.3) Random Matrices.

. There are systems with limit laws non-Gaussian. One class of such models share the same limit laws (on a mesoscopic or macroscopic) as Random Matrices.

. Consider the following example: $N \times N$ real symmetric matrices with independent entries :
$$\begin{cases} H_{i,i} \sim \mathcal{N}(0,2N) & , 1 \le i \le N \\ H_{i,j} = H_{j,i} \sim \mathcal{N}(0,N), & 1 \le i < j \le N \end{cases}$$

. Often the quantity of interest are the eigenvalues, and not the specific entries, since they are independent of the basis used to describe the system.

. In this case, we have N real eigenvalues : $\lambda_1, \lambda_2, \dots, \lambda_N$ distributed as
$$p(\lambda_1, \dots, \lambda_N)\, d\lambda_1 \dots d\lambda_N = \text{const} \times \underbrace{\prod_{1\le i<j\le N} |\lambda_i - \lambda_j|}_{\doteq \Delta(\lambda) \text{ the Vandermonde determinant}} \cdot \prod_{i=1}^{N} \left(e^{-\frac{\lambda_i^2}{2N}} d\lambda_i\right).$$

. Key feature of random matrices : eigenvalue's repulsion, due to the $\Delta(\lambda)$ term.

. What can one analyze? $\rightarrow$ Statistical properties.

(a) Macroscopic behavior : For $N \gg 1$, the density of eigenvalues around λ is given by the Wigner semi-circle law:

(1.4)
$$\rho(\lambda) \cong \begin{cases} \frac{1}{\pi} \cdot \sqrt{1 - \left(\frac{\lambda}{2N}\right)^2}, & \lambda \in [-2N, 2N], \text{ (Figure 1.)} \\ 0 & , \lambda \notin [-2N, 2N]. \end{cases}$$

This is not a universal quantity, but depends on the details (like D).

. Some universal quantities are the following:

(b) Nearest-neighbor spacing (in the bulk): see below.

(c) Fluctuations of the largest eigenvalue (see later part of the lecture).

. Question: Is the system behaving like a random matrix?

. Roughly speaking, we say that a system is modeled by a random matrix theory if it behaves statistically as the eigenvalues of large matrices.

. Suppose that a scientist makes an experiment and get some data as output (e.g., the neutron resonance spectrum of heavy nuclei). (Figure 3.)

Step 1: Centering and rescaling

Exp. Data	Eigenvalues
$\{a_k\}$ around A	$\{\lambda_k\}$ around E
$\downarrow$ centering	$\downarrow$ centering
$\{a_k - A\}$	$\{\lambda_k - E\}$
$\downarrow$ rescale to density one	$\downarrow$
$\left\{\frac{a_k - A}{\gamma_A}\right\}$	$\left\{\frac{\lambda_k - E}{\gamma_E}\right\}$

. Now we have two comparable sets of data.

Step 2: Comparison with R.M.

. The scientist can compare the statistical properties of the two set of data and if the fit is good, he concludes that the system is well-modeled by a R.M. Theory.

. Examples: Nearest-neighbor spacing statistics.

(a) R.M. considered above : "GOE" : Analytic

(b) Nuclei resonances : Experimental.

(c) Spectrum of a free particle in a stadium: Numerics.

(d) Zeros of Riemann-zeta function : Deterministic! (This is GUE-like)

A List of Figures:

(a) GOE = nearest-neighbor spacing for eigenvalues: Figures 1, 2.

(b) Nuclei resonances: Figures 3, 4.2.

(c) Stadium spectrum: Figure 4.3.

($Solve\ -\Delta\psi = E\psi,\ \psi(\partial\text{stadium}) = 0$).

(d) Riemann-ζ function: Figures 5.1, 5.2, 5.3.

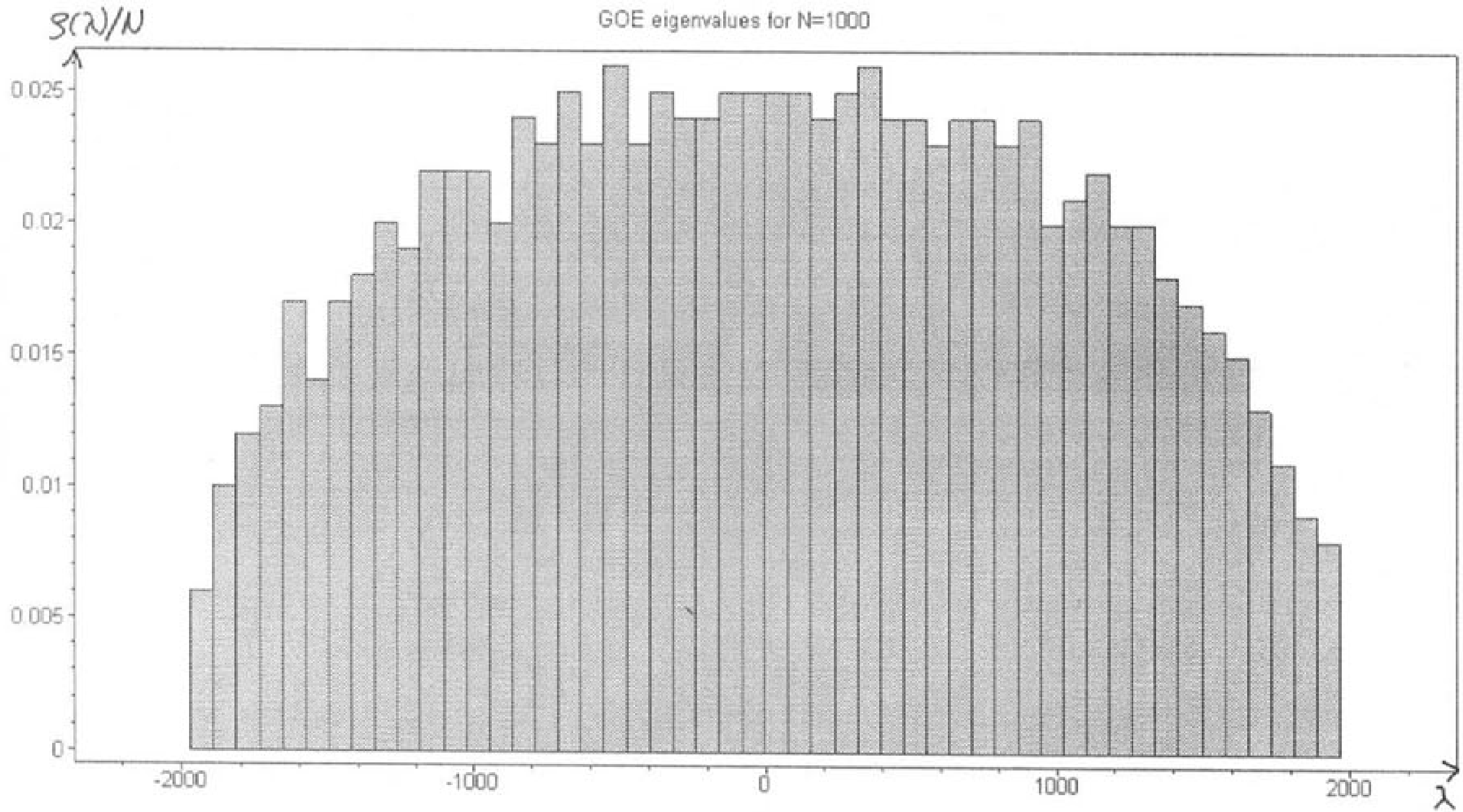
$$\zeta(z) \doteq \sum_{n=1}^{\infty} \frac{1}{n^z} = \prod_{\text{primes } p} \frac{1}{1-\frac{1}{p^z}}, \quad \mathrm{Re}\, z > 1$$

$\oplus$ Analytic continuation for $\mathrm{Re}\, z \leq 1$.

Look at $\{\gamma_n\}_{n\geq 1}$ s.t. $\zeta(\frac{1}{2} \pm i\gamma_n) = 0$. (non-trivial zeros).

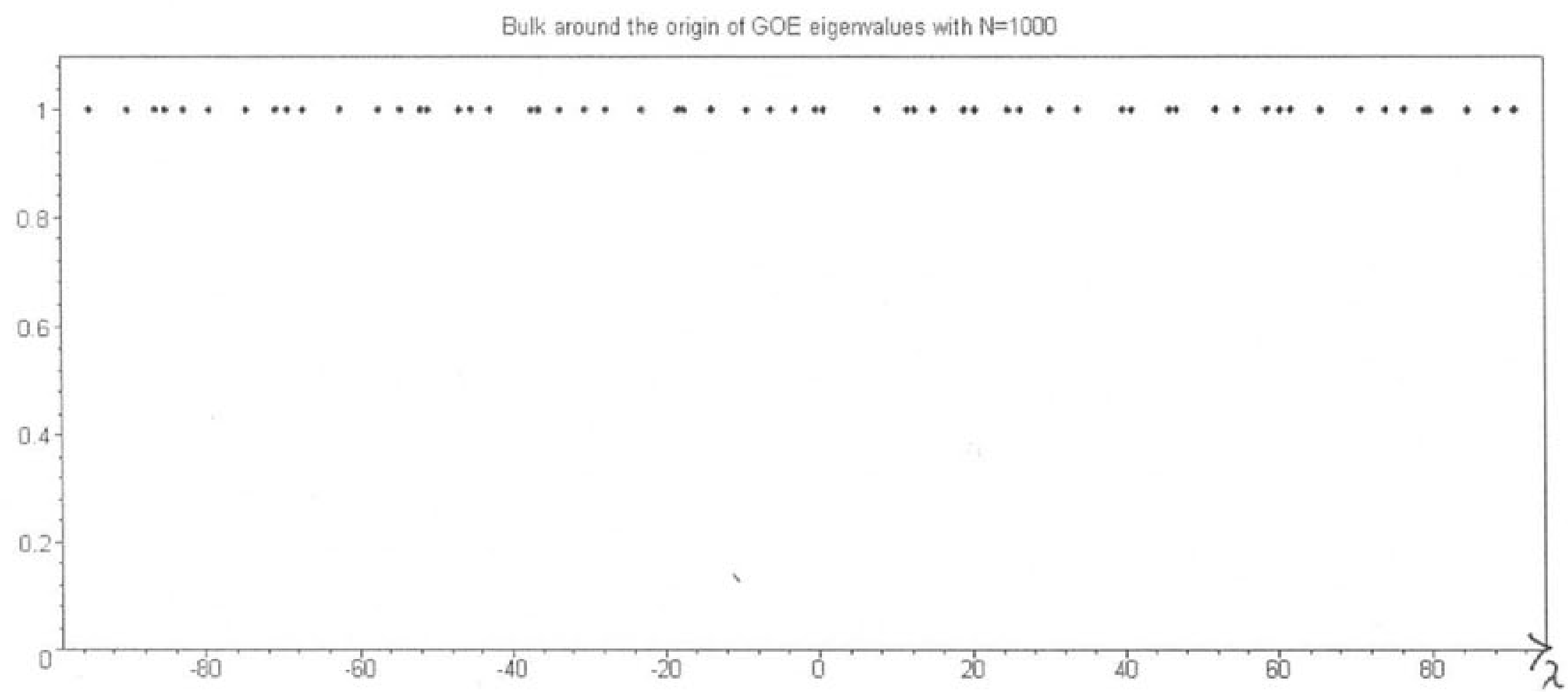
$\cong$ GUE instead of GOE

Figure 1



Eigenvalue density of a $N \times N$ matrix in the GOE ensemble, $N=1000$.

Figure 2



Eigenvalues in the bulk of the $N \times N$ GOE matrix, $N = 1000$,

Figure 3

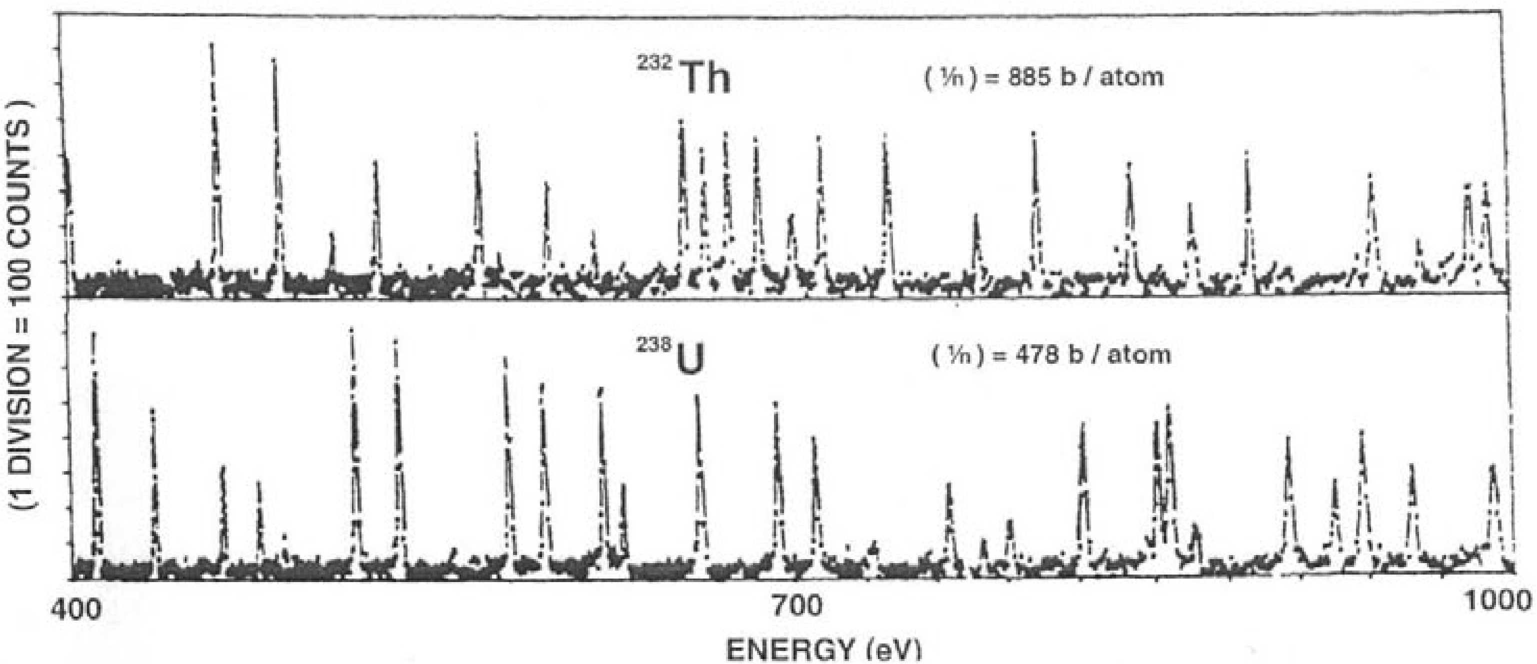


Figure 1.1. Slow neutron resonance cross-sections on thorium 232 and uranium 238 nuclei. Reprinted with permission from The American Physical Society, Rahn et al., Neutron resonance spectroscopy, X, *Phys. Rev. C* 6, 1854–1869 (1972).

. Taken from Mehta book "Random Matrices", page 2.

Figure 4.3

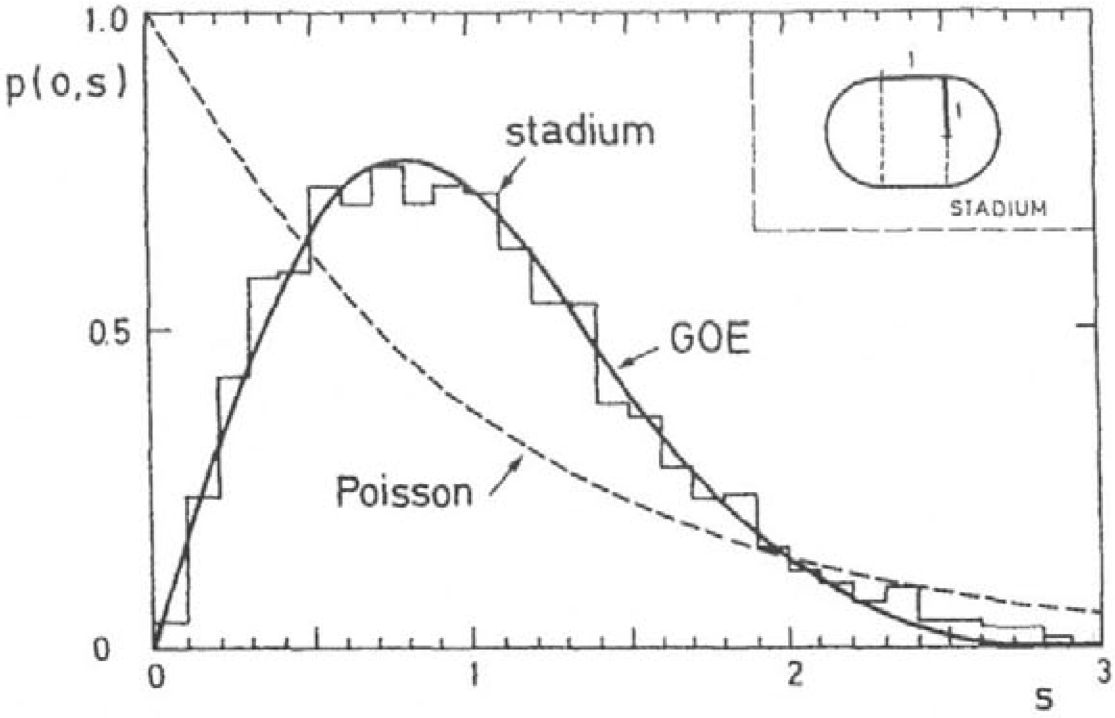


Figure 7.7. Empirical probability density of the nearest neighbor spacings of the possible energies of a particle free to move on the stadium consisting of a rectangle of size 1×2 with semi-circular caps of radius 1, depicted in the right upper corner. The stadium can be defined by the inequalities $|y| \leqslant 1$, and either $|x| \leqslant 1/2$ or $(x \pm 1/2)^2 + y^2 \leqslant 1$. The solid curve represents Eq. (7.3.19) corresponding to the Gaussian orthogonal ensemble (GOE), while the dashed curve is for the Poisson process corresponding to no correlations. Supplied by O. Bohigas, from Bohigas et al. (1984a).

. Mehta book, page 172.

Figure 4.2

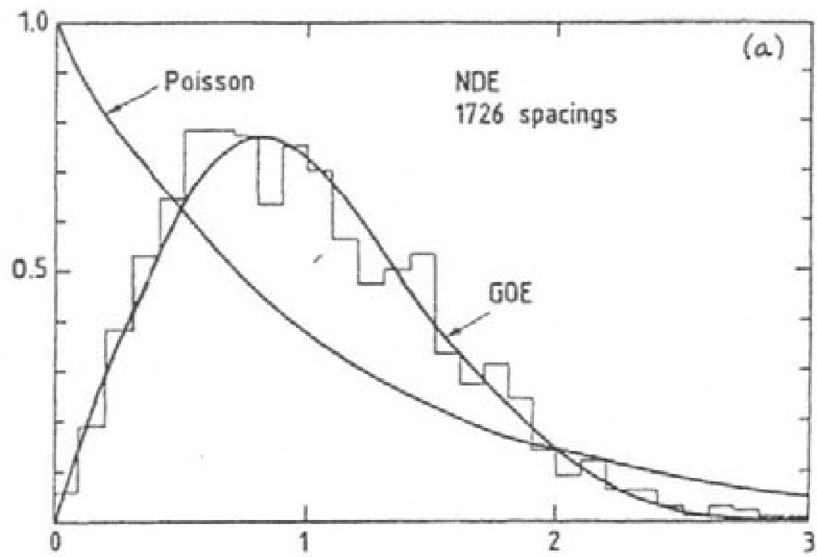


Figure 1.4. Level spacing histogram for a large set of nuclear levels, often referred to as nuclear data ensemble. The data considered consists of 1407 resonance levels belonging to 30 sequences of 27 different nuclei: (i) slow neutron resonances of Cd(110, 112, 114), Sm(152, 154), Gd(154, 156, 158, 160), Dy(160, 162, 164), Er(166, 168, 170), Yb(172, 174, 176), W(182, 184, 186), Th(232) and U(238); (1146 levels); (ii) proton resonances of Ca(44) ($J = 1/2+$), Ca(44) ($J = 1/2-$), and Ti(48) ($J = 1/2+$); (157 levels); and (iii) (n, γ)-reaction data on Hf(177) ($J = 3$), Hf(177) ($J = 4$), Hf(179) ($J = 4$), and Hf(179) ($J = 5$); (104 levels). The data chosen in each sequence is believed to be complete (no missing levels) and pure (the same angular momentum and parity). For each of the 30 sequences the average quantities (e.g. the mean spacing, spacing/mean spacing, number variance μ_2, etc., see Chapter 16) are computed separately and their aggregate is taken weighted according to the size of each sequence. The solid curves correspond to the Poisson distribution, i.e. no correlations at all, and that for the eigenvalues of a real symmetric random matrix taken from the Gaussian orthogonal ensemble (GOE). Reprinted with permission from Kluwer Academic Publishers, Bohigas O., Haq R.U. and Pandey A., Fluctuation properties of nuclear energy levels and widths, comparison of theory with experiment, in: *Nuclear Data for Science and Technology*, Bökhoff K.H. (Ed.), 809–814 (1983).

. Mehta book, page 13.

Figuro 5.1

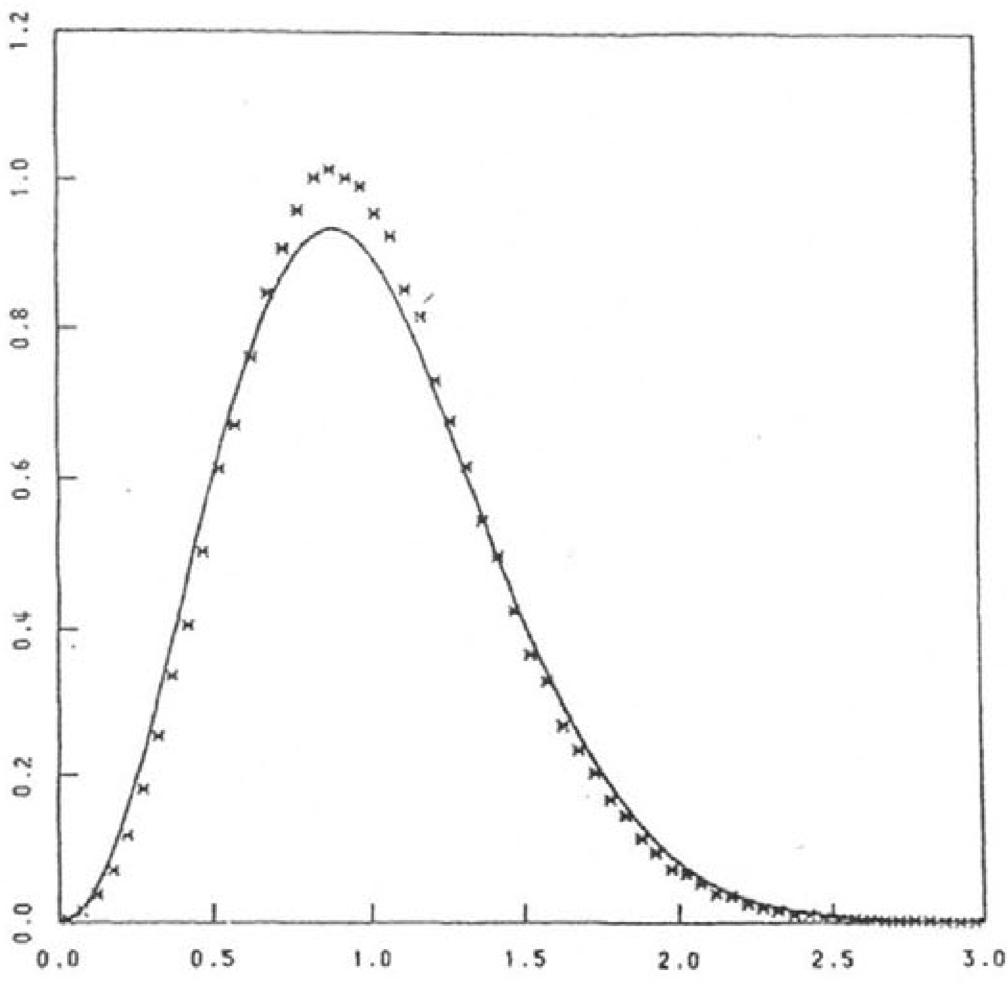


Figure 1.12. Plot of the density of normalized spacings for the zeros $0.5 \pm i\gamma_n$, γ_n real, of the Riemann zeta function on the critical line. $1 < n < 10^5$. The solid curve is the spacing probability density for the Gaussian unitary ensemble, Eq. (6.4.32). From Odlyzko (1987). Reprinted from "On the distribution of spacings between zeros of the zeta function," *Mathematics of Computation* (1987), pages 273–308, by permission of The American Mathematical Society.

. Mehta book, page 24.

Figure 5.2

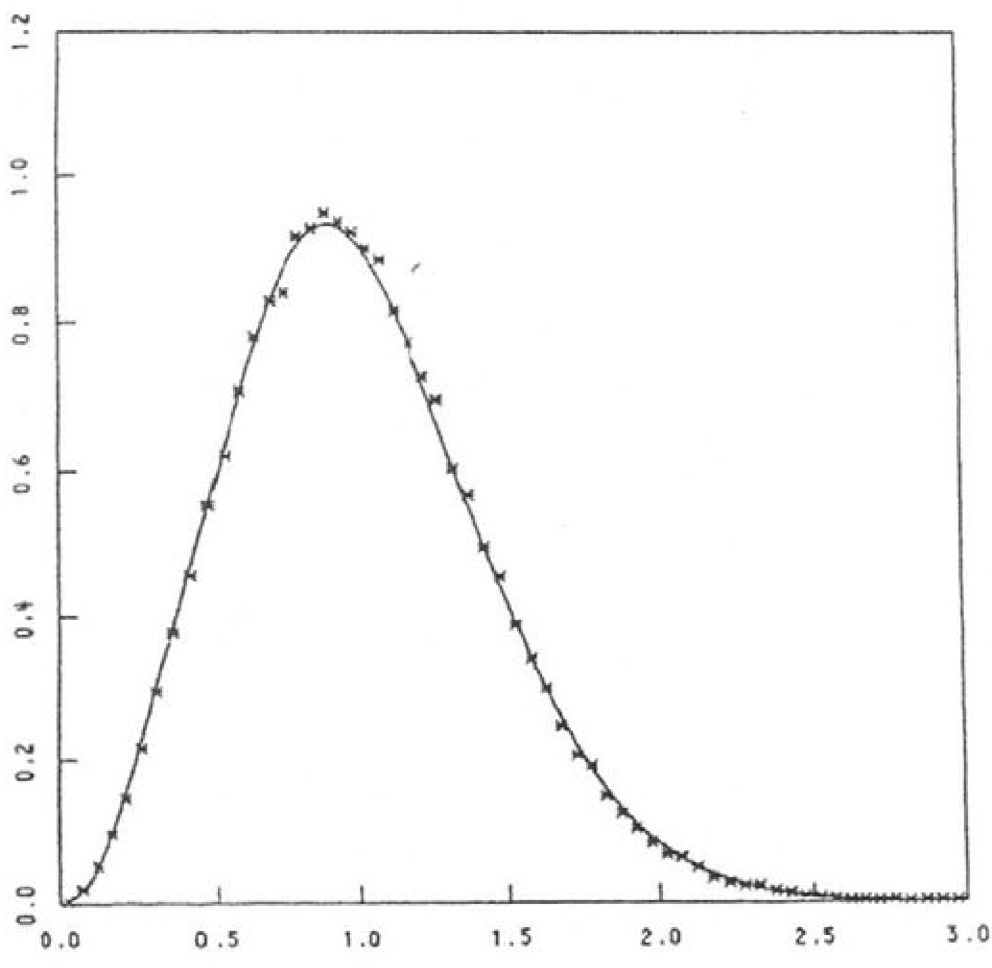


Figure 1.13. The same as Figure 1.12 with $10^{12} < n < 10^{12} + 10^{5}$. Note the improvement in the fit. From Odlyzko (1987). Reprinted from "On the distribution of spacings between zeros of the zeta function," *Mathematics of Computation* (1987), pages 273–308, by permission of The American Mathematical Society.

. Mehta book, page 25.

Figura 5.3

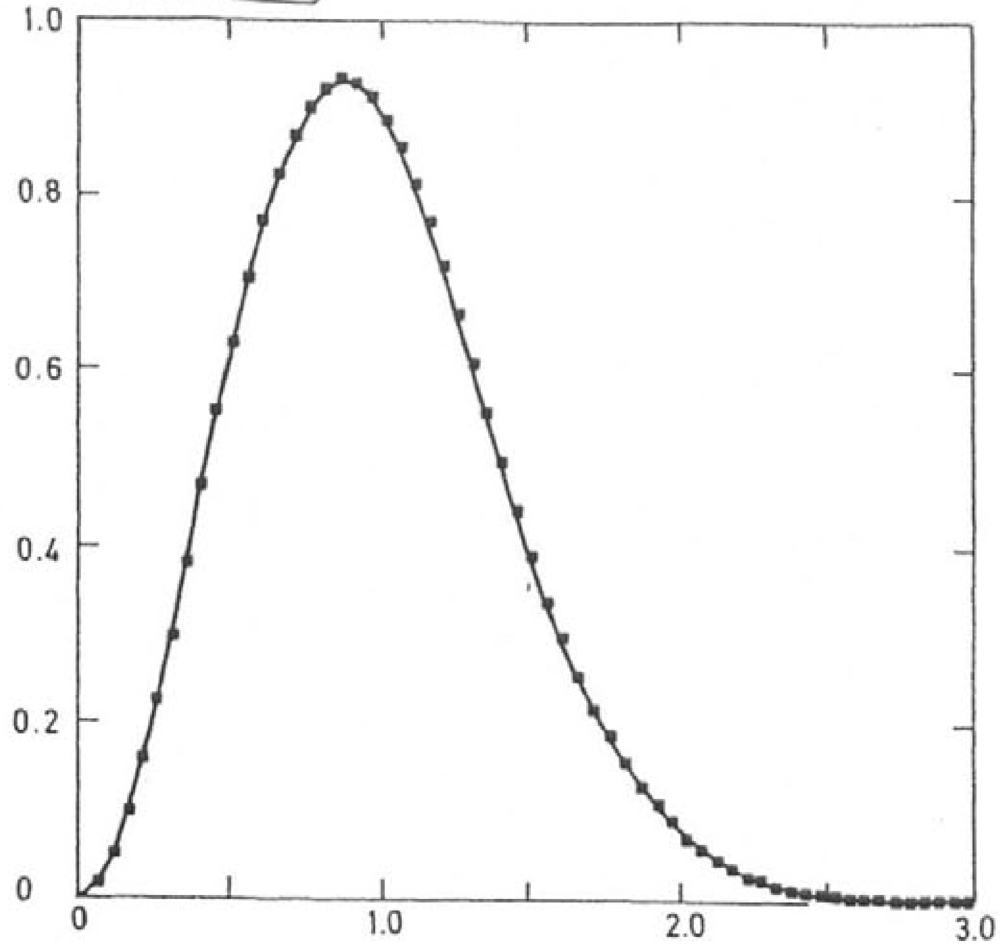


Figure 1.14. The same as Figure 1.12 but for the 79 million zeros around the 10^{20}th zero. From Odlyzko (1989). Copyright © 1989, American Telephone and Telegraph Company, reprinted with permission.

. Mehta book, page 26.

2) Gaussian Unitary / Orthogonal Ensemble.

2.1) Definitions.

. The Gaussian Ensembles of random matrices have been introduced by physicists (Dyson, Wigner...) in the 60's to model statistical properties of heavy nuclei resonance spectrum. The different ensembles are related with intrinsic symmetries of the system.

. A real symmetric matrix can a priori describe a system with : → time reversal and (rotation invariant or integer magnetic momentum).

[. A real quaternionic matrix (basis ≡ Pauli matrices) : time reversal and half-integer magnetic momentum.]

. A complex hermitian matrix : not time-reversal (e.g. with external magnetic field).

Definition 1: (GOE random matrices).

. The Gaussian Orthogonal Ensemble of random matrices is a measure P on the set of $N\times N$ real symmetric matrices given by:

$$P(H)\,dH = \text{const}\times\exp\left(-\frac{\mathrm{Tr}(H^2)}{4N}\right)dH, \tag{2.1}$$

where dH is the flat reference measure: $dH = \prod_{1\le i\le j\le N} dH_{ij}$

Remark: . "Orthogonal" because (2.1) is invariant under orthogonal transformation.

Definition 2: (GUE). The Gaussian Unitary Ensemble of random matrices is a measure P on the set of $N\times N$ complex hermitian matrices given by:

$$P(H)\,dH = \text{const}\times\exp\left(-\frac{\mathrm{Tr}(H^2)}{2N}\right)dH, \tag{2.2}$$

where $dH = \prod_{i=1}^{N} dH_{ii} \prod_{1\le i<j\le N} d\,\mathrm{Re}H_{ij}\, d\,\mathrm{Im}H_{ij}$.

<u>Remark:</u> . The original definition is different: $p(H)$ is defined to be : (1) invariant under the change of basis (the group of symmetry: $O(N)$, $U(N)$).

$\Rightarrow p(H) = \text{fct}(\text{Tr}(H^k), k=1,\dots,N)$.

(2) the entries of the matrices are independent random variables (up to the imposed symmetry).

$\Rightarrow p(H) \propto \exp\left(-a \cdot \text{Tr}(H^2) + b\,\text{Tr}(H) + c\right)$, $a>0$, $b, c \in \mathbb{R}$.

→ Shifting the zero of the energy, one can get rid of b, while c is just a normalisation constant.

2.2) Eigenvalues' distributions.

. Often one is interested in the eigenvalues of the matrix, since they are independent of the choice of basis.

<u>Proposition 3:</u> Let $\lambda_1, \lambda_2, \dots, \lambda_N$ be the eigenvalues of a GOE/GUE. Then, the joint distribution of eigenvalues is given by

$$(2.3) \qquad p(\lambda)\underset{(d\lambda_1 \cdots d\lambda_N)}{\underset{\shortparallel}{d\lambda}} = \text{const} \times \Bigg[\underbrace{\prod_{1\le i<j\le N} (\lambda_j - \lambda_i)}_{\equiv \Delta_N(\lambda)}\Bigg]^{\beta} \cdot \prod_{i=1}^{N} \left(e^{-\frac{\lambda_i^2 \beta}{4N}} d\lambda_i\right)$$

where: $\begin{cases} \beta = 1 \text{ for GOE}, \\ \beta = 2 \text{ for GUE}. \end{cases}$

<u>Remark:</u> $\prod_{1\le i<j\le N} (\lambda_j - \lambda_i) = \det\left(\lambda_i^{j-1}\right)_{1\le i,j\le N}$, so it is a determinant, the Vandermonde determinant.

. One can prove by induction:

$$\det\left(\lambda_i^{j-1}\right)_{1\le i,j\le N} = \begin{vmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{N-1} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{N-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \lambda_N & \lambda_N^2 & \cdots & \lambda_N^{N-1} \end{vmatrix} = \begin{vmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{N-1} \\ 0 & \lambda_2 - \lambda_1 & \lambda_2^2 - \lambda_1^2 & \cdots & \lambda_2^{N-1} - \lambda_1^{N-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & \lambda_N - \lambda_1 & \lambda_N^2 - \lambda_1^2 & \cdots & \lambda_N^{N-1} - \lambda_1^{N-1} \end{vmatrix}$$

$$= \prod_{i=2}^{N} (\lambda_i - \lambda_1) \cdot \underbrace{\begin{bmatrix} 1 & \lambda_2+\lambda_1 & \cdots & \lambda_2^{N-2}+\cdots+\lambda_1^{N-2} \\ \vdots & & & \\ 1 & \lambda_N+\lambda_1 & \cdots & \lambda_N^{N-2}+\cdots+\lambda_1^{N-2} \end{bmatrix}}_{\overset{\text{lin. comb.}}{=} \Delta_{N-1}(\lambda_2,\ldots,\lambda_N)} \quad \#$$

<u>Proof of Proposition 3:</u> Let us prove it for GOE. The GUE case is proven similarly.

. Given a matrix H, real symmetric, $\exists\, g \in O(N)$ st.

$$H = g \Lambda g^{-1}, \quad \text{with } \Lambda_{ij} = \lambda_i \cdot \delta_{ij}, \quad 1 \le i,j \le N.$$

. Since we have $\frac{N(N+1)}{2}$ independent entries of H and only N eigenvalues, it remains $\frac{N(N-1)}{2}$ "angular variables" in g's.

(a). An infinitesimal transformation of H gives:

$$dH = dg \cdot \Lambda \cdot g^{-1} + g \cdot \Lambda \cdot dg^{-1} + g\, d\Lambda \cdot g^{-1}$$

Since: $\mathbb{1} = g \cdot g^{-1} \Rightarrow dg \cdot g^{-1} = -g \cdot dg^{-1} \Rightarrow dg^{-1} = -g^{-1} dg\, g^{-1}$

$$\Rightarrow dH = g \cdot [g^{-1} dg \Lambda - \Lambda g^{-1} dg + d\Lambda] \cdot g^{-1}$$

$$= g \cdot d\tilde{H} \cdot g^{-1} \quad \text{where } d\tilde{H} = d\Lambda + [\underbrace{g^{-1}dg}_{\doteq d\Omega \,:\, \text{angular variables}}, \Lambda]$$

$\Rightarrow$ Jacobian $H \to \tilde{H}$ is one.

(b) Jacobian $\tilde{H} \to (\Lambda, \Omega)$:

In components: $d\tilde{H}_{ij} = d\lambda_i \cdot \delta_{ij} + \sum_{i=1}^{N} (d\Omega_{ik} \delta_{kj} \lambda_j - \lambda_i \delta_{ik} d\Omega_{kj})$

$$= d\lambda_i \cdot \delta_{ij} + d\Omega_{ij} (\lambda_j - \lambda_i).$$

$\Rightarrow$ Jacobian $\tilde{H} \to (\Lambda, \Omega)$ equal to

$$\left| \det \left[\frac{\partial(\tilde{H}_{11}, \ldots, \tilde{H}_{NN}; \tilde{H}_{12}, \ldots, \tilde{H}_{1N}, \ldots, \tilde{H}_{N-1,N})}{\partial(\lambda_1, \ldots, \lambda_N; \Omega_{12}, \ldots, \Omega_{1N}, \ldots, \Omega_{N-1,N})} \right] \right| = \quad /.$$

$$= \left| \det \begin{pmatrix} \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} & & & \\ & \lambda_1 - \lambda_2 & & O \\ & & \lambda_1 - \lambda_N & \\ O & & & \ddots \\ & & & & \lambda_{N-1} - \lambda_N \end{pmatrix} \right| = |\Delta_N(\lambda)|.$$

. Therefore: $dH = |\Delta_N(\lambda)| d\lambda \cdot \underbrace{d\Omega}_{\text{Haar measure on } SO(N)}$.

$$(2.1) \Rightarrow p(H)\, dH = \text{const} \times \exp\left(-\frac{\beta}{4N} \cdot \mathrm{Tr}(H^2)\right) \cdot \Delta_N(\lambda)\, d\lambda\, d\Omega$$
$$= \text{const} \cdot d\Omega \cdot |\Delta_N(\lambda)|^{\beta} \prod_{i=1}^{N} \left(e^{-\frac{\beta}{4N}\lambda_i^2} d\lambda_i\right).$$

. (c) Integrate out the angular variables, leads to (2.3).

. For the GUE ensemble, the only difference is that for the upper-diagonal components we have $\mathrm{Re}\, H_{ij}$ and $\mathrm{Im}\, H_{ij}$, and in the angular variables too: $\mathrm{Re}\, \Omega_{ij}$, $\mathrm{Im}\, \Omega_{ij}$. This gives each factor $\lambda_i - \lambda_j$ twice. #

2.3) Correlation functions for GUE.

2.3.1) Generalities.

. Consider a measure like (2.3) and take any bounded disjoints Borel sets $A_1, \dots, A_n$ of $\mathbb{R}$. Then, denote

$$(2.4) \qquad M_n(A_1, \dots, A_n) = \mathbb{E}\left(\prod_{i=1}^{n} (\#\text{eigenvalues in } A_i)\right),$$

where $\mathbb{E}$ is the expectation under the measure.

Definition 4: (Correlation functions). If M_n is absolutely continuous with respect to a reference measure μ^n on $\mathbb{R}^n$, i.e., if $M_n(A_1, \dots, A_n) = \int_{A_1 \times \dots \times A_n} d\mu(x_1) \dots d\mu(x_n)\, \rho^{(n)}(x_1, \dots, x_n)$, $\forall$ Borel sets A_i in $\mathbb{R}$, then we call $\rho^{(n)}$ the n-point correlation function.

Remarks: • $\rho^{(n)}(x_1,\dots,x_n) = \rho^{(n)}(x_{\sigma_1},\dots,x_{\sigma_n})$, $\forall \sigma \in S_n$ (symmetry).

• It makes sense to speak about $\rho^{(n)}$ only specifying the reference measure.

If $\mu(dx) = dx$ • Probabilistic interpretation: In the case where a.s. no double points occurs (i.e., for simple point processes), we have the following probabilistic interpretation:

. Let $[x_i, x_i+\Delta x_i]$, $i=1,\dots,n$ be disjoint infinitesimally small sets. Then, we will have at most one point in each $[x_i, x_i+\Delta x_i]$ and

$$\rho^{(n)}(x_1,\dots,x_n) = \lim_{\substack{\Delta x_i \to 0 \\ i=1,\dots,n}} \frac{\mathbb{P}(\text{one point in each } [x_i,x_i+\Delta x_i],\ 1\le i\le n)}{\Delta x_1 \cdots \Delta x_n}$$

[To see it: Take $A_i = [x_i, x_i+dx_i)$ in (2.4).]

• $\rho^{(1)}(x)$ is the density of points at x.

Lemma 5: Consider the case (like GUE) where the probability density $P_N(x_1,\dots,x_N)$ is symmetric on $\mathbb{R}^N$.

Then,

$$\rho^{(n)}(x_1,\dots,x_n) = \frac{N!}{(N-n)!} \int_{\mathbb{R}^{N-n}} dx_{n+1}\cdots dx_N \cdot P_N(x_1,\dots,x_N). \qquad (2.6)$$

Proof: $\rho^{(n)}(x_1,\dots,x_n)$ is the probability density of finding a particle at $x_1,\dots$, a particle at x_n, but it does not say which of the N particle is at which x_i's.

By the symmetry of P_N, each choice gives a contribution

$$\int_{\mathbb{R}^{N-n}} dx_{n+1}\cdots dx_N P_N(x_1,\dots,x_N),$$

and there are $n!\binom{N}{n} = \frac{N!}{(N-n)!}$ possible choices. #

2.3.2) $P_N(x_1,\dots,x_N)$ and orthogonal polynomials for GUE.

. Consider the weight $w(x) = \exp\left(-\frac{x^2}{2N}\right)$ and define the orthogonal polynomials $\{q_k(x), k = 0,\dots,N-1\}$ by the following conditions:

(1) $q_k(x)$ is of degree k with $q_k(x) = u_k x^k + \dots$, $u_k > 0$.

(2) they are orthonormal:

$$\int_{\mathbb{R}} dx\, w(x)\, q_k(x)\, q_\ell(x) = \delta_{k\ell}.$$

. Notice that: $\det\left(\lambda_i^{j-1}\right)_{1\le i,j\le N} = \text{const} \times \det\left(q_{j-1}(\lambda_i)\right)_{1\le i,j\le N}$.

Thus, the eigenvalues' measure for GUE ((2.3) for $\beta=2$) is

$$P_N(\lambda_1,\dots\lambda_N) = \frac{1}{Z_N} \times \left(\prod_{i=1}^N w(\lambda_i)\right) \cdot \left(\det\left(q_{i-1}(\lambda_j)\right)_{1\le i,j\le N}\right)^2$$

$$\underset{\text{multiplications}}{\overset{\text{matrix}}{=\!=\!=}} \frac{1}{Z_N}\left(\prod_{i=1}^N w(\lambda_i)\right) \cdot \det\left(\sum_{k=1}^N q_{k-1}(\lambda_i)\, q_{k-1}(\lambda_j)\right)_{1\le i,j\le N}.$$

. Define the kernel: $\boxed{K_N(x,y) = w(x)^{1/2}\, w(y)^{1/2} \cdot \sum_{k=1}^N q_{k-1}(x)\, q_{k-1}(y).}$

(2.7) Then, $P_N(\lambda_1,\dots,\lambda_N) = \frac{1}{Z_N} \cdot \det\left[K_N(\lambda_i,\lambda_j)\right]_{1\le i,j\le N}$.

. Two properties of K_N: (proof follows from orthonormality property of q_k's).

(2.8) • $\displaystyle\int_{\mathbb{R}} dx\, K_N(x,x) = N.$

(2.9) • $\displaystyle\int_{\mathbb{R}} dz\, K_N(x,z)\, K_N(z,y) = K_N(x,y).$

⚠ Orthogonal polynomials plays a central role here!

Proposition 6: The n-point correlation functions of GUE are given by:

(2.10) $\rho^{(n)}(\lambda_1,\dots,\lambda_n) = \det\left(K_N(\lambda_i,\lambda_j)\right)_{1\le i,j\le n}$, with respect to the flat measure $d\lambda$.

Proof of Proposition 6:

(a) Determination of Z_N in (2.7).

- For $n=N$, by Lemma 5, $\rho^{(N)}(\lambda_1,\dots,\lambda_N) = P_N(\lambda_1,\dots,\lambda_N)\, N!$

$$\Rightarrow \int_{\mathbb{R}^N} d\lambda_1 \dots d\lambda_N\, \rho^{(N)}(\lambda_1,\dots,\lambda_N) = N! \quad \text{, i.e.,}$$

$$\int_{\mathbb{R}^N} d\lambda_1 \dots d\lambda_N\, P_N(\lambda_1 \dots \lambda_N) = 1$$

$$\begin{array}{c} \parallel\! | \end{array}$$

$$\frac{1}{Z_N} \int_{\mathbb{R}^N} d\lambda_1 \dots d\lambda_N\, w(\lambda_1) \dots w(\lambda_N)\, \det\left(q_{i-1}(\lambda_j)\right) \det\left(q_{i-1}(\lambda_j)\right)$$

$$= \frac{1}{Z_N} \cdot N! \cdot \det\Bigg(\underbrace{\int_{\mathbb{R}^N} d\lambda\, w(\lambda)\, q_{i-1}(\lambda)\, q_{j-1}(\lambda)}_{=\delta_{i,j}} \Bigg) = \frac{N!}{Z_N} \Rightarrow \underline{Z_N = N!}$$

$\uparrow$

<u>Used Cauchy-Binet (Heine) identity:</u>

$$\int_{\Lambda^N} d\lambda(x_1) \dots d\lambda(x_N)\, \det\left(\phi_i(x_j)\right) \det\left(\psi_i(x_j)\right) = N! \det\left(\int_{\Lambda} d\lambda(x)\, \phi_i(x)\, \psi_j(x) \right)$$

(b) By Lemma 5:

$$\rho^{(n)}(\lambda_1,\dots,\lambda_n) = \frac{1}{(N-n)!} \cdot \int_{\mathbb{R}^{N-n}} d\lambda_{n+1} \dots d\lambda_N\, \det\left(K_N(\lambda_i,\lambda_j)\right)_{1\le i,j\le N}.$$

- We integrate $N-n$ variables. Consider the case when we have a $m\times m$ matrix

$$\int_{\mathbb{R}} dx_m\, \det\left(K_N(x_i,x_j)\right)_{1\le i,j\le m} = \int_{\mathbb{R}} K_N(x_m,x_m) \cdot \det\left(K_N(x_i,x_j)\right)_{1\le i,j\le m-1} dx_m \qquad \boxed{\text{Develop on last column.}}$$

$$+ \int \sum_{k=1}^{m-1} (-1)^{m-k} \cdot K_N(x_k,x_m) \det\begin{bmatrix} K_N(x_i,x_j) \\ \hline K_N(x_m,x_j) \end{bmatrix}_{\substack{1\le i,j\le m-1 \\ i\neq k}} dx_m$$

$$\overset{\text{linearity}}{=} \int_{\mathbb{R}} dx_m\, K_N(x_m,x_m) \cdot \det\left(K_N(x_i,x_j)\right)_{1\le i,j\le m-1}$$

$$+ \sum_{k=1}^{m-1} (-1)^{m-k} \cdot \det\begin{bmatrix} K_N(x_i,x_j) \\ \hline \int K_N(x_k,x_m) \cdot K_N(x_m,x_j)\, dx_m \end{bmatrix}$$

$$\overset{(2.8),(2.9)}{=} (N-(m-1)) \cdot \det\left(K_N(x_i,x_j)\right)_{1\le i,j\le m-1}.$$

- By iteration we get: $\rho^{(n)}(\lambda_1,\dots,\lambda_n) = \frac{\cancel{1\cdot 2 \cdots (N-n)}}{\cancel{(N-n)!}} \cdot \det\left(K_N(\lambda_i,\lambda_j)\right)_{1\le i,j\le n}$. #

2.3.3) Different representations of the Kernel.

(A) $K_N(x,y) = w(x)^{1/2} w(y)^{1/2} \cdot \sum_{k=1}^{N} q_{k-1}(x)\, q_{k-1}(y).$

with $w(x) = \exp\left(-\frac{x^2}{2N}\right)$ and q_k are the orthogonal polynomials, degree k, satisfying $\int_{\mathbb{R}} dx\, w(x)\, q_k(x)\, q_\ell(x) = \delta_{k,\ell}$.

. q_k's can be written in terms of the standard Hermite polynomials:

(2.11) $$H_k(x) := e^{x^2} \frac{d^k}{dx^k} e^{-x^2} \text{ satisfying } \int_{\mathbb{R}} H_k(x) H_\ell(x)\, e^{-x^2} dx = \sqrt{\pi}\cdot 2^k k!\, \delta_{k,\ell}.$$

with $H_k(x) = 2^k x^k + \ldots$

. A simple calculation gives:

(2.12) $$q_k(x) = \frac{1}{\sqrt[4]{2\pi N}} \cdot \frac{1}{\sqrt{2^k \cdot k!}} \cdot H_k\left(\frac{x}{\sqrt{2N}}\right) \Rightarrow u_k = \frac{1}{\sqrt[4]{2\pi N}} \cdot \frac{1}{\sqrt{2^k k!}} \cdot \left(\frac{2}{N}\right)^{k/2}.$$

(B) Using Christoffel-Darboux formula:

(2.13) $$\sum_{k=0}^{N-1} q_k(x)\, q_k(y) = \begin{cases} \frac{u_{N-1}}{u_N} \cdot \frac{q_N(x) q_{N-1}(y) - q_{N-1}(x) q_N(y)}{x-y}, & \text{for } x \neq y, \\ \frac{u_{N-1}}{u_N} \cdot \left[q_N'(x) q_{N-1}(x) - q_{N-1}'(x) q_N(x)\right], & \text{for } x = y. \end{cases}$$

. Then, (A) can be rewritten as:

(2.14) $$K_N(x,y) = \begin{cases} N \cdot e^{-\frac{x^2+y^2}{4N}} \cdot \frac{q_N(x) q_{N-1}(y) - q_{N-1}(x) q_N(y)}{x-y}, & \text{for } x \neq y, \\ N \cdot e^{-\frac{x^2}{2N}} \cdot \left[q_N'(x) q_{N-1}(x) - q_{N-1}'(x) q_N(x)\right], & \text{for } x = y. \end{cases}$$

2.4) Bulk and edge scaling limits.

2.4.1) Wigner's semicircle law.

. With the chosen rescaling: $p(H) = e^{-\frac{Tr(H^2)}{2N}}$, the largest eigenvalue is around $2N$ and the smallest eigenvalue around $-2N$. Since there are exactly N eigenvalues, the density will be of order one, without extra rescaling.

. Let $\rho_N(\lambda)$ be the density of eigenvalues around $2N.\lambda$.
Then,

$$\rho_\infty(\lambda) := \lim_{N\to\infty} \rho_N(\lambda) = \lim_{N\to\infty} K_N(2N\lambda, 2N\lambda) = \begin{cases} \frac{1}{\pi}.\sqrt{1-\lambda^2}, & \lambda \in [-1,1], \\ 0 &, \text{otherwise.} \end{cases} \tag{2.16}$$

. This is called Wigner's semicircle law.

2.4.2) Bulk scaling limit: Sine kernel.

. Let us consider the eigenvalues in the bulk around $2N\lambda$ for some $\lambda \in (-1,1)$ and rescale space to have density one:

$$\begin{cases} X = 2N\lambda + \frac{\xi_1}{\rho_\infty(\lambda)} \\ Y = 2N\lambda + \frac{\xi_2}{\rho_\infty(\lambda)} \end{cases} \tag{2.17}$$

Since the Kernel is related to correlation functions ($\sim$ densities), we also have to rescale it accordingly:

$$K_N^{resc}(\xi_1,\xi_2) := \frac{1}{\rho_\infty(\lambda)} K_N\left(2N\lambda + \frac{\xi_1}{\rho_\infty(\lambda)}, 2N\lambda + \frac{\xi_2}{\rho_\infty(\lambda)}\right). \tag{2.18}$$

<u>Proposition 7:</u>

$$\underline{\lim_{N\to\infty} K_N^{resc}(\xi_1,\xi_2) = \frac{\sin(\pi\cdot(\xi_1-\xi_2))}{\pi(\xi_1-\xi_2)}}. \tag{2.19}$$

<u>This is called the Sine Kernel.</u>

Proof of Proposition 7: Recall the form of the Kernel:

$$K_N^{resc}(\xi_1,\xi_2) = \frac{N\cdot}{\rho_{sc}(\lambda)}\, \frac{q_N(X)\,q_{N-1}(Y) - q_{N-1}(X)\,q_N(Y)}{(\xi_1-\xi_2)\,\frac{1}{\rho_{sc}(\lambda)}} \cdot e^{-\frac{X^2}{4N}} \cdot e^{-\frac{Y^2}{4N}}$$

. The asymptotics of the Hermite polynomials are known, from which:

$$\sqrt{N}\cdot q_{N-m}(x)\cdot e^{-\frac{x^2}{4N}} \underset{x = 2N\lambda + \frac{\xi}{\rho_{sc}(\lambda)}}{\cong} \frac{1}{\pi\cdot\sqrt{\rho_{sc}(\lambda)}}\cdot \sin\left[\alpha_0 N + \pi\cdot\xi + m\cdot\varphi\right]$$

with $\varphi = a\cos(\lambda)$.

$$\Rightarrow K_N^{resc}(\xi_1,\xi_2) \underset{N\to\infty}{\cong} \frac{1}{\pi^2\cdot\rho_{sc}(\lambda)(\xi_1-\xi_2)}\cdot\Big\{\sin(\alpha_0 N+\pi\xi_1)\sin(\alpha_0 N+\pi\xi_2+\varphi) - \sin(\alpha_0 N+\pi\xi_1+\varphi)\cdot\sin(\alpha_0 N+\pi\xi_2)\Big\}$$

. Use the identity: $\sin(a)\cdot\sin(b+\varphi) - \sin(a+\varphi)\sin(b) = \sin(\varphi)\cdot\sin(a-b)$

and obtain:

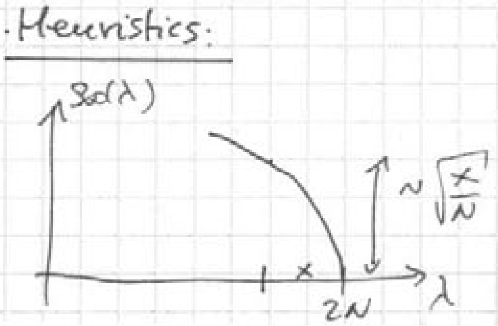
$$\lim_{N\to\infty} K_N^{resc}(\xi_1,\xi_2) = \frac{1}{\pi^2(\xi_1-\xi_2)}\cdot\frac{1}{\frac{1}{\pi}\sqrt{1-\lambda^2}}\cdot\sin(\pi(\xi_1-\xi_2))\cdot\underbrace{\sin(a\cos(\lambda))}_{=\sqrt{1-\lambda^2}}$$

$$= \frac{\sin(\pi(\xi_1-\xi_2))}{\pi(\xi_1-\xi_2)}. \quad \#$$

2.4.3) Edge scaling limit:

. The largest eigenvalue is around $2N$ and its fluctuations are on a $N^{1/3}$ scale. The $1/3$ exponent is connected with the square root behavior of the density at the edge of the spectrum.

. Heuristics:



$\#$ e.v. $\geq 2N-x \approx N\cdot\left(\frac{x}{N}\right)^{3/2} = \frac{x^{3/2}}{\sqrt{N}}$ over distance x.

$\Rightarrow \#$ e.v. $\geq 2N-x$ is of order one for $x\sim N^{1/3}$, i.e., the top eigenvalues fluctuates over distances $\mathcal{O}(N^{1/3})$.

. Therefore, the scaling limit is as follows:

(2.20) $$\begin{cases} x = 2N + \xi_1 N^{1/3} \\ y = 2N + \xi_2 N^{1/3} \end{cases}$$ and the rescaled kernel is

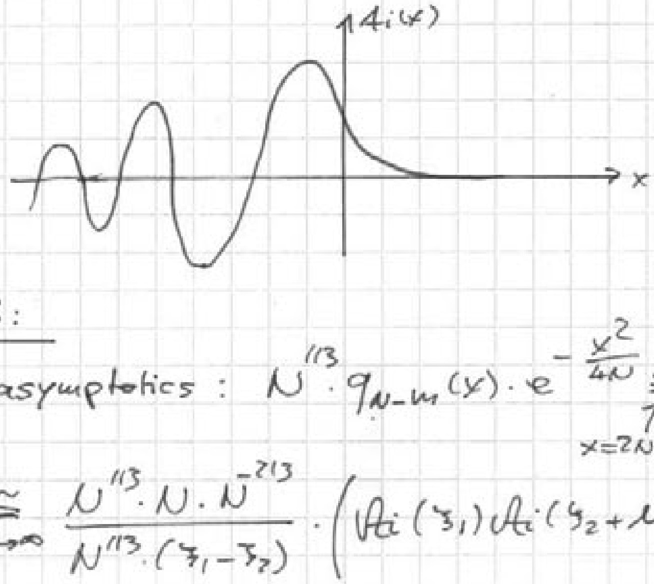
(2.21) $$K_N^{resc}(\xi_1, \xi_2) = N^{1/3} K_N(x, y).$$

. Proposition 8:

(2.22) $$\lim_{N\to\infty} K_N^{resc}(\xi_1, \xi_2) = \frac{Ai(\xi_1)Ai'(\xi_2) - Ai'(\xi_1)Ai(\xi_2)}{\xi_1 - \xi_2}$$

. This is called the Airy kernel: $K_{Ai}(\xi_1, \xi_2)$

. Remark: . $Ai(x)$ is the Airy function, solution of $y''(x) = x \cdot y(x)$ and with asymptotic behavior: $y(x) \sim \frac{e^{-\frac{2}{3}x^{3/2}}}{2\sqrt{\pi}\, x^{1/4}}$ as $x \to +\infty$.



Proof of Proposition 8:

. One uses the asymptotics: $N^{1/3} \cdot q_{N-m}(x) \cdot e^{-\frac{x^2}{4N}} \underset{x = 2N + \xi N^{1/3}}{\cong} Ai(\xi + m \cdot N^{-1/3})$.

$$\Rightarrow K_N^{resc}(\xi_1, \xi_2) \underset{N\to\infty}{\cong} \frac{N^{1/3} \cdot N \cdot N^{-2/3}}{N^{1/3} \cdot (\xi_1 - \xi_2)} \cdot \left(Ai(\xi_1) Ai(\xi_2 + N^{-1/3}) - Ai(\xi_1 + N^{-1/3}) Ai(\xi_2) \right)$$

$$\cong \frac{N^{1/3}}{(\xi_1 - \xi_2)} \left(Ai(\xi_1)(Ai(\xi_2) + N^{-1/3} Ai'(\xi_2)) - (\xi_1 \leftrightarrow \xi_2) \right)$$

$$= \frac{Ai(\xi_1)Ai'(\xi_2) - Ai'(\xi_1)Ai(\xi_2)}{\xi_1 - \xi_2}. \#$$

. Properties of the Airy kernel, $K_{\mathcal{A}_2}$:

① $K_{\mathcal{A}_2}(u,v) = \int_0^\infty d\lambda\, Ai(u+\lambda) Ai(v+\lambda)$.

Proof.: $(u-v) \cdot \int_0^\infty d\lambda\, Ai(u+\lambda) Ai(v+\lambda) = \int_0^\infty d\lambda\, (u+\lambda) Ai(u+\lambda) Ai(v+\lambda)$
$- \int_0^\infty d\lambda\, Ai(u+\lambda)(v+\lambda) Ai(v+\lambda)$

$Ai''(x) = x Ai(x)$

$$= \int_0^\infty d\lambda\, Ai''(u+\lambda) Ai(v+\lambda) - (u \leftrightarrow v)$$

$$= Ai(v+\lambda) Ai'(u+\lambda)\Big|_0^\infty - \int_0^\infty d\lambda\, Ai'(u+\lambda) Ai'(v+\lambda)$$

$$- (u \leftrightarrow v)$$

$$= Ai(u) Ai'(v) - Ai'(u) Ai(v). \quad \#$$

② $K_{\mathcal{A}_2}^2 = K_{\mathcal{A}_2}$ $\quad \int_0^\infty d\lambda \int_0^\infty d\mu\, Ai(u+\lambda) \Big(\underbrace{\int_{\mathbb{R}} dz\, Ai(\lambda+z) Ai(\mu+z)}_{=\delta(\mu-\lambda)} \Big) Ai(v+\mu) = A_2(u,v)$.

③ let $H = -\frac{d^2}{dx^2} + x$ the Airy operator.

then: $H \psi_\lambda = \lambda \psi_\lambda$ with $\psi_\lambda(x) = Ai(x-\lambda)$.

$\Rightarrow K_{\mathcal{A}_2}$ is the spectral projection of H on $\{H \leq 0\}$.

④ $K_{\mathcal{A}_2}$ is locally trace-class and $\|\mathbb{1}_{[s,\infty)} K_{\mathcal{A}_2} \mathbb{1}_{[s,\infty)}\| < 1$, $\forall s > -\infty$.

•

3) Determinantal point processes.

3.1) Point processes.

Definition 9: A point process η on Λ is a random point measure, where point measures are measures which are locally finite sum of Dirac measures.

Definition 10: A point process is simple if

$$\mathbb{P}\left(\eta(\{x\}) \leq 1, \forall x \in \Lambda\right) = 1,$$

i.e., no double points.

Remark: A simple point process can be identified with the support of the random point measure.

. Let us do two examples:

(1) Poisson point process on $\mathbb{R}^d$ with intensity ρ:

. Take $\Lambda = \mathbb{R}^d$ and $\mathbb{P}$ the probability measure s.t., $\forall B_1, B_2 \subset \Lambda$, bounded and $\forall n_1, n_2 \geq 0$:

$$\mathbb{P}(\# B_1 = n_1) = \frac{(\rho\, |B_1|)^n}{n!} \cdot e^{-\rho |B|}$$

and if $B_1 \cap B_2 = \phi \Rightarrow \mathbb{P}(\# B_1 = n_1, \# B_2 = n_2) = \mathbb{P}(\# B_1 = n_1)\mathbb{P}(\# B_2 = n_2)$.

(2) GUE eigenvalues:

. In this case, $\Lambda = \mathbb{R}$ and $\mathbb{P}$ is the probability of the GUE ensemble. Then, the point process η associated with the GUE eigenvalues is

$$\eta(x) = \sum_{i=1}^{N} \delta(x - \lambda_i). \tag{3.1}$$

3.2) Correlation functions.

. For a point process η on Λ, the total mass (points with multiplicity, in the support of η) in a set $A \subset \Lambda$ is given by

$$\eta(\mathbb{1}_A), \quad \mathbb{1}_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases} \tag{3.2}$$

. For a general function, we denote:

$$\eta(f) = \int_\Lambda f(x)\,\eta(x)\,dx \tag{3.3}$$

. <u>Moments:</u> Let $\rho^{(n)}(x_1, .., x_n)$ be the n-point correlation functions of the point process η with respect to a measure μ.

<u>Lemma 11:</u> Let $A \subset \Lambda$ a subset. Then,

$$\int_{A^n} d\mu(x_1) \dots d\mu(x_n)\, \rho^{(n)}(x_1, .., x_n) = \mathbb{E}\left(\frac{\eta(\mathbb{1}_A)!}{(\eta(\mathbb{1}_A) - n)!} \right) \tag{3.4}$$

<u>Proof.:</u> . For $n=1$: $\eta(\mathbb{1}_A) = \sum_i \mathbb{1}_{[x_i \in A]} \Rightarrow \mathbb{E}(\eta(\mathbb{1}_A)) = \mathbb{E}(\# x_i \in A) = \int_A d\mu(x) \rho^{(1)}(x)$

. For $n=2$: notice that $\eta(\mathbb{1}_A)(\eta(\mathbb{1}_A) - 1) = \sum_i \mathbb{1}_{[x_i \in A]} \sum_{j \neq i} \mathbb{1}_{[x_j \in A]}$,
because if $i \notin A \Rightarrow \sum_{j \neq i} (\dots)$ is irrelevant, while if $i \in A \Rightarrow \sum_{j \neq i} (\dots) = (\eta(\mathbb{1}_A) - 1)$

. For general n:

$$\eta(\mathbb{1}_A)(\eta(\mathbb{1}_A) - 1) \cdots (\eta(\mathbb{1}_A) - n + 1) =$$

$$= \sum_{i_1} \sum_{i_2 \neq i_1} \cdots \sum_{i_n \neq \{i_1, .., i_{n-1}\}} \prod_{k=1}^{n} \left(\mathbb{1}_{[x_{i_k} \in A]} \right)$$

$$\Rightarrow \mathbb{E}(\quad) = \int_{A^n} d\mu(x_1) \dots d\mu(x_n)\, \rho^{(n)}(x_1, .., x_n). \quad \#$$

In particular: (a) $\mathbb{E}(\eta(\mathbb{1}_A)) = \int_A \rho^{(1)}(x)\, d\mu(x)$

(3.5) (b) $$\mathrm{Var}(\eta(\mathbb{1}_A)) = \int_{A^2} \rho^{(2)}(x,y)\, d\mu(x)\, d\mu(y) + \int_A \rho^{(1)}(x)\, d\mu(x) - \left(\int_A \rho^{(1)}(x)\, d\mu(x)\right)^2.$$

3.3) Determinantal class of point processes.

Definition 12: A point process is determinantal if the n-point correlation functions are given by

(3.6) $$\rho^{(n)}(x_1, \dots, x_n) = \det\left(K(x_i, x_j)\right)_{1 \le i,j \le n}$$

where $K(x,y)$ is a kernel of an integral operator:
$K : L^2(\Lambda, d\mu) \to L^2(\Lambda, d\mu)$, non-negative, locally trace-class).

Theorem 13: (Macchi; Soshnikov). In the case of Hermitian K, K defines a determinantal point process iff $0 \le K \le 1$.

- If the corresponding point process exists, then it is unique.

Remarks:
- A determinantal point process is simple.
- The # of points is n with proba. 1 iff K is an orthogonal projection with $\mathrm{rank}(K) = n$.

Example:
- GUE eigenvalues' point process!

3.4) Hole probability.

- One of the quantity interest is the hole probability of a set A, i.e., the probability that no points is in A.

- $$\mathbb{P}(\gamma(\mathbb{1}_A)=0) = \mathbb{E}\left(\prod_i (1-\mathbb{1}_A(x_i))\right), \quad \text{because } \prod_i (1-\mathbb{1}_A(x_i)) = \begin{cases} 0, \exists i \text{ s.t. } x_i \in A, \\ 1, A \text{ is empty.} \end{cases}$$

$$= \sum_{n=0}^{\infty} (-1)^n \mathbb{E}\left(\sum_{i_1<\dots<i_n} \prod_{k=1}^{n} \mathbb{1}_A(x_{i_k})\right)$$

(3.7)
$$\overset{\text{symmetry}}{=} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \mathbb{E}\left(\sum_{\substack{i_1,\dots,i_n \\ \text{all different}}} \prod_{k=1}^{n} \mathbb{1}_A(x_{i_k})\right)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{A^n} d\mu(x_1)\dots d\mu(x_n)\, \rho^{(n)}(x_1,\dots,x_n).$$

- For determinantal point processes:

(3.8)
$$\underline{\mathbb{P}(\gamma(\mathbb{1}_A)=0) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{A^n} d\mu(x_1)\dots d\mu(x_n) \cdot \det\left(K(x_i,x_j)\right)_{1\le i,j\le n}}$$

$$\underline{\equiv \det(\mathbb{1}-K)_{L^2(A,d\mu)}} \quad \left(= \det(\mathbb{1}-\mathbb{1}_A K \mathbb{1}_A)_{L^2(\Lambda,d\mu)}\right)$$

- This is called <u>Fredholm determinant.</u>

<u>Remark:</u> There is a whole theory on Fredholm determinants of operators, but here we do not need it. As soon as the series is well defined it is fine.

Example: $\mu(A)$ finite, and $|K(x,y)| \le C$ in A. Then,

$$\left|\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{A^n} d\mu(x_1)\dots d\mu(x_n) \det(K(x_i,x_j))\right| \le \sum_{n=0}^{\infty} \frac{1}{n!} C^n \mu(A)^n \underset{\text{Hadamard bound!}}{n^{n/2}} < \infty.$$

Application: Distribution of the largest eigenvalue for GUE.

$$\mathbb{P}(\lambda_{max}^{GUE,N} \leq s) = \mathbb{P}(\eta^{GUE,N}(\mathbb{1}_{(s,\infty)})=0) = \det(\mathbb{1} - \mathbb{1}_{(s,\infty)} K_N^{GUE} \mathbb{1}_{(s,\infty)}) \tag{3.9}$$

where K_N^{GUE} is the Kernel for GUE eigenvalues of $N\times N$ matrices.

3.5) When a measure defines a determinantal point process?

. We have seen that for GUE eigenvalues, the measure

$$\frac{1}{Z}\left(\det(\lambda_i^{j-1})\right)^2 \prod_{i=1}^{N} \left(e^{-\frac{\lambda_i^2}{2N}} d\lambda_i\right)$$ induces a determinantal

point process. This is a particular case of the following theorem.

Theorem 14: (Borodin; Tracy-Widom for GUE).

Consider a measure of the form

$$\frac{1}{Z_N} \det_{1\leq i,j\leq N}(\phi_i(x_j)) \cdot \det_{1\leq i,j\leq N}(\psi_i(x_j))\, d\mu(x_1)\cdots d\mu(x_N) \tag{3.10}$$

with $Z_N \neq 0$. Then, (3.10) defines a determinantal point process with kernel

$$K_N(x,y) = \sum_{i=1}^{N} \psi_i(x) [A^{-1}]_{ij}\, \varphi_j(y) \tag{3.11}$$

where: $A_{ij} = \int_\Lambda \varphi_i(s)\psi_j(s)\, d\mu(s)$.

Proof.: Notation: $\langle a | b\rangle \doteq \int_\Lambda a(x) b(x)\, d\mu(x)$, $\langle x|b\rangle \doteq b(x)$, $\langle a|x\rangle \doteq a(x)$

. Suppose that we can find functions $\tilde{\phi}_i, \tilde{\psi}_i$, $i=1,\dots,N$ such that:

(a) $\mathrm{vect}(\{\phi_i\}) = \mathrm{vect}(\{\tilde{\phi}_i\})$

(b) $\mathrm{vect}(\{\psi_i\}) = \mathrm{vect}(\{\tilde{\psi}_i\})$

and (c) $\langle \tilde{\phi}_k | \tilde{\psi}_\ell\rangle = \delta_{k,\ell}$.

[(a), (b) possible by Gram-Schmidt; (c) is possible since $Z_N \neq 0$].

. Then: $(3.10) = \text{const.} \det\left(\tilde{\phi}_i(x_j)\right)_{1\le i,j\le N} \cdot \det\left(\tilde{\psi}_i(x_j)\right)_{1\le i,j\le N} d^N\mu(x)$

as for GUE

$= \text{const} \cdot \det\left(K_N(x_i, x_j)\right)_{1\le i,j\le N} d^N\mu(x)$

with $K_N(x,y) = \sum_{k=1}^N \tilde{\psi}_k(x)\tilde{\phi}_k(y)$.

. Using $\langle\tilde{\phi}_k, \tilde{\psi}_l\rangle = \delta_{kl}$, one verifies:

$$\int_\Lambda d\mu(x) K_N(x,x) = N$$

$$\int_\Lambda d\mu(z) K_N(x,z)K_N(z,y) = K_N(x,y).$$

. These two properties are the only two used in the GUE case.

Thus: $\rho^{(n)}(x_1 ... x_n) = \det\left(K_N(x_i,x_j)\right)_{1\le i,j\le n}$.

. $K_N = \sum_{k=1}^N |\tilde{\psi}_k\rangle\langle\tilde{\phi}_k|$.

let S and T the change of basis matrices:

$$\phi_i = \sum_{j=1}^N S_{ij}\tilde{\phi}_j \quad ; \quad \psi_i = \sum_{j=1}^N T_{ij}\tilde{\psi}_j$$

$$\Rightarrow K_N = \sum_{k=1}^N |\tilde{\psi}_k\rangle\langle\tilde{\phi}_k| = \sum_{k=1}^N \sum_{i,j=1}^N (T^{-1})_{ki}|\psi_i\rangle (S^{-1})_{kj}\langle\phi_j|$$

$$= \sum_{i,j=1}^N |\psi_i\rangle \underbrace{\left((T^t)^{-1}\cdot S^{-1}\right)_{ij}}_{=(S\cdot T^t)^{-1}_{ij}} \langle\phi_j|$$

. Define $A = S\cdot T^t$ and compute

$$\langle\phi_i|\psi_j\rangle = \sum_{kl} S_{ik}T_{jl}\cdot\underbrace{\langle\tilde{\phi}_k,\tilde{\psi}_l\rangle}_{=\delta_{kl}} = \sum_k S_{ik}\cdot T_{jk} = A_{ij} \ \#.$$

<u>Remark:</u> . A det. p.p. with Kernel $K(x,y)$ is the same as the one with Kernel $\tilde{K}(x,y) = \frac{f(x)}{f(y)}K(x,y)$ for any $f(x)$ with $f(x)\neq 0, x\in\Lambda$.

. One say that <u>K and $\tilde{K}$ are conjugate Kernels.</u>

4) Edge scaling and Tracy-Widom distributions.

4.1) TW distribution.

. <u>Reminder:</u> . Consider the GUE ensembles of $N \times N$ matrices.

. At the edge:

$$\lambda = 2N + \xi N^{1/3} \tag{4.1}$$

the rescaled kernel converges to the Airy kernel:

$$\lim_{N\to\infty} K_N^{resc}(\xi_1,\xi_2) = K_{Ai_2}(\xi_1,\xi_2) \tag{4.2}$$

$$= \int_0^\infty d\lambda\, Ai(\xi_1+\lambda) Ai(\xi_2+\lambda).$$

. <u>Eigenvalues' rescaling:</u>

$$. \lambda_i^{resc,N} \doteq \frac{\lambda_i - 2N}{N^{1/3}} \tag{4.3}$$

Gap probability.

$$\Rightarrow \mathbb{P}\left(\lambda_{max}^{resc,N} \leq s\right) = \det\left(\mathbb{1} - P_s \cdot K_N^{resc} \cdot P_s\right)_{L^2(\mathbb{R},dx)} \tag{4.4}$$

where $P_s(x) = \begin{cases} 1, & x > s, \\ 0, & x \leq s. \end{cases}$

. The convergence of the kernel K_N^{resc} to the Airy kernel (plus some bounds for large values of ξ s.t. the $\lim_{N\to\infty}$ can be taken inside the Fredholm series expansion by using dominated convergence) implies that:

$$\underline{F_2(s)} := \lim_{N\to\infty} \mathbb{P}\left(\lambda_{max}^{resc,N} \leq s\right) = \underline{\det\left(\mathbb{1} - P_s \cdot K_{Ai_2} \cdot P_s\right)_{L^2(\mathbb{R},dx)}} \tag{4.5}$$

$\uparrow$

. This is called the <u>(GUE) Tracy-Widom distribution</u>.

. A rewriting on a fixed Hilbert space.

. Let $B_s(u,v) := Ai(u+v+s)$, then

$$(4.6) \qquad F_2(s) = \det\left(\mathbb{1} - B_s^2\right)_{L^2(\mathbb{R}_+, dx)}.$$

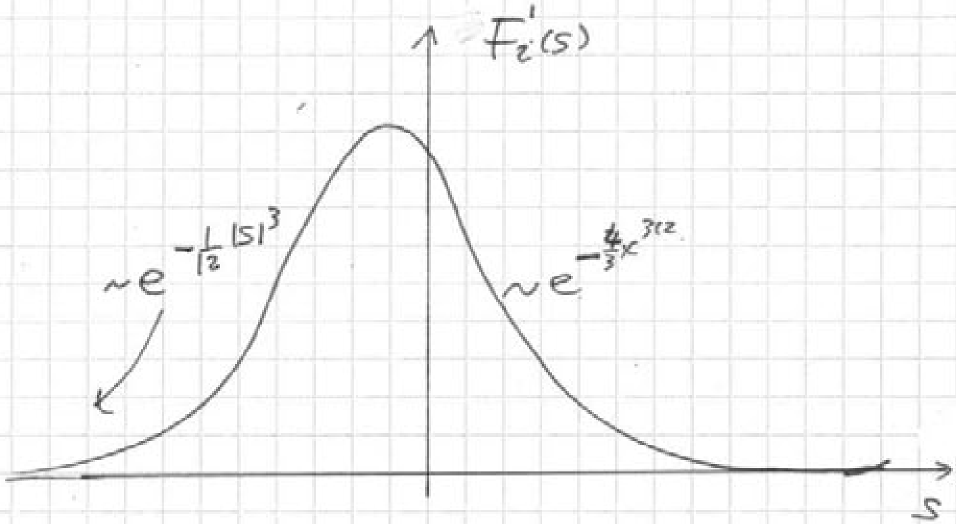
Indeed, $B_s^2(u,v) = \int_0^\infty d\lambda\, B_s(u,\lambda) B_s(\lambda,v) = \int_0^\infty d\lambda\, Ai(u+\lambda+s)\, Ai(v+\lambda+s)$

$$\Rightarrow \det\left(\mathbb{1} - B_s^2\right)_{L^2(\mathbb{R}_+, dx)} \underset{\substack{u+s\to u \\ v+s \to v}}{=} \det\left(\mathbb{1} - K_{Ai_2}\right)_{L^2((s,\infty), dx)} = F_2(s).$$

Remarks: ① B_s is not a positive operator.

② B_s is HS on $L^2(\mathbb{R}_+, dx)$, $\forall s > -\infty$

③ B_s^2 is trace-class on $L^2(\mathbb{R}_+, dx)$ $\forall s > -\infty$.



. GOE Tracy-Widom distribution: a det. formula (Ferrari–Spohn).

. With the same rescaling as (4.3) but for GOE eigenvalues, the limit distribution function is called (GOE) Tracy-Widom distribution, denoted by $F_1(s)$.

$$(4.7). \quad F_1(s) = \det\left(\mathbb{1} - B_s\right)_{L^2(\mathbb{R}_+, dx)}.$$

4.2) F_2 and Painlevé-II equations.

Theorem 15: (Tracy-Widom)

(4.8) $$F_2(s) = \exp\left(- \int_s^\infty dx\,(x-s)\, q^2(x) \right),$$

where $q(x)$ is the unique solution of the Painlevé-II equation: $q''(x) = x\,q(x) + 2q^3(x)$ satisfying the asymptotic condition: $q(s) \cong Ai(s)$ for $s \to +\infty$.

Remarks: ① Also $F_1(s)$ can be written in terms of q.

(4.9) $$F_1(s) = \exp\left(-\frac{1}{2} \int_s^\infty dx\, q(x) \right) \cdot F_2(s)^{1/2}.$$

② The importance of the relation to the P-II equation was also that it was providing a way to numerically obtain explicit quantities, like moments, for F_1, F_2.

. A recent work of Bornemann(*) shows however that there exists efficient numerical evaluation of the Fredholm determinant for Analytic Kernels, which is the case for F_2 and F_1 (see (4.5) and (4.7)).

. Thus the importance of the P-II connection is relative and will be presented only if time permits.

(*) arXiv: 0804.2543 "On the Numerical Evaluation of Fredholm Determinants" by F. Bornemann.

Proof of Theorem 15:

. We use the expression: $F_2(s) = \det(\mathbb{1} - B_s^2)$ and the space will be always $L^2(\mathbb{R}_+, dx)$. $\boxed{K_s \doteq B_s^2.}$

. Let $A_s(x) \doteq Ai(x+s)$. Then,

$$\frac{\partial}{\partial s} K_s(x,y) = \frac{\partial}{\partial s}\left(\int_0^\infty d\lambda\, Ai(x+\lambda+s)\, Ai(y+\lambda+s)\right)$$

$$= \int_0^\infty d\lambda\, Ai'(x+\lambda+s) Ai(y+\lambda+s) + \int_0^\infty d\lambda\, Ai(x+\lambda+s) Ai'(y+\lambda+s)$$

$$= Ai(x+\lambda+s) Ai(y+\lambda+s)\Big|_0^\infty - \cancel{(\;)} + \cancel{(\;)}$$

$$= - A_s(x) A_s(y).$$

. In bra-ket notations:

$$\frac{\partial}{\partial s} K_s = -|A_s\rangle\langle A_s| \qquad (4.10)$$

. let $u(s) := \frac{\partial}{\partial s} \ln\left(\det(\mathbb{1} - K_s)\right)$ (4.11)

Using the identity: $\det(\mathbb{1}+A) = \exp(\mathrm{Tr}(\ln(\mathbb{1}+A)))$, we get

$$\underline{u(s)} = \frac{\partial}{\partial s} \mathrm{Tr}\left(\ln(\mathbb{1}-K_s)\right)$$

$$= -\mathrm{Tr}\left((\mathbb{1}-K_s)^{-1} \frac{\partial}{\partial s} K_s\right) \quad \text{: used cyclicity of trace}$$

$$= \mathrm{Tr}\left((\mathbb{1}-K_s)^{-1} |A_s\rangle\langle A_s|\right) \quad \text{: rank-one operator}$$

$$= \underline{\langle A_s | (\mathbb{1}-K_s)^{-1} A_s\rangle} \qquad (4.12)$$

. Another expression for $u(s)$ is:

$$\underline{u(s)} = \frac{\partial}{\partial s} \sum_{n\geq 1} \frac{-1}{n} \mathrm{Tr}(K_s^n) = -\sum_{n\geq 1} \frac{1}{n}\cdot n \cdot \mathrm{Tr}\left(K_s^{n-1} \frac{\partial}{\partial s} K_s\right)$$

$$= \sum_{n\geq 1} \mathrm{Tr}(K_s^{n-1} |A_s\rangle\langle A_s|) = \sum_{n\geq 1} \langle A_s | K_s^{n-1} A_s\rangle$$

$$|A_s\rangle = B_s|\delta_0\rangle \quad = \sum_{n\geq 1} \langle \delta_0 | K_s^n \delta_0\rangle = \underline{\langle \delta_0 | K_s (\mathbb{1}-K_s)^{-1} \delta_0\rangle}. \qquad (4.13)$$

. Define: $\begin{cases} q(s) = \langle \delta_0 | (\mathbb{1} - K_s)^{-1} A_s \rangle \\ p(s) = \langle \delta_0 | (\mathbb{1} - K_s)^{-1} A'_s \rangle \\ v(s) = \langle A_s | (\mathbb{1} - K_s)^{-1} A'_s \rangle \end{cases}$

Lemma 16: ⓐ $\dfrac{\partial u(s)}{\partial s} = -q^2(s)$

ⓑ $q^2(s) = u^2(s) - 2v(s)$

ⓒ $\dfrac{\partial q(s)}{\partial s} = p(s) - q(s)u(s)$

ⓓ $\dfrac{\partial p(s)}{\partial s} = s q(s) - 2q(s)v(s) + p(s)u(s).$

. By Lemma 16, we have:

$$\begin{aligned}\frac{\partial^2 q(s)}{\partial s^2} &= p'(s) - q'(s)u(s) - q(s)u'(s) \\ &= s.q(s) - 2q(s)v(s) + \cancel{p(s)u(s)} - \cancel{p(s)u(s)} + q(s)u^2(s) + q^3(s) \\ &= sq(s) + q(s)\Big(\underbrace{q^2(s) + u^2(s) - 2v(s)}_{=2q^2(s)}\Big).\end{aligned}$$

. Moreover, as $s \to \infty$, $(\mathbb{1} - K_s)^{-1} \to \mathbb{1} \Rightarrow q(s) \to Ai(s)$.

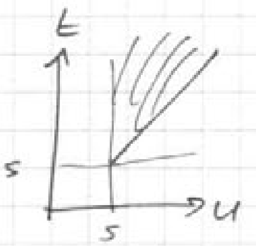
. To get the final formula we need to integrate twice:

$$\frac{\partial u(s)}{\partial s} = \frac{\partial^2}{\partial s^2} \ln(F_2(s)) = -q^2(s)$$

$$\Rightarrow -\int_s^\infty dt\, q^2(t) = \int_s^\infty dt \frac{d^2}{dt^2}(\ln F_2(s)) = \frac{d}{dt} \ln F_2(t)\Big|_s^\infty = -\frac{d}{ds} \ln F_2(s)$$

$$\Rightarrow -\int_s^\infty du \int_u^\infty dt\, q^2(t) = -\int_s^\infty du \frac{d}{du} \ln F_2(u) = -\ln F_2(u)\Big|_s^\infty = \underline{\ln F_2(s)}.$$

$$\lfloor\lfloor\; -\int_s^\infty dt\, q^2(t) \int_s^t du = \underline{-\int_s^\infty dt\, (t-s) q^2(t)}. \qquad \#$$



Proof of Lemma 16: (a) $\frac{\partial u(s)}{\partial s} = \frac{\partial}{\partial s}\langle \delta_0 | (\mathbb{1}-K_s)^{-1}\delta_0\rangle$

$$\frac{d}{d\alpha}(\mathbb{1}-K)^{-1} \overset{\circledast}{=} (\mathbb{1}-K)^{-1}\frac{dK}{d\alpha}(\mathbb{1}-K)^{-1} \qquad = \langle \delta_0 | (\mathbb{1}-K_s)^{-1}\frac{\partial K_s}{\partial s}(\mathbb{1}-K_s)^{-1}\delta_0\rangle \overset{(4.10)}{=} -(\langle \delta_0 | (\mathbb{1}-K_s)^{-1} A_s\rangle)^2$$
$$= -q^2(s).$$

(b) From (4.12): $\frac{\partial u(s)}{\partial s} = \frac{\partial}{\partial s}\langle A_s | (\mathbb{1}-K_s)^{-1} A_s\rangle$

$$\overset{\circledast}{=} 2.\langle A_s | (\mathbb{1}-K_s)^{-1} A_s'\rangle - \langle A_s | (\mathbb{1}-K_s)^{-1} A_s\rangle\langle A_s | (\mathbb{1}-K_s)^{-1} A_s\rangle$$
$$= 2v(s) - u(s)^2 \underset{(a)}{=} -q^2(s).$$

(c) $$\frac{\partial q(s)}{\partial s} \underset{(4.12)}{\overset{\circledast}{=}} \langle \delta_0 | (\mathbb{1}-K_s)^{-1} A_s'\rangle - \langle \delta_0 | (\mathbb{1}-K_s)^{-1} A_s\rangle\langle A_s | (\mathbb{1}-K_s)^{-1} A_s\rangle$$
$$= p(s) - q(s)\cdot u(s).$$

(d) We also need: $[L, (\mathbb{1}-K)^{-1}] = (\mathbb{1}-K)^{-1}[L, K](\mathbb{1}-K)^{-1}$ (4.14)

and $[Q, K_s] = |A_s\rangle\langle A_s'| - |A_s'\rangle\langle A_s|$ (4.15)

where Q is the operator multiplication by the position.

$$\Rightarrow \frac{\partial p(s)}{\partial s} \underset{(4.12)}{\overset{\circledast}{=}} -\langle \delta_0 | (\mathbb{1}-K_s)^{-1} A_s\rangle\langle A_s | (\mathbb{1}-K_s)^{-1} A_s'\rangle + \langle \delta_0 | (\mathbb{1}-K_s)^{-1} A_s''\rangle$$

Using: $Ai''(x+s) = (x+s)Ai(x+s)$: $A_s'' = (Q+s)A_s$

$$\Rightarrow \langle \delta_0 | (\mathbb{1}-K_s)^{-1} A_s''\rangle = s.\langle \delta_0 | (\mathbb{1}-K_s)^{-1} A_s\rangle + \langle \delta_0 | (\mathbb{1}-K_s)^{-1} Q A_s\rangle$$

$$= s\cdot q(s) + \underbrace{\langle \delta_0 | Q(\mathbb{1}-K_s)^{-1} A_s\rangle}_{=0} - \langle \delta_0 | [Q, (\mathbb{1}-K_s)^{-1}] A_s\rangle$$

$$\underset{(4.15)}{\overset{(4.14)}{=}} s\cdot q(s) - \langle \delta_0 | (\mathbb{1}-K_s)^{-1} A_s\rangle\langle A_s' | (\mathbb{1}-K_s)^{-1} A_s\rangle$$
$$+ \langle \delta_0 | (\mathbb{1}-K_s)^{-1} A_s'\rangle\langle A_s | (\mathbb{1}-K_s)^{-1} A_s\rangle$$

$$= s\cdot q(s) - q(s)v(s) + p(s)u(s)$$

$$\Rightarrow \frac{\partial p(s)}{\partial s} = sq(s) - 2q(s)v(s) + p(s)u(s) \qquad \#$$

5) Extended determinantal point processes.

5.1) Karlin-McGregor theorem.

. The original work of Karlin and McGregor was in continuous time but discrete space. Here we present the analogue for Brownian Motions.

. Consider N Brownian Motions starting from $X_i(0) = Y_i$ and arriving at $X_i(t) = x_i$, $1 \leq i \leq N$.

. Let $x_1 > x_2 > \dots > x_N$ and $Y_1 > Y_2 > \dots > Y_N$. Denote by $\mathbb{P}_{\text{non-int}}(A)$ the probability of A and that the Brownian Motions do not intersect.

Theorem 17:

$$(5.1) \qquad \mathbb{P}_{\text{non-int}}\left(X_1(t) = x_1, \dots, X_N(t) = x_N \mid X_1(0) = Y_1, \dots, X_N(0) = Y_N\right) =$$

$$= \det\left[\mathbb{P}(X(t) = x_i \mid X(0) = Y_j)\right]_{1 \leq i,j \leq N}.$$

where $\mathbb{P}(X(t) = x_i \mid X(0) = Y_j)$ is the transition density for a single Brownian Motion.

Proof: . Ingredients: ⓐ Continuity of paths

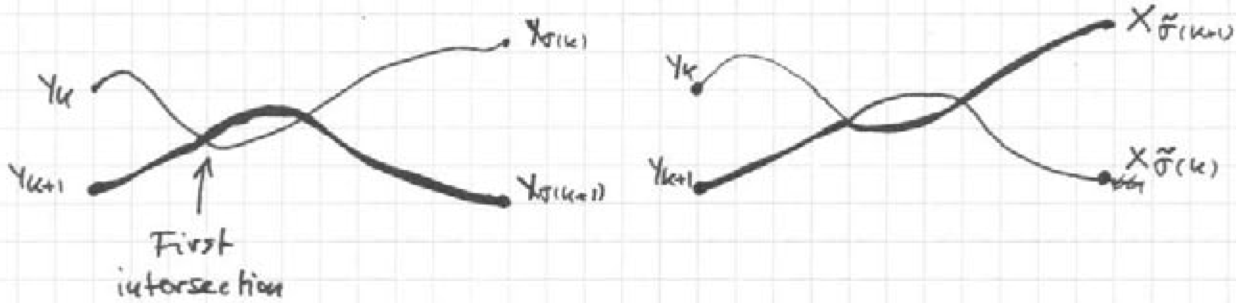
ⓑ Reflection principle.

. Consider the R.H.S. of (5.1): Let $p(x_i, Y_j) \doteq \mathbb{P}(X(t) = x_i \mid X(0) = Y_j)$.

$$\Rightarrow \det\left(p(x_i, Y_j)\right)_{1 \leq i,j \leq N} = \sum_{\sigma \in S_N} (-1)^{|\sigma|} \cdot \underbrace{\prod_{k=1}^{N} p(x_k, Y_{\sigma(k)})}_{\text{Transition density from } x_1, \dots, x_N \text{ to } Y_{\sigma(1)}, \dots, Y_{\sigma(N)} \text{ with intersections.}}.$$

. We have to see that the contributions of the paths with intersections is exactly cancelled.

. Consider paths with intersections. Let us focus at the <u>first intersection</u>.



<u>Prefactor:</u> $\underbrace{(-1)^{|\sigma|} \qquad\qquad (-1)^{|\tilde\sigma|}}_{\text{The contributions cancels, since have the same weight.}} = (-1)^{|\sigma|}\cdot(-1).$ #

. A first generalisation is known on Graphs. It is called the Lindström-Gessel-Viennot theorem (LGV).

5.2) LGV theorem.

. Consider a <u>directed</u> graph (V,E) of vertices V and edges E.

. A <u>path</u> π is a sequence of consecutive vertices joined by directed edges. We denote by $\mathcal{P}(u,v)$ = set of all paths from u to v, $u,v\in V$.

. Two paths π and π' <u>intersects</u> if they have a common vertex.

<u>.Weights:</u> . To every edge $e\in E$ assign a weight $w(e)$.

. The weight of a path π is then $w(\pi) = \prod_{e\in\pi} w(e)$.

. The total weights of paths from u to v is

$$h(u,v) = \sum_{\pi\in\mathcal{P}(u,v)} w(\pi).$$

. <u>Assumption:</u> Given points $(u_1,\dots,u_m)$ and $(v_1,\dots,v_m)$, $\exists$ at <u>most</u> one $\sigma\in S_m$ s.t. one can connect u_i to $v_{\sigma(i)}$, $i=1,\dots,m$, <u>without intersections</u>.
⊕ No loops in the graph.

. If such σ exists, $(u_1,\dots,u_m)$ and $(v_1,\dots,v_m)$ are <u>compatible</u>, and we can relabel them to have $\sigma = \mathbb{1}$.

Theorem (LGV) 18 Denote by $\mathcal{P}_{\text{non-int}}(\vec{u},\vec{v})$ the set of non-intersecting m-tuples of paths from $\vec{u}=(u_1,\dots,u_m)$ to $\vec{v}=(v_1,\dots,v_m)$. Then,

$$w\left(\mathcal{P}_{\text{non-int}}(\vec{u},\vec{v})\right)=\sum_{\substack{(\pi_1,\dots,\pi_m)\\ \in\mathcal{P}_{\text{non-int}}(\vec{u},\vec{v})}} w(\pi_1)\dots w(\pi_m)=\det\left(h(u_i,v_j)\right)_{1\le i,j\le N}. \tag{5.2}$$

Remark: The proof uses the same ingredients of Karlin-McGregor Theorem, namely weight = product of local weights (Markov) and the fact that to exchange their position they first have to intersect.

5.3) Non-intersecting Brownian Motions.

. Consider N Brownian motions with fixed initial and final positions.

Focus at m intermediate times: $\underset{\tau_0}{0} < \tau_1 < \tau_2 < \dots < \tau_m < \underset{\tau_{m+1}}{T}$

. Denote $x_k^n := x_k(\tau_n)$, $0\le n\le m+1$, $1\le k\le N$

. Let $\phi_{n,n+1}(x,y)=\mathbb{P}(X.(\tau_{n+1})=y \mid X.(\tau_n)=x)$, the transition density from time τ_n to time τ_{n+1}.

Then, by Theorem 17, the measure on $\{x_k^n,\ 1\le k\le N,\ 1\le n\le m\}$ is given by

$$\frac{1}{Z_{N,m}}\cdot\det\left(\phi_{0,1}(x_i^0,x_j^1)\right)_{1\le i,j\le N}\cdot\left(\prod_{n=1}^{m-1}\det\left(\phi_{n,n+1}(x_i^n,x_j^{n+1})\right)_{1\le i,j\le N}\right)\cdot\det\left(\phi_{m,m+1}(x_i^m,x_j^{m+1})\right)_{1\le i,j\le N}, \tag{5.3}$$

where $Z_{N,m}$ is the normalisation constant.

- Assume that we have a measure of the form (5.3) with $Z_{N,m}\neq 0$.

Then, the space-time correlation functions have determinantal form.

Rem.: x_i^0's and x_j^{m+1}'s are fixed.

Proposition 19: The space-time correlation functions of (5.3) are given by:

(5.4)
$$S^{(n)}(x_1,t_1;\dots;x_n,t_n) = \det\big(K(x_i,t_i;x_j,t_j)\big)_{1\le i,j\le n}, \text{ where}$$

$x_i \in \mathbb{R}$, $t_i \in \{\tau_1,\dots,\tau_m\}$ and the kernel is given by

(5.5)
$$K(x,\tau_r;y,\tau_s) = -\phi_{r,s}(x,y) + \sum_{i,j=1}^{N} \phi_{r,m+1}(x,x_i^{m+1})\cdot[A^{-1}]_{ij}\,\phi_{0,s}(x_j^0,y)$$

with $\phi_{r,s}(x,y) = \begin{cases} (\phi_{r,r+1} * \cdots * \phi_{s-1,s})(x,y), & \text{if } r<s, \\ 0, & \text{if } r\ge s. \end{cases}$

and $A_{ij} = \phi_{0,m+1}(x_i^0, x_j^{m+1})$.

$\boxed{(\phi_{r,r+1} * \phi_{r+1,r+2})(x,y) \equiv \int dz\, \phi_{r,r+1}(x,z)\,\phi_{r+1,r+2}(z,y)}$ Notation ($*$)

Remark: Applying Cauchy-Binet ~~m~~ times one obtains $Z_{N,m} = \det(A)$, so since by assumption $Z_{N,m} \neq 0$, A is invertible.

Proof of Proposition 19: We prove it for $m=2$. The proof for any m can be made on the same line.

First a small **Lemma**; Lemma 20: Let $(x_1,\dots,x_N;y_1,\dots,y_N) \equiv (X,Y)$ and consider the matrix M with blocs $\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22}\end{pmatrix}$ and entries.

$$\langle x,y|M(x,y\rangle \equiv \begin{pmatrix} \langle X, M_{11} X\rangle & \langle X, M_{12} Y\rangle \\ \langle Y, M_{21} X\rangle & \langle Y, M_{22} Y\rangle \end{pmatrix}. \quad \text{Then,}$$

(5. 6)
$$\underline{\det\big(\langle x,y|M|XY\rangle\big) = \det\big(\langle x,y|M\cdot Q|XY\rangle\big)} \text{ for any } Q$$
of the form $\begin{pmatrix} \mathbb{1} & 0 \\ A & \mathbb{1}\end{pmatrix}$.

Proof:
$$\det\big(\langle x,y|M|x,y\rangle\big) = \det\begin{pmatrix} \langle X, M_{11} X\rangle & \langle X, M_{12} Y\rangle \\ \langle Y, M_{21} X\rangle & \langle Y, M_{22} Y\rangle \end{pmatrix} \cdot \int dz\,dw\, \langle z, \mathbb{1} X\rangle \langle w, \mathbb{1} Y\rangle$$
$$= \int dz\,dw\, \det\begin{pmatrix} \langle X, M_{11} Z\rangle & \langle X, M_{12} W\rangle \\ \langle Y, M_{21} Z\rangle & \langle Y, M_{22} W\rangle \end{pmatrix} \cdot \det\begin{pmatrix} \langle Z, \mathbb{1} X\rangle & 0 \\ \langle Z, A X\rangle & \langle W, \mathbb{1} Y\rangle \end{pmatrix}$$
$$= \det\big(\langle x,y|\; M\cdot Q\,|x,y\rangle\big) \quad \#$$

. For $m=2$, (5.3) becomes:

$$(5.7)\quad \frac{1}{Z}\cdot \det\left(\phi_{0,1}(x_i^0,x_j^1)\right)\det\left(\phi_{12}(x_i^1,x_j^2)\right)\cdot \det\left(\phi_{2,3}(x_i^2,x_j^3)\right).$$

. Let $A_{ij} = (\phi_{0,1}*\phi_{12}*\phi_{2,3})(x_i^0,x_j^3)$. Then, $\underline{Z=\det A}\neq 0$.

. Notations:
. $\Psi_j^2(x) \equiv \phi_{2,3}(x,x_j^3)$
. $\Psi_j^1(x) := (\phi_{12} * \Psi_j^2)(x)$

Define : . $\Phi_j^1(x) = \sum_{k=1}^{N} A^{-1}_{jk}\cdot \phi_{0,1}(x_k^0,x)$ and set $\Phi_j^2(x) := (\Phi_j^1 * \phi_{12})(x)$.

Then, (5.7) becomes:

$$(5.7) = \frac{1}{\cancel{\det A}}\cdot \cancel{\det A}\cdot \det\left(\Phi_i^1(x_j^1)\right)\cdot \det\left(\Psi_i^2(x_j^2)\right)\cdot \det\left(\phi_{12}(x_i^1,x_j^2)\right).$$

$$(5.8)\qquad = \det\begin{pmatrix} 0 & -\phi_{1,2}(x_i^1,x_j^2) \\ \sum_{k=1}^{N}\Psi_k^2(x_i^2)\,\Phi_k^1(x_j^1) & 0 \end{pmatrix}$$

In operator form: $\det\left(\langle x^1,x^2|\,M\,|x^1,x^2\rangle\right)$ with

$$M = \begin{pmatrix} 0 & -\phi_{12} \\ \sum_{k=1}^{N}|\Psi_k^2\rangle\langle\Phi_k^1| & 0 \end{pmatrix}$$

. Consider $\tilde{M} = \begin{pmatrix} \mathbb{1} & \phi_{12} \\ 0 & \mathbb{1}\end{pmatrix}\cdot M\cdot \begin{pmatrix} \mathbb{1} & \phi_{12} \\ 0 & \mathbb{1}\end{pmatrix} = \begin{pmatrix} \sum_{k=1}^{N}|\phi_{12}*\Psi_k^2\rangle\langle\Phi_k^1| & -\phi_{12}+\sum_{k=1}^{N}|\phi_{12}*\Psi_k^2\rangle\langle\Phi_k^1*\phi_{12}| \\ \sum_{k=1}^{N}|\Psi_k^2\rangle\langle\Phi_k^1| & \sum_{k=1}^{N}|\Psi_k^2\rangle\langle\Phi_k^1*\phi_{12}| \end{pmatrix}$

By Lemma 20,

$$(5.8) = \det\left(\langle x^1,x^2|\,\tilde{M}\,|x^1,x^2\rangle\right).$$

So, we have obtained:

$$P(x_1^1,\dots,x_N^1;\, x_1^2,\dots,x_N^2) = \det\begin{pmatrix} K_{11}(x_i^1,x_j^1) & K_{12}(x_i^1,x_j^2) \\ K_{21}(x_i^2,x_j^1) & K_{22}(x_i^2,x_j^2)\end{pmatrix}$$

. Replacing the definitions of Ψ and Φ we get:

$$(5.9)\begin{cases} K_{11}(x,y) = \sum_{i,j=1}^{N} \phi_{1,3}(x,x_i^3)\, A^{-1}_{ij}\, \phi_{0,1}(x_j^0,y), \\ K_{12}(x,y) = -\phi_{1,2}(x,y) + \sum_{i,j=1}^{N} \phi_{1,3}(x,x_i^3)\, A^{-1}_{ij}\, \phi_{0,2}(x_j^0,y), \\ K_{2,1}(x,y) = \sum_{i,j=1}^{N} \phi_{2,3}(x,x_i^3)\, A^{-1}_{ij}\cdot \phi_{0,1}(x_j^0,y), \\ K_{2,2}(x,y) = \sum_{i,j=1}^{N} \phi_{2,3}(x,x_i^3)\, A^{-1}_{ij}\cdot \phi_{0,2}(x_j^0,y). \end{cases}$$

. What remains to do is to integrate over some of the variable and see that the form for the correlations is maintained.

. <u>An orthogonal relation:</u> $\boxed{\Phi^n_k * \Psi^n_e = \delta_{k,e} \quad , 1 \le k,e \le N \quad , n=1,2.}$

Let us verify for $n=1$: $\Phi^1_k * \Psi^1_e = \sum_{j=1}^{N} A^{-1}_{kj}\cdot \underbrace{\phi_{0,1}(x_j^0,\cdot) * \phi_{1,2} * \phi_{2,3}(\cdot,x_e^3)}_{= \phi_{0,3}(x_j^0,x_e^3) = A_{j,e}}$

$$= \sum_{j=1}^{N} A^{-1}_{kj} A_{je} = \delta_{ke}. \#$$

. The correlation function for n_1 points at time τ_1 and n_2 points at time τ_2 is then given by:

$$S^{(n_1,n_2)}(x^1_1,\dots,x^1_{n_1};\ x^2_1,\dots,x^2_{n_2}) = \frac{N}{(N-n_1)!}\cdot\frac{N}{(N-n_2)!}\int dx^1_{n_1+1}\dots dx^1_N\, dx^2_{n_2+1}\dots dx^2_N \cdot$$
$$\cdot \det\begin{pmatrix} K_{11}(x^1_i,x^1_j) & K_{12}(x^1_i,x^2_j) \\ K_{21}(x^2_i,x^1_j) & K_{22}(x^2_i,x^2_j) \end{pmatrix}$$

. Using the orthogonal relation it is very easy to verify that:

$$\left.\begin{array}{ll} \cdot\, K_{22} * K_{21} = K_{21} & ;\ K_{22} * K_{22} = K_{22} \\ \cdot\, K_{12} * K_{21} = 0 & ;\ K_{12} * K_{22} = 0 \\ \cdot\, K_{11} * K_{11} = K_{11} & ;\ K_{11} * K_{12} = 0 \\ \cdot\, K_{21} * K_{12} = 0 & ;\ K_{21} * K_{11} = K_{21}. \end{array}\right\}$$

The analogous of (2.9), see page (12).

and: $\int dx\, K_{ii}(x,x) = N$: Analogous of (2.8).

Remark:

. The generalisation to any m is made using the following matrix multiplications: (ex. for $m=4$)

$$M = \begin{pmatrix} 0 & -\phi_{12} & 0 & 0 \\ 0 & 0 & -\phi_{23} & 0 \\ 0 & 0 & 0 & -\phi_{34} \\ \sum_k |\Psi_k^4\rangle\langle\Phi_k^1| & 0 & 0 & 0 \end{pmatrix}$$

$$\tilde{M} = \begin{pmatrix} 1 & 0 & 0 & \phi_{14} \\ 0 & 1 & 0 & \phi_{24} \\ 0 & 0 & 1 & \phi_{34} \\ 0 & 0 & 0 & 0 \end{pmatrix} . M \cdot \begin{pmatrix} 1 & \phi_{12} & \phi_{13} & \phi_{14} \\ 0 & 1 & \phi_{23} & \phi_{24} \\ 0 & 0 & 1 & \phi_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- The final step is to use these relations and develop the determinant along the column with the variable to integrate out, exactly like in the proof of Proposition 6 (see page 13) and one gets the result. #

5.4) Application to Brownian Bridges.

- From Brownian Motions to Brownian Bridges one does the usual limit procedure: $x_i^0 = x_i^{m+1} = -\varepsilon \cdot i$ and send $\varepsilon \to 0$.

- <u>First term:</u> $\det\left(\phi_{0,1}(x_i^0, x_j^1)\right)$:

$$\phi_{0,1}(-\varepsilon k, x) = \text{const} \times e^{\frac{-x^2}{2\tau_1}} \cdot \underbrace{e^{-\frac{\varepsilon^2 k^2}{2\tau_1}}}_{\xrightarrow[\varepsilon\to 0]{} 1} \cdot \underbrace{e^{-\frac{\varepsilon k x}{\tau_1}}}_{= 1 - \frac{k\varepsilon x}{\tau_1} + \dots + \frac{(-k\varepsilon x)^{N-1}}{(N-1)!\,\tau_1^{N-1}} + \sigma(\varepsilon^N)}$$

Linear combinations

$$\Rightarrow \det\left(\phi_{0,1}(x_i^0, x_j^1)\right) = \text{const} \cdot \det\begin{pmatrix} 1 + \sigma(\varepsilon^N) \\ \varepsilon x_j^1 + \sigma(\varepsilon^N) \\ \vdots \\ (\varepsilon x_j^1)^{N-1} + \sigma(\varepsilon^N) \end{pmatrix}$$

$$= \varepsilon^{\frac{N(N-1)}{2}} \cdot \text{cte} \cdot \underbrace{\det\begin{pmatrix} 1 + \sigma(\varepsilon^N) \\ x_j^1 + \sigma(\varepsilon^{N-1}) \\ \vdots \\ (x_j^1)^{N-1} + \sigma(\varepsilon) \end{pmatrix}}_{\xrightarrow[\varepsilon\to 0]{} \Delta_N(x_1^1, \dots, x_N^1)}.$$

- The last term is analogous and the factors $\varepsilon^{\frac{N(N-1)}{2}}$ enters in the normalization.
- Also, we can freely choose two sets of polynomials $\{\Phi_{i-1}^1(x), i=1,\dots,N\}$ and $\{\Psi_{i-1}^m(x), i=1,\dots,N\}$ with $\Phi_{i-1}^1(x)$, $\Psi_{i-1}^m(x)$ of degree $i-1$, instead of the monomials in the Vandermonde determinant.

Notations:

(5.10)
$$\begin{cases} \cdot\ \phi_{n,m}(x,y) = \dfrac{\exp\left(-\frac{(y-x)^2}{2(\tau_m-\tau_n)}\right)}{\sqrt{2\pi(\tau_m-\tau_n)}} \cdot \mathbb{1}_{[\tau_m>\tau_n]} \\ \cdot\ \Phi_i^r(x) = \dfrac{\sqrt[4]{2\pi T}}{\sqrt{i!\,2^i}} \cdot \left(\dfrac{T-\tau_r}{\tau_r}\right)^{i/2} \cdot H_i\left(\dfrac{x}{\sqrt{2\tau_r(T-\tau_r)/T}}\right) \cdot \dfrac{e^{-\frac{x^2}{2\tau_r}}}{\sqrt{2\pi\tau_r}} \\ \cdot\ \Psi_j^s(x) = \dfrac{\sqrt[4]{2\pi T}}{\sqrt{j!\,2^j}} \cdot \left(\dfrac{\tau_s}{T-\tau_s}\right)^{j/2} \cdot H_j\left(\dfrac{x}{\sqrt{2\tau_s(T-\tau_s)/T}}\right) \cdot \dfrac{e^{-\frac{x^2}{2(T-\tau_s)}}}{\sqrt{2\pi(T-\tau_s)}} \end{cases}$$

with $H_i(x)$ the standard Hermite polynomial of degree i.

. These functions satisfy:

(5.11)
$$\begin{cases} \int_{\mathbb{R}} dx\ \Phi_i^r(x)\phi_{r,s}(x,y) = \Phi_i^s(y), \text{ for } s>r. \\ \int_{\mathbb{R}} dx\ \phi_{r,s}(x,y)\Psi_j^s(y) = \Psi_j^r(x), \text{ for } r<s. \end{cases}$$

. Moreover, $\begin{cases} \text{vect}\{\Phi_i^1(x), i=0,..,N-1\} = \text{vect}\{e^{-\frac{x^2}{2\tau_1}}\cdot x^i, i=0,..,N-1\} \\ \text{vect}\{\Psi_j^m(x), i=0,..,N-1\} = \text{vect}\{e^{-\frac{x^2}{2(T-\tau_m)}}\cdot x^i, i=0,..,N-1\}. \end{cases}$

. Therefore, as $\varepsilon\to 0$, the measure on $\{x_k^n, 1\le k\le N, n=1,..,m\}$ for Brownian Bridges from 0 at time 0 to 0 at time T, is given by:

(5.12)
$$\frac{1}{Z}\cdot \det\left(\Phi_i^1(x_j^1)\right)\left(\prod_{k=1}^{m-1}\det\left(\phi_{k,k+1}(x_i^k,x_j^{k+1})\right)\right)\cdot\det\left(\Psi_i^m(x_j^m)\right).$$

This measure is exactly of the form (5.3). Therefore, the space-time correlations are <u>determinantal</u> <u>(extended determinantal point process)</u>, with kernel:

(5.13)
$$K_N(x,\tau_r;y,\tau_s) = -\phi_{r,s}(x,y) + \sum_{k=0}^{N-1}\Psi_k^r(x)\Phi_k^s(y).$$

<u>Remark:</u> This for $\tau_r=\tau_s$ is also the GUE kernel!

5.5) Edge scaling and Airy$_2$ process.

- We consider the following edge scaling:

$\bullet\ T = 4N \Rightarrow$ at $t \approx 2N$, the top B.B is around $2N$.

$$(5.14)\quad \bullet \begin{cases} \tau_i = 2N + 2u_i N^{2/3} & , u_1, u_2, \dots, u_m \text{ fixed.} \\ x_i = \underbrace{2N - u_i^2 \cdot N^{1/3}}_{\text{position from the limit shape}} + \underbrace{s_i N^{1/3}}_{\text{Fluctuations: } \frac{1}{3} \text{ exponent as in R.M.!}} & \left(\equiv H_N(\tau_i) + s_i N^{1/3} + O(1)\right). \end{cases}$$

$$\boxed{\text{Limit shape}:\ x_1(\tau) \cong \sqrt{\tau(4N-\tau)}} =: H_N(\tau)$$

- Rescaled Kernel: $K_N^{edge}(u_1 s_1; u_2, s_2) \doteq N^{1/3} \cdot K_N(x_1, \tau_1; x_2, \tau_2)$.

- One uses the asymptotics for Ψ and Φ, setting $k = N - \lambda N^{1/3}$, $\lambda \in \frac{\mathbb{N}}{N^{1/3}}$.

$$(5.15)\ \begin{cases} \bullet\ \Psi^{\tau_i}_{N-\lambda N^{1/3}}(x_i) \cong N^{-1/3} \cdot Ai(s_i + \lambda) \cdot e^{\lambda u_i} \cdot \underbrace{f(s_i, u_i)}_{= \exp\left(-\frac{u_i^3}{3} + s_i u_i\right)} . \\ \bullet\ \Phi^{\tau_i}_{N-\lambda N^{1/3}}(x_i) \cong N^{-1/3} Ai(s_i + \lambda) \cdot e^{-\lambda u_i} \cdot f(s_i, u_i)^{-1} \end{cases}$$

For $\tau_j > \tau_i$: $\bullet\ \phi_{\tau_i, \tau_j}(x, y) \cong N^{-1/3} \cdot \dfrac{f(s_j, u_j)}{f(s_i, u_i)} \cdot \dfrac{1}{\sqrt{4\pi(u_j - u_i)}} \cdot \exp\left(-\dfrac{(s_j - s_i)^2}{4(u_j - u_i)} + \dfrac{1}{12} \cdot (u_j - u_i)^3 - \dfrac{(u_j - u_i)(s_i + s_j)}{2}\right)$

$$\sum_{k=0}^{N-1} \Rightarrow \frac{1}{N^{1/3}} \sum_{\lambda \in \frac{\mathbb{N}}{N^{1/3}}} \rightarrow \int_0^\infty d\lambda .$$

- Final result: $\boxed{\text{Theorem 21:}}$ Let N Brownian Bridges from $t=0$ to $t = 4N$ be conditioned on non-intersecting (excepts at $t=0$, $t=4N$).

Let $\begin{cases} \tau = 2N + 2u N^{2/3}, \\ x = 2N - u^2 N^{1/3} + s N^{1/3} \end{cases}$

Then, in the $N \to \infty$ limit, the extended determinantal point process has kernel given by:

$$K_{\mathcal{A}_2}(u,s;u',s') = \begin{cases} \int_0^\infty d\lambda\, Ai(s+\lambda) Ai(s'+\lambda)\cdot e^{(u'-u)\lambda}, & \text{if } u'\leq u, \\ -\int_{-\infty}^0 d\lambda\, Ai(s+\lambda) Ai(s'+\lambda)\cdot e^{(u'-u)\lambda}, & \text{if } u'>u. \end{cases} \tag{5.16}$$

. This is called the extended Airy kernel.

Remark: $\cdot \int_{\mathbb{R}} d\lambda\, Ai(s+\lambda) Ai(s'+\lambda)\, e^{\lambda u} \overset{u>0}{=} \frac{1}{\sqrt{4\pi u}} \cdot \exp\left(-\frac{(s'-s)^2}{4u} + \frac{1}{12}u^3 - u\frac{(s'+s)}{2}\right).$

Corollary 22: Let $x_1(t)$ be trajectory of the top Brownian Bridge. Define the rescaled process:

$$Y_N(u) := \frac{x_1(2N + 2uN^{2/3}) - (2N - u^2 N^{1/3})}{N^{1/3}}. \tag{5.17}$$

Then, $\lim_{N\to\infty} Y_N(u) = \mathcal{A}_2(u)$

in the sense of finite-dimensional distributions, where $\mathcal{A}_2(u)$ is called the Airy$_2$ process.

Definition 23: The Airy$_2$ process is defined via the finite-dimensional distributions given by:

$$\mathbb{P}\left(\bigcap_{k=1}^{m} \mathcal{A}_2(u_k)\leq s_k\right) = \det(\mathbb{1} - \chi_s K_{\mathcal{A}_2} \chi_s)_{L^2(\{u_1,\dots,u_m\}\times\mathbb{R})} \tag{5.18}$$

with $\chi_s(u_k,x) = \mathbb{1}[x>s_k]$ and $K_{\mathcal{A}_2}$ is the extended Airy kernel given above.

. Compact formula:

$$K_{\mathcal{A}_2}(u,s;u',s') = -\left(e^{-(s'-s)H_{Ai}}\right)(u,u')\cdot \mathbb{1}_{[s'>s]} + \left(e^{sH_{Ai}} \cdot \overline{K}_{\mathcal{A}_2} \cdot e^{-s'H_{Ai}}\right)(u,u'), \text{ with}$$

$\overline{K}_{\mathcal{A}_2}(s,s') = \int_0^\infty d\lambda\, Ai(\lambda+s) Ai(\lambda+s')$, and H_{Ai} is the Airy operator: $(H_{Ai}\psi)(x) = -\frac{d^2}{dx^2}\psi(x) + x\psi(x)$.

Eigenfunctions: $H_{Ai}\, Ai(x-\lambda) = \lambda\, Ai(x-\lambda)$.

5.6) Dyson's Brownian Motion.

. It is a Brownian Motion on space of Matrices.

. let us denote by H_μ, $\mu = 1, \dots, \frac{N(N+1)}{2} = p$, for GOE, the
$\mu = 1, \dots, N^2 = p$, for GUE

independent entries of the symmetric/hermitian $N \times N$ matrices.

• let these H_μ perform independent Brownian Motion in a quadratic potential: $dH_\mu = -\gamma \cdot H_\mu dt + \sigma_\mu \underset{\substack{\uparrow \\ \text{standard B.M.} \\ (\mathrm{Var}(B_\mu(t)) = t)}}{dB_\mu}$, with $\sigma_\mu = \begin{cases} 1, & \mu = (ii), \\ \frac{1}{2}, & \text{otherwise}. \end{cases}$

. Then, $P(H_1, \dots, H_p; t)$ be the probability density of H at time t, it satisfies the Smoluchowski equation:

$$\frac{\partial P}{\partial t} = \sum_{\mu=1}^{p} \left[\frac{1}{2} \sigma_\mu \frac{\partial^2}{\partial H_\mu^2} P + \gamma \cdot \frac{\partial}{\partial H_\mu} (H_\mu P) \right]. \tag{5.19}$$

. Its stationary solution is the GOE/GUE measure:

$$P^{stat}(H) = \frac{1}{Z} \cdot e^{-\gamma \mathrm{Tr}(H^2)}. \tag{5.20}$$

. The <u>stationary process</u> is called <u>Dyson's Brownian Motion</u>, and has transition probability density

$$P(H(t) \mid H(0)) = \frac{\mathrm{cte}}{(1 - q_t^2)^{p/2}} \cdot \exp\left(- \frac{\gamma \cdot \mathrm{Tr}\,(H(t) - q_t \cdot H(0))^2}{1 - q_t^2} \right) \tag{5.21}$$

where $q_t = \exp(-\gamma \cdot t)$.

• The evolution of the eigenvalues satisfy then (one need some computations): $d\lambda_i = \left[-\gamma \lambda_i + \frac{\beta}{2} \cdot \sum_{j \neq i} \frac{1}{\lambda_j - \lambda_i} \right] dt + db_i$, $i = 1, \dots, N$

where db_i are standard independent B.M.

5.6.1) $\beta=2$ Dyson's Brownian Motion.

- Consider GUE ($\beta=2$) case and set $\gamma = \frac{1}{2N}$.
  Let $P(H(t=0)) = P^{stat}(H(t=0))$.

- Consider m times $\underset{t_0}{\overset{\parallel}{0}} < t_1 < t_2 < \dots < t_m$.

Then, the multi-time measure is, with $H_i \doteq H(t_i)$, given by

$$(5.22) \qquad \frac{1}{Z} \cdot e^{-\frac{\mathrm{Tr}(H_0^2)}{2N}} \cdot \prod_{j=0}^{m-1} e^{-\frac{\mathrm{Tr}(H_{j+1} - q_j H_j)^2}{2N(1-q_j^2)}} \, dH_0 \dots dH_m$$

where $q_j \doteq \exp\left(-\frac{t_{j+1}-t_j}{2N}\right)$.

- Denote by $\alpha_k \doteq \frac{1}{2N(1-q_k^2)}$, $\gamma_k = \frac{1-q_{k-1}^2 q_k^2}{2N(1-q_{k-1}^2)(1-q_k^2)}$, $\beta_k = \frac{q_k}{N(1-q_k^2)}$.

Then, one verifies that (5.22) can be rewritten as

$$(5.23) \qquad \frac{1}{Z} \cdot e^{-\alpha_0 \cdot \mathrm{Tr}(H_0^2)} \left(\prod_{k=1}^{m-1} e^{-\gamma_k \cdot \mathrm{Tr}(H_k^2)}\right) e^{-\alpha_{m-1} \mathrm{Tr}(H_m^2)} \cdot$$

$$\cdot \prod_{k=0}^{m-1} e^{\beta_k \cdot \mathrm{Tr}(H_k \cdot H_{k+1})} \, dH_0 \dots dH_m .$$

- What about the measure on their eigenvalues?
  For GUE we have seen that $dH_k = \Delta_N^2(\lambda^k) d\lambda^k \cdot \underbrace{dU_k}_{\text{Haar measure on } U(N)}$
  with $\lambda^k = (\lambda_1^k, \dots, \lambda_N^k)$, and $H_k = U_k \cdot \Lambda_k^{-1} U_k^{-1}$, with

$$\Lambda_k = \begin{pmatrix} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \lambda_N^k \end{pmatrix}.$$

- Thus, $\prod_{k=0}^{m-1} e^{\beta_k \mathrm{Tr}(H_k H_{k+1})} dH_0 \dots dH_m = \prod_{k=0}^{m-1} e^{\beta_k \mathrm{Tr}(U_k \Lambda_k U_k^{-1} U_{k+1} \Lambda_{k+1} U_{k+1}^{-1})} \cdot$

$$\cdot \Delta_N^2(\lambda^1) \dots \Delta_N^2(\lambda^m) d\lambda^0 \dots d\lambda^m \, dU_0 \dots dU_m$$

- Defining $U_k^{-1} U_{k+1} \doteq V_k$, we get the terms of type

$$(5.24) \qquad \int_{U(N)^m} \prod_{k=0}^{m-1} e^{\beta_k \mathrm{Tr}(\Lambda_k V_k \Lambda_{k+1} V_k^{-1})} dV_k$$

to evaluate.

Lemma 24 [Harish-Chandra / Itzykson-Zuber formula]:

$$(5.25)\qquad \int_{U(N)} dU\, e^{\gamma \operatorname{Tr}(\Lambda_1 U \Lambda_2 U^{-1})} = \frac{1}{\gamma^{N(N-1)/2}} \cdot \frac{\left(\prod_{p=1}^{N-1} p!\right) \cdot \det\left(e^{\gamma \lambda_i^1 \lambda_j^2}\right)_{1\le i,j\le N}}{\Delta_N(\lambda_1)\, \Delta_N(\lambda_2).}$$

. Using this, the measure on the eigenvalues $\{\lambda_i^k, i=1,\dots,N, k=0,\dots,M\}$ is given by

$$(5.26)\qquad \frac{1}{Z} \cdot \left(\prod_{i=1}^{N} e^{-\alpha_0 (\lambda_i^0)^2}\right) \cdot \left(\prod_{k=1}^{M-1} \prod_{i=1}^{N} e^{-\gamma_k (\lambda_i^k)^2}\right) \cdot \prod_{i=1}^{N} e^{-\alpha_{M-1} (\lambda_i^M)^2} d\lambda^0 \dots d\lambda^M$$
$$\cdot \Delta_N(\lambda^0) \left(\prod_{k=0}^{M-1} \det\left(e^{\beta_k \lambda_i^k \lambda_j^{k+1}}\right)_{1\le i,j\le N} \right) \cdot \Delta_N(\lambda^M).$$

. This measure has the form (5.3), so one can apply Proposition 19, and finds:

- The $\beta=2$ Dyson's Brownian Motion's eigenvalues form an extended determinantal point process with kernel (Extended Hermite):

$$(5.27)\qquad \boxed{K_N(x_1,t_1;x_2,t_2) = \begin{cases} \sum_{k=0}^{N-1} e^{-\frac{k(t_2-t_1)}{2N}} \cdot P_k(x) P_k(y)\, e^{-\frac{x^2+y^2}{4N}}, & \text{for } t_1 \le t_2, \\ -\sum_{k=N}^{\infty} e^{-\frac{k(t_2-t_1)}{2N}} \cdot P_k(x) P_k(y) \cdot e^{-\frac{x^2+y^2}{4N}}, & \text{for } t_1 > t_2, \end{cases} \quad \text{where } P_k(x) = \frac{1}{\sqrt[4]{2\pi N}\sqrt{2^k k!}} \cdot H_k\left(\frac{x}{\sqrt{2N}}\right)}$$

. Edge scaling: $\lambda_i(t)$ be the i^{th} largest e.v.

$$\lambda_{i,N}^{edge}(s) := \frac{\lambda_i(2 \cdot s \cdot N^{2/3}) - 2N}{N^{1/3}}$$

. The rescaled kernel converges to the $\mathcal{A}iry_2$ kernel as $N \to \infty$.

$$\Rightarrow \lim_{N\to\infty} \lambda_{N,N}^{edge}(s) = \mathcal{A}_2(s) : \text{ the } \mathcal{A}iry_2 \text{ process.}$$

6.a) Application to the polynuclear growth model in the droplet geometry.

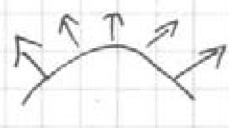
6.a.1) Generalities on KPZ class in one dimension.

- Kardar-Parisi-Zhang (KPZ) wrote what they considered be an effective equation describing a stochastically growing interface $x \mapsto h(x,t)$:

$$\frac{\partial h(x,t)}{\partial t} = \nu \Delta h(x,t) + \frac{1}{2}\lambda (\nabla h(x,t))^2 + \eta(x,t), \qquad |\nabla h| \ll 1. \tag{6.1}$$

where : $\nu > 0$: is responsible for the smoothing (surface tension)

$\lambda > 0$: lateral growth



η : space-time local noise

- This equation comes from Taylor expansion of $v(\nabla h)$ for $|\nabla h| \ll 1$:

<u>let $u = \nabla h$:</u>

$$v(\nabla h) = v(0) + \frac{\partial v(0)}{\partial u} \cdot \nabla h + \frac{1}{2} \underbrace{\frac{\partial^2 v(0)}{\partial u^2}}_{\to \pm\lambda} \cdot (\nabla h)^2 + \cdots$$

Setting: $\tilde{h}(x,t) = h(x - \frac{\partial v(0)}{\partial u} \cdot t, t) - v(0) \cdot t$, this term disappears.

- Higher order terms do not matters for large t.
- (6.1) is the simplest continuous equation for an <u>irreversible</u>, <u>local</u>, <u>non-linear</u>, and <u>random</u> growth.

<u>Macroscopic behavior:</u> The smoothing mechanism makes the surface "macroscopically deterministic"; i.e.,

$$\lim_{t \to \infty} \frac{h(\xi \cdot t, t)}{t} = h_{ma}(\xi) \quad \text{is } \underline{\text{non-random}} \text{ (called limit shape).} \tag{6.2}$$

- <u>Fluctuations:</u> $\quad \underline{H(x,t) := h(x,t) - t \cdot h_{ma}(x/t).}$
- Fluctuation exponent : $1/3$: $\quad |H(x,t)| \sim t^{1/3}$
- Correlation exponent : $2/3$: $\quad |H(x,t) - H(x + t^{2/3}, t)| \sim t^{1/3}$.

. Scaling limit:

(6.3) $$h_t^{\text{resc}}(u) := \frac{h(\xi t + u t^{2/3}, t) - t \cdot h_{ma}\left(\frac{\xi t + u t^{2/3}}{t}\right)}{t^{1/3}}.$$

Q.: What is $\lim_{t\to\infty} h_t^{\text{resc}}(u)$? Does the limit process depends on the initial profile or not?

. To answer to these questions, one considers simplified models and try to get insights from the results proven.

Universality hypothesis: The statistical properties of the interface, for large growth time t, they should depends only on few global properties of the dynamics like:

- substrate dimension,
- locality of growth,
- symmetries,
- conservation laws.

. We consider the polynuclear growth model.

. The results so far proven indicates that, starting with an initial substrate without noise, e.g. $h(x,0)=0$, $\forall x$, the limit process

(6.4)
$$\begin{cases} \lim_{t\to\infty} h_t^{\text{resc}}(u) = \kappa_v \cdot \mathcal{A}_2(u/\kappa_h), & \text{if } h_{ma}''(\xi) \neq 0 \text{ (assuming } h_{ma}'' \text{ smooth)}. \\ \lim_{t\to\infty} h_t^{\text{resc}}(u) = \kappa_v \cdot \mathcal{A}_1(u/\kappa_h), & \text{if } h_{ma}''(\xi) = 0 \text{ in a } \xi\text{-neighborhood}. \end{cases}$$

where κ_v, κ_h are vertical/horizontal scaling coefficients,

$\mathcal{A}_2$ is the Airy$_2$ process and $\mathcal{A}_1$ is the Airy$_1$ process (which we did not see yet).

6.a.2) Discrete polynuclear growth model.

. It is a growth model with space and time discrete.

. The height function is integer-valued:

(6.5) $$h(x,t) \in \mathbb{Z}, \quad x \in \mathbb{Z}, t \in \mathbb{N}.$$

. In the "droplet geometry" or "corner growth" is given as follows.

. Initial condition: $h(x,0) = 0, \; x \in \mathbb{Z}$.

Dynamics: $$h(x,t+1) = \max\{h(x-1,t), h(x,t), h(x+1,t)\} + \omega(x,t+1).$$

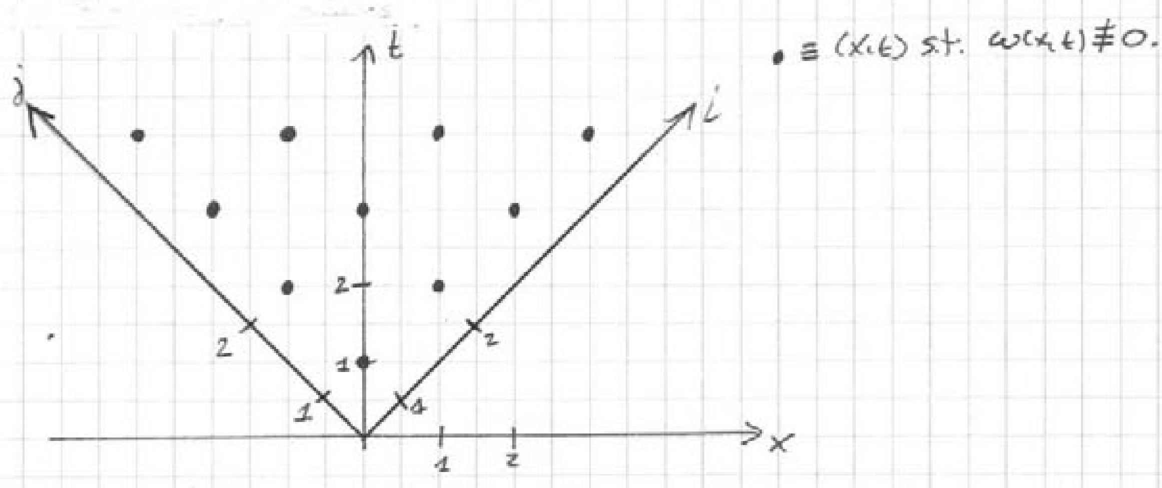
. The following special case is exactly solvable:

$$\omega(x,t) = 0 \text{ if } t-x \text{ is odd or } |x|>t.$$

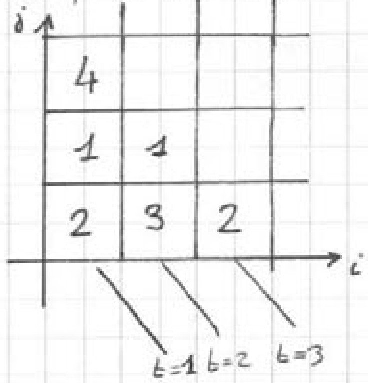
Otherwise: $\omega(i-j, i+j-1) = w(i,j)$, $i,j \in \mathbb{N}_+^2$ are independent geometric random variables with parameter $0 < a_i b_j < 1$: i.e.,

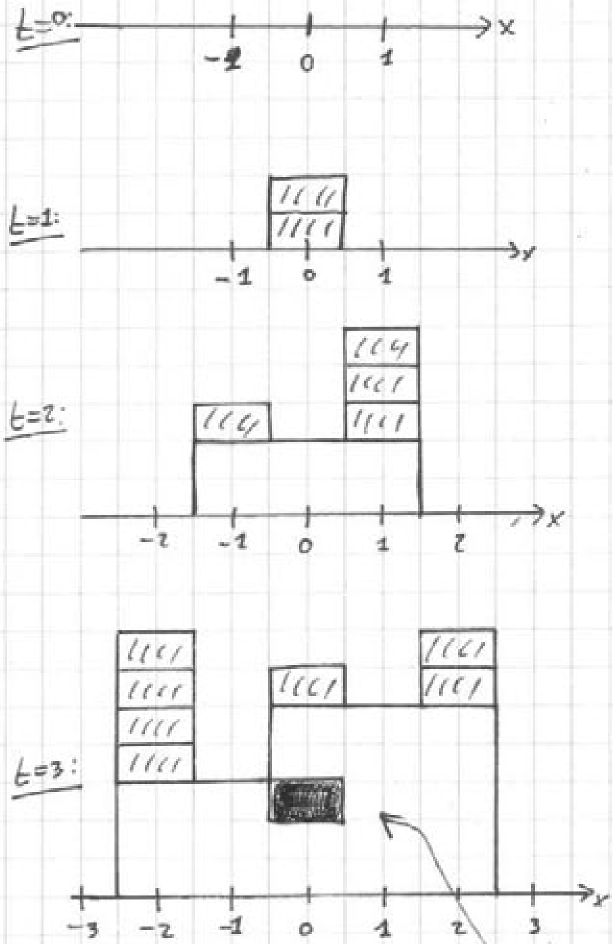
(6.6) $$\mathbb{P}(w(i,j) = k) = (1-a_i b_j)\cdot(a_i b_j)^k, \quad k \geq 0.$$

. Later we will consider $a_i = b_j = \sqrt{q}$. Also, by taking $q \to 0$ and rescale space and time by $\frac{1}{\sqrt{q}}$ one gets Poisson points $\equiv$ continuous time PNG model.



. let us see a couples of iterations.



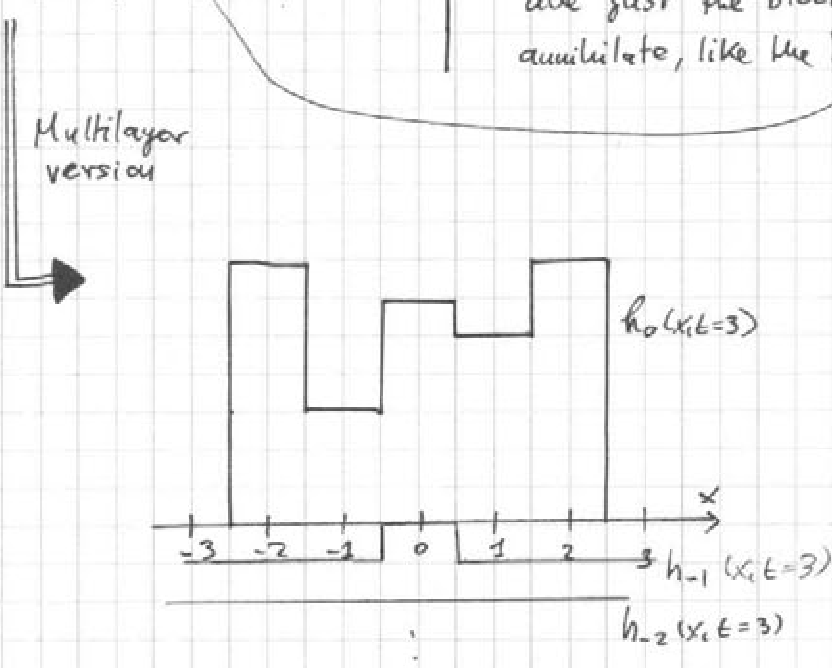


. Multilayer PNG: . One sees that already at time $t=3$ two "islands" meet and "information" is lost: One can not recover the $w(i,j)$'s. So, even a nice measure on $w(i,j)$'s will not translate into a nice measure on the height function $h(x,t)$.

. This is avoided as follows:
instead of a single height line, one has a set of lines $h_\ell(x,t)$, $\ell \le 0$.

. $h_0(x,t) \equiv h(x,t)$; $h_\ell(x,0) = -\ell$, $\ell \le 0$.

. $h_\ell(x,t)$ evolves like the dynamics of h_0 but the "nucleations", the $w_\ell(i,j)$, are just the blocks at level $\ell+1$ which annihilate, like the .



. Let us see how the <u>measure</u> on the $w(i,j)$ translates into the line ensemble.

. $w(1,1)$: $(a_1 \cdot b_1)^{w(1,1)}$

. $w(1,2)$: $(a_1 \cdot b_2)^{w(1,2)}$

. $w(2,1)$: $(a_2 \cdot b_1)^{w(2,1)}$

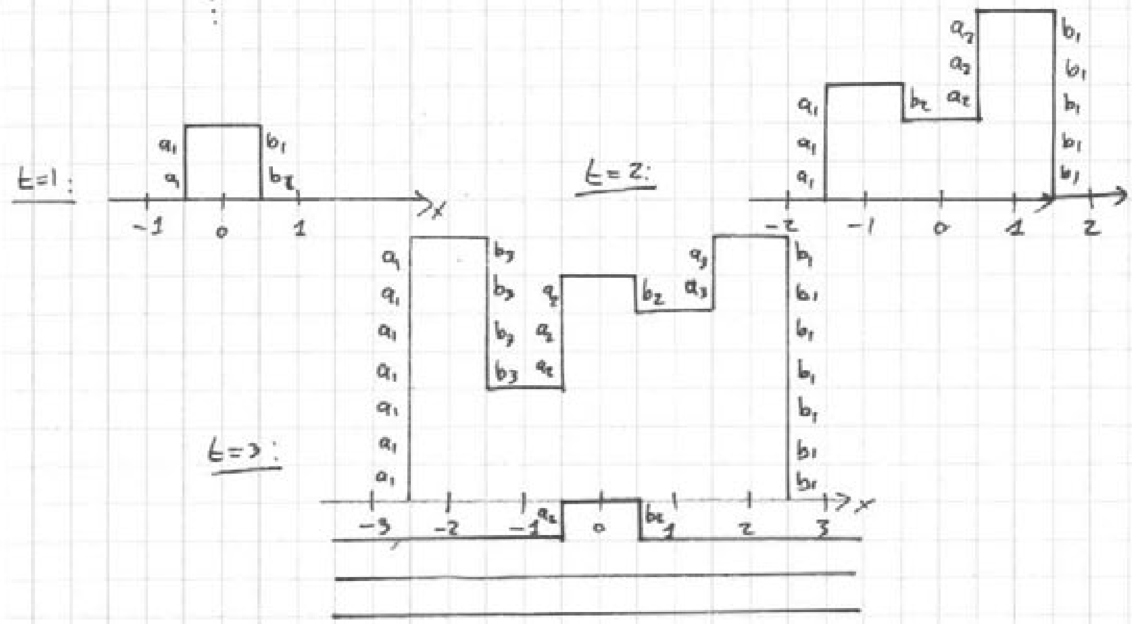
$\vdots$

<u>After time t:</u>

<u>$t=1$</u> $(a_1 \cdot b_1)^{w(1,1)}$

<u>$t=2$:</u> $(a_1 \cdot b_1)^{w(1,1)} \cdot (a_1 b_2)^{w(1,2)} \cdot (a_2 \cdot b_1)^{w(2,1)}$

<u>$t=3$:</u> . . .



• So, we see that the mapping to the line ensemble keep a simple measure.

. By running the dynamics backwards, one can see that every non-intersecting line ensemble with $h_\ell(-t,t) = h_\ell(-t,t) = \ell$, $\ell \le 0$, it corresponds a unique configuration of the realization of the $\{w(i,j),\ i+j \le t+1\}$.

. <u>LGV scheme:</u> One can describe the line ensembles in the LGV scheme, as follows. The directed graph is between the vertices $(\mathbb{Z}+\frac{1}{2}) \times \mathbb{Z}$. It is a "square graph" with the horizontal edges directed to the right and with weight 1, while the vertical ones alternating directed $\uparrow$ and $\downarrow$, with weights

$$\boxed{\begin{array}{ll} -t+\frac{1}{2}+2k: \ a_{k+1}, & k=1,\dots,t \\ t-\frac{1}{2}-2k: \ b_{k+1}, & k=1,\dots,t \end{array}}$$

as indicated in the above picture.

. To apply the LGV scheme, leading to a determinantal point process, we need to start with a finite number of lines, say N.

. At time t, they starts at $(-t,-\ell)_{\ell=0,\dots,N-1}$ and ends at $(t,\ell)_{\ell=0,\dots,N-1}$.

. Remarks: Actually there will be "a copy" of lines not straight at the bottom, but as $N\to\infty$, they becomes independent of the above ones (actually starting e.g. from $N>2t$).

. Now, we do not carry out the computations in this case. For $a_i=b_j=\sqrt{q}$, the details have been made by Johansson [math/0206208].

. 6.a.3) Continuous time PNG droplet.

. To get back to the continuous time PNG, one considers

$$a_i=b_j=\sqrt{q}\,,\ \text{so},\ \mathbb{P}(w(i,j)=k)=(1-q)q^k,\ k\geq 0.$$

In particular $\mathbb{E}(w(i,j))=\frac{q}{1-q}=q+\mathcal{O}(q^2)$.

. Rescale space and time by $\frac{1}{\sqrt{q}}$:

$$x=\left[\frac{X}{\sqrt{q}}\right],\ t=\left[\frac{T}{\sqrt{q}}\right]. \tag{6.7}$$

. Then, $\mathbb{E}\left(\sum_{i,j\leq\frac{c}{\sqrt{q}}} w(i,j)\right)=c^2(1+o(q))$ as $q\to 0$.

. Most of the time, $w(i,j)=0$, rarely, $w(i,j)=1$ and essentially never $w(i,j)\geq 2$ (in any region $i,j\sim\frac{c}{\sqrt{q}}$).

So, in the limit $q\to 0$, and rescaling space and time by $\frac{1}{\sqrt{q}}$, one has Poisson points with intensity one.

. The LGV scheme describing the multilayer PNG droplet is the following:

. Consider N lines starting from:

$$(-t, \ell) \quad , \quad \ell = 0, \dots, -N+1$$

and arriving at

$$(t, \ell) \quad , \quad \ell = 0, \dots, -N+1,$$

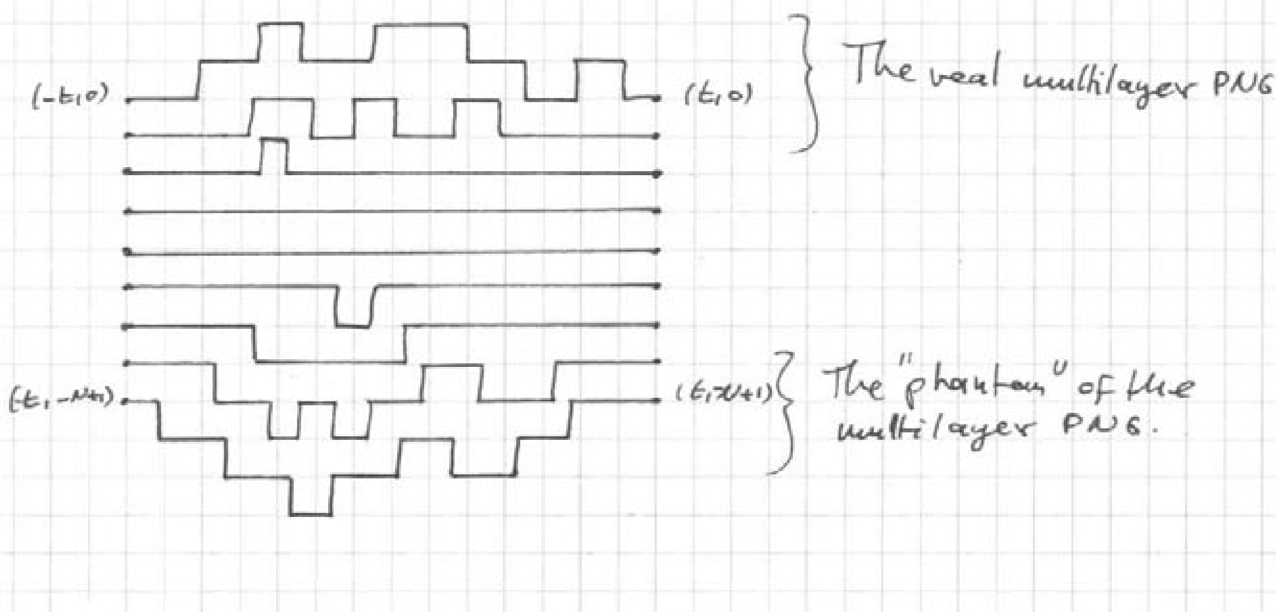
with one-particle transition during "time interval" τ given by

$$P_\tau(x,y) = \langle y, e^{-\tau H} x \rangle \text{ , with} \tag{6.8}$$

$$H\psi(n) = -(\psi(n+1) + \psi(n-1)) \quad , \quad \psi \in \ell^2(\mathbb{Z}).$$

. In other words, $P_\tau(x,y) = \frac{1}{2\pi i} \oint_{\Gamma_0} \frac{dz}{z^{y-x+1}} \cdot e^{\tau(z + \frac{1}{z})}$. (6.9)

. <u>A picture of the LGV scheme:</u>



. When N becomes large, the real and the "phantom" multilayers become independent. So, if we focus in any region bounded from below, after the $N \to \infty$ limit we have <u>exactly</u> the multilayer PNG with ∞-many layers.

Measure: m-points at $-t < \tau_1 < \tau_2 < \dots < \tau_m < t$ is given by

$$(6.10) \quad \text{const} \times \det\left(P_{t+\tau_1}(-i+1, x_j^1)\right)_{1\le i,j\le N} \cdot \left(\prod_{k=1}^{m-1} \det\left(P_{\tau_{k+1}-\tau_k}(x_i^k, x_j^{k+1})\right)_{1\le i,j\le N}\right) \cdot \det\left(P_{t-\tau_m}(x_i^m, -j+1)\right)_{1\le i,j\le N}.$$

- By Proposition 19, we have the Kernel

$$(6.11) \quad K_N(t_1 x_1; t_2 x_2) = -P_{t_2-t_1}(x_1, x_2)\cdot \mathbb{1}_{[t_2>t_1]} + \sum_{i,j=1}^{N} P_{t-t_2}(x_2, -i+1)\cdot [A_N^{-1}]_{i,j}\cdot P_{t+t_1}(-j+1, x_1)$$

where $[A_N]_{i,j} = P_{2t}(-i+1, -j+1)$, $1\le i,j\le N$.

- **Remarks on the $N\to\infty$ limit:**
  ⓐ $P_{t-t_2}(x_2, -i+1)$ goes exponentially to zero as $i\to\infty$,
  ⓑ $P_{t+t_1}(-j+1, x_1)$ " " " $j\to\infty$,
  ⓒ $A_N(i,j)$ " " " $|i-j|\to\infty$.

$$A_N = \begin{pmatrix} A_N(1,1) & \cdots & A_N(1,N) \\ \vdots & & \vdots \\ A_N(N,1) & \cdots & A_N(N,N) \end{pmatrix}$$

- From ⓐ and ⓑ, one need to prove that for any given m, the $m\times m$ block of $[A_N^{-1}]_{1\le i,j\le m} \xrightarrow{N\to\infty} [A_\infty^{-1}]_{1\le i,j\le m}$, and that the remainder of the inverse is not exploding.

- The half-infinite matrix has an inverse which we can compute. One then uses $A_N\cdot A_\infty^{-1} = \mathbb{1} + R_N \Rightarrow A_N^{-1} = (\mathbb{1}+R_N)^{-1}A_\infty$.
  (the $N\times N$ matrix-block of the A_∞^{-1} matrix)
- One has to compute A_∞^{-1}, then $A_N\cdot A_\infty^{-1}$ and see that R_N is small as $N\to\infty$.
- We do not give more details here, but in this case can be done.

. After having taken the $N \to \infty$ we have a kernel given by:

(6.12)
$$K(t_1x_1;t_2x_2) = -P_{t_2-t_1}(x_1,x_2)\,\mathbb{1}_{[t_2>t_1]} + \sum_{i,j=1}^{\infty} P_{t-t_2}(x_2,-i+1)\cdot [A_\infty^{-1}]_{ij} \cdot P_{t+t_1}(-j+1,x_1).$$

(6.13) . $A_\infty = [A]_{1\le i,j<\infty}$ with $A_{ij} = \frac{1}{2\pi i}\oint_{\Gamma_0} dz \frac{e^{2t(z+z^{-1})}}{z^{i-j+1}}$.

. Its inverse is given by:

(6.14)
$$[A_\infty^{-1}]_{ij} = \sum_{k\geq 1} \frac{1}{(2\pi i)^2}\oint_{\Gamma_0} dw \frac{e^{-2tw}}{w^{i-k+1}} \oint_{\Gamma_0} dz \frac{e^{-2tz^{-1}}}{z^{k-j+1}}.$$

. To obtain this, remark that A is the product of two triangular matrices:

$$A_{ij} = \sum_{k\geq 1} \left(\frac{1}{2\pi i}\oint_{\Gamma_0} dz \frac{e^{2tz^{-1}}}{z^{i-k+1}}\right)\left(\frac{1}{2\pi i}\oint_{\Gamma_0} dw \frac{e^{2tw}}{w^{k-j+1}}\right)$$

$$= \frac{1}{(2\pi i)^2}\cdot \underbrace{\oint_{\Gamma_0} dz \oint_{\Gamma_0} dw}_{|z|<|w|} \frac{e^{2tz^{-1}}\cdot e^{2tw}}{z^{i+1}}\, w^{j-1} \underbrace{\sum_{k\geq 1}\left(\frac{z}{w}\right)^k}_{=\frac{z}{w}\cdot\frac{w}{w-z}}$$

$$\overset{\substack{\text{Simple pole}\\ \text{at } w=z}}{=} \frac{1}{2\pi i}\oint_{\Gamma_0} dz \frac{e^{2t(z^{-1}+z)}}{z^{i-j+1}}.$$

. Thus, the inverse is the product of the inverses with exchanged order:
$$[A_\infty^{-1}]_{ij} = \sum_{k\geq 1} \frac{1}{2\pi i}\oint_{\Gamma_0} dw \frac{e^{-2tw}}{w^{i-k+1}}\cdot \frac{1}{2\pi i}\oint_{\Gamma_0} dz \frac{e^{-2tz^{-1}}}{z^{k-j+1}}.$$

. Therefore, the kernel is given by:

(6.15)
$$\boxed{K(t_1x_1;t_2x_2) =} -\frac{1}{2\pi i}\oint_{\Gamma_0} dz\, e^{(t_2-t_1)(z+z^{-1})}\,\mathbb{1}_{[t_2>t_1]}$$
$$+ \sum_{k\geq 0} \frac{1}{2\pi i}\oint_{\Gamma_0} dz \frac{e^{(t-t_2)z^{-1}-(t+t_2)z}}{z^{-k-x_2+1}}\cdot \frac{1}{2\pi i}\oint_{\Gamma_0} dw \frac{e^{(t+t_1)w-(t-t_1)/w}}{w^{x_1+k+1}}$$

$$= \underline{-\frac{1}{2\pi i}\oint_{\Gamma_0} dz\, e^{(t_2-t_1)(z+z^{-1})}\,\mathbb{1}_{[t_2>t_1]}} + \underline{\frac{1}{(2\pi i)^2}\cdot\oint_{\Gamma_0} dz \oint_{\Gamma_{0,z}} dw \frac{e^{(t-t_2)z^{-1}-(t+t_2)z}}{e^{(t-t_1)w^{-1}-(t+t_1)w}}\cdot\frac{z^{x_2-1}}{w^{x_1}}\cdot\frac{1}{w-z}}.$$

- Kernel given in terms of Bessel functions.

  - For $b \geq a$ and $n \in \mathbb{Z}$: $\displaystyle \frac{1}{2\pi i} \oint_{\Gamma_0} \frac{dz}{z} \frac{e^{b(z-z^{-1})} \cdot e^{a(z+z^{-1})}}{z^n} = \left(\frac{b+a}{b-a}\right)^{n/2} J_n\left(2\sqrt{b^2-a^2}\right)$

where J_n are the standard Bessel functions.

- Then, the kernel writes:

$$(6.16)\quad K(t_1 x_1; t_2 x_2) = \begin{cases} \displaystyle\sum_{\ell \leq 0} \left(\frac{t-t_2}{t+t_2}\right)^{\frac{x_2-\ell}{2}} \cdot \left(\frac{t+t_1}{t-t_1}\right)^{\frac{x_1-\ell}{2}} \cdot J_{x_2-\ell}\left(2\sqrt{t^2-t_2^2}\right) J_{x_1-\ell}\left(2\sqrt{t^2-t_1^2}\right), & \text{for } t_2 \leq t_1, \\ \displaystyle\sum_{\ell > 0} \quad '' & \text{for } t_2 > t_1. \end{cases}$$

6.a.4) Edge scaling and Airy_2 process.

- We want to study: $\displaystyle h_t^{edge}(u) := \frac{h(u t^{2/3}, t) - 2t + u^2 t^{1/3}}{t^{1/3}}$

- So, the rescaled point process of $\eta_t(x, j)$, the one associated to the multilayer, is $\eta_t^{edge}(u, s) = t^{1/3} \eta_t\left(u t^{2/3}, [2t - u^2 t^{1/3} + s t^{1/3}]\right)$,

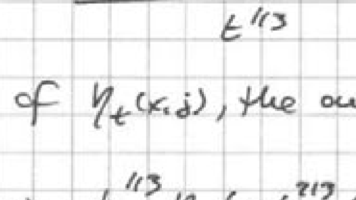
and the associated kernel:

$$K_t^{edge}(u_1 s_1; u_2 t_2) = t^{1/3} K\left(u_1 t^{2/3}, [2t - u_1^2 t^{1/3} + s_1 t^{1/3}]; u_2 t^{2/3}, [2t - u_2^2 t^{1/3} + s_2 t^{1/3}]\right)$$

- One analyzes K_t^{edge} with steep descent method and obtain:

$$(6.17)\quad \underline{\lim_{t\to\infty} K_t^{edge}(u_1 s_1; u_2 s_2)} \overset{\substack{(\text{conjugation}) \\ \downarrow}}{=} \frac{+1}{(2\pi i)^2} \int_{\gamma_1} dw \int_{\gamma_2} dz \, \frac{e^{\frac{z^3}{3} + u_2 z^2 + (u_2^2 - s_2) z}}{e^{\frac{w^3}{3} + u_1 w^2 + (u_1^2 - s_1) w}} \cdot \frac{1}{z-w}$$

where



$$- \frac{\mathbb{1}[u_1 < u_2]}{2\pi i} \cdot \int_{i\mathbb{R}} dw \, e^{(u_2 - u_1) w^2 + (u_2^2 - u_1^2 - s_2 + s_1) w}$$

$$\underline{= K_{\mathcal{A}_2}(u_1 s_1; u_2 s_2).}$$

. By appropriate control in the tails (largest), so that the Fredholm determinant converges too, one then gets:

Theorem 25: In the sense of finite-dimensional distributions,

$$\lim_{t\to\infty} h_t^{edge}(u) = \mathcal{A}_2(u). \qquad (6.19)$$

. Remark: . As shown by Johansson in the context of discrete time PNG,

$$\mathbb{P}\left(\sup_{u\in\mathbb{R}}\left(\mathcal{A}_2(u)-u^2\right)\leq s\right) = F_1(s).$$

. So, if the process h_t^{edge} is tight, then

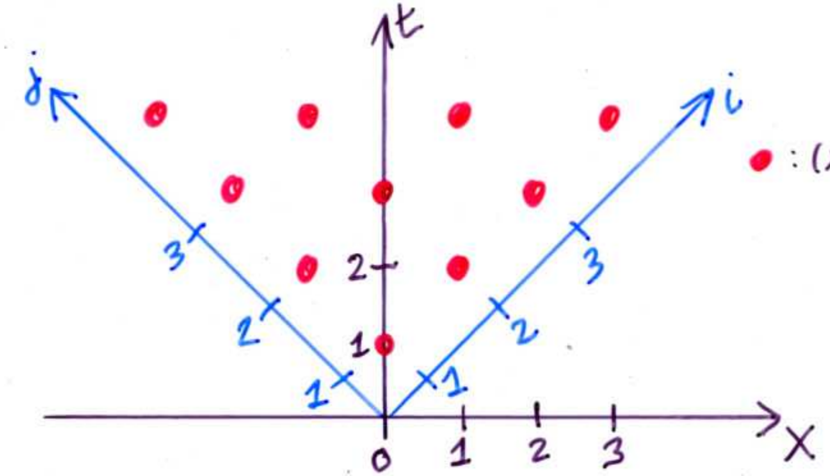
$$\lim_{t\to\infty} \mathbb{P}\left(\sup_{u\in\mathbb{R}} \frac{h(u\ \ ,t)-2t}{t^{1/3}} \leq s\right) = F_1(s).$$

. Remark: The analogue of the $\mathcal{A}iry_2$ process for flat PNG, defined by removing the constraint that the poisson points occurs only on $|x|\leq t$.

$\Rightarrow h_{ma}(\xi) = 2$ and

$$\lim_{t\to\infty} h_t^{edge}(u) = 2^{1/3}\mathcal{A}_1(2^{-2/3}u)$$: the $\mathcal{A}iry_1$ Process given in the next section.

Discrete time PNG $\leftrightarrow$ Directed Polymers



$\bullet$: (x,t) s.t. $\omega(x,t) \neq 0$.

I.C.: $h(x, t=0) = 0, \; \forall x \in \mathbb{Z}$

Dynamics: $h(x, t+1) = \max\{h(x-1,t), h(x,t), h(x+1,t)\} + \omega(x, t+1)$

"energy" $\updownarrow$ $\equiv$ "length" of the longest directed polymer.

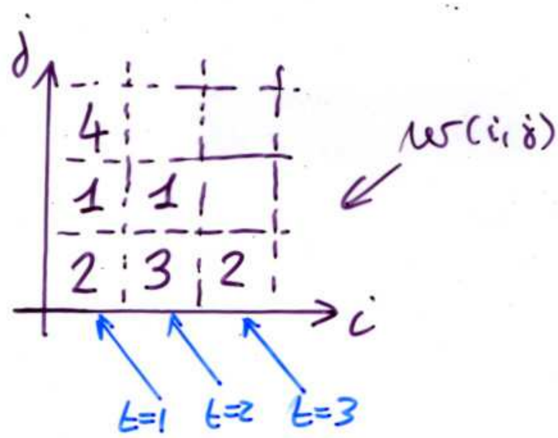
. Particular (solvable) case:

$$\begin{cases} \omega(x,t) = 0 & \text{if } t-x \text{ is odd or } |x| > t \\ \omega(x,t) \neq 0 & \text{otherwise, with} \end{cases}$$

$\omega(i-j, i+j-1) = w(i,j)$, $i,j \in \mathbb{N}_+^2$ i.i.d. geometric random variables:

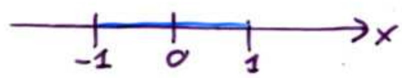
$$\mathbb{P}(w(i,j) = k) = (1 - a_i b_j) \cdot (a_i b_j)^k, \; k \geq 0.$$

Example:

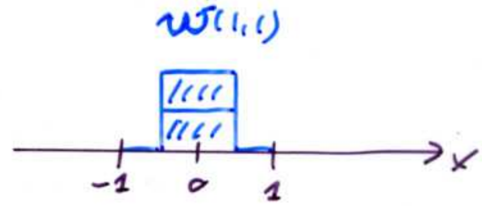


Height function:

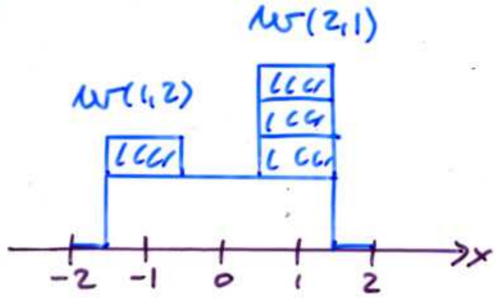
$t=0$:



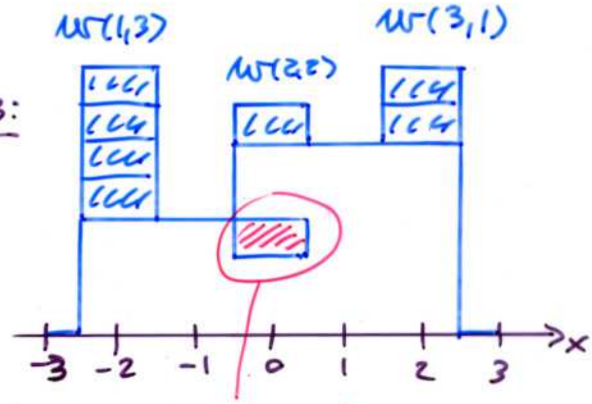
$t=1$:



$t=2$:

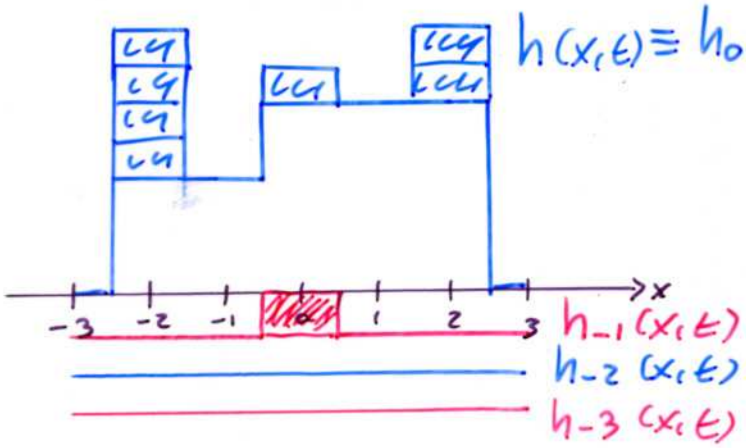


$t=3$:



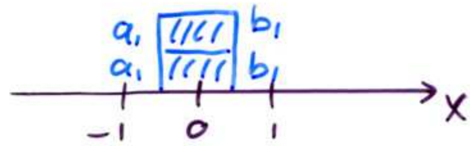
Intersection
≡
Lost of information.

Multilayer PNG:



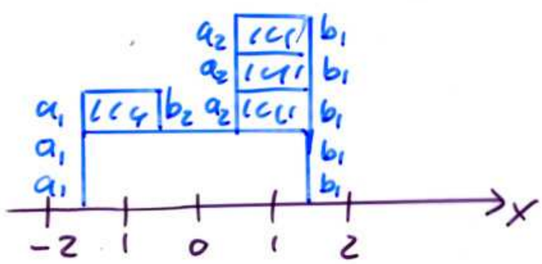
Measure: from $w(i,j)$ to multilayer.

t=1: $w(1,1) \Rightarrow \text{weight } (a_1 \cdot b_1)^{w(1,1)} = a_1^2 \cdot b_1^2$

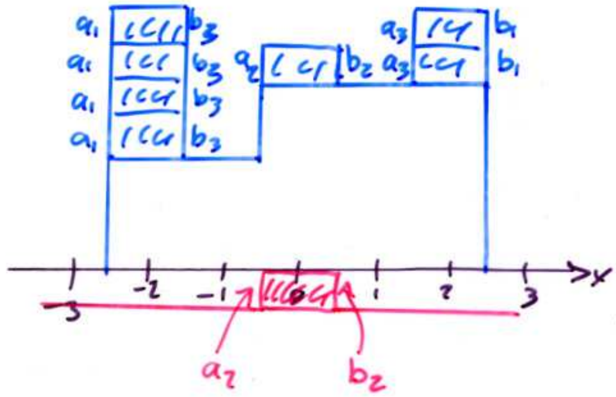


t=2: $\text{weight } (a_1 b_1)^{w(1,1)} \cdot (a_1 b_2)^{w(1,2)} \cdot (a_2 b_1)^{w(2,1)}$

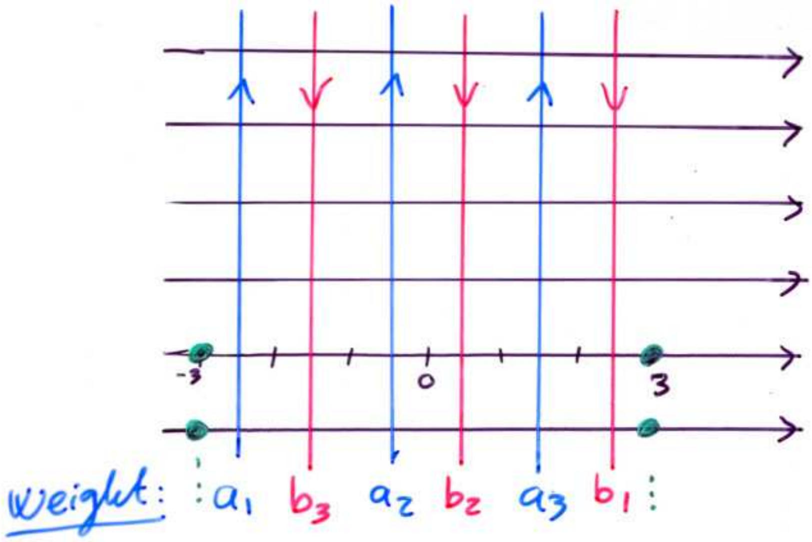
$$= a_1^3 \cdot b_2^1 \cdot a_2^3 \cdot b_1^5$$



t=3: $\text{weight} = (\text{weight at } t=2) \cdot (a_1 b_3)^4 \cdot (a_2 b_2)^1 \cdot (a_3 b_1)^2$



LGV scheme : example for $t=3$.



$\longrightarrow$: oriented to the right, weight $\equiv 1$.

- Non-intersecting lines from
  $(-t, e)$, $e \leq 0$
  to (t, e), $e \leq 0$.

- Steps: 1) Consider N non-intersecting lines on LGV and compute the Kernel.
  2) Take $N \to \infty$ for finite t
  3) Take $t \to \infty$ under appropriate scaling limit.

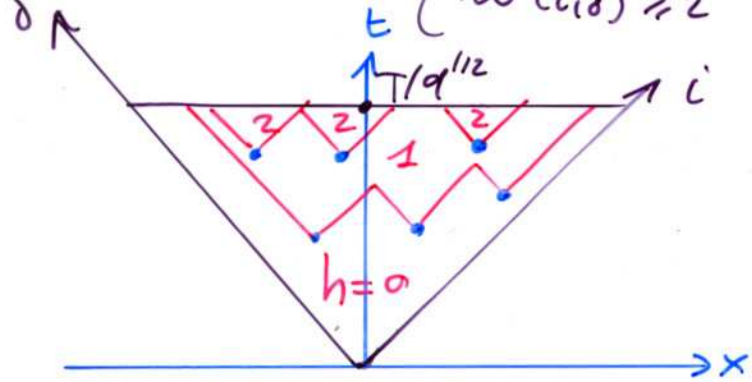
- For $a_i = b_i = \sqrt{q}$: Top line $\equiv$ PNG height function converges to the Airy$_2$ Process [Johansson]

Continuous time PNG

- $a_i = b_i = \sqrt{q} \Rightarrow \mathbb{P}(w(i,j)=k) = (1-q)q^k, \quad k \geq 0.$
- Take $q \to 0$ and rescale space and time:

$$x = \left[\frac{X}{\sqrt{q}}\right], \quad t = \left[\frac{T}{\sqrt{q}}\right].$$

- When $q \to 0$: $\begin{cases} w(i,j) = 0 \text{ most of the time,} \\ w(i,j) = 1 \text{ with proba.} \sim q, \\ w(i,j) \geq 2 \quad \text{"} \quad \text{"} \quad q^2: \underline{\text{irrelevant}}. \end{cases}$



$\bullet \equiv w(i,j) = 1$: Poisson Points with intensity one.

Results: [Prähofer–Spohn'02]

(1) $\mathbb{P}\left(\bigcap_{i=1}^{m}\{h(X_i,T) \leq a_i\}\right) = \det(\mathbb{1} - \chi_a K_T^{PNG} \chi_a)$

$L^2(\{X_1, \ldots, X_m\} \times \mathbb{R})$

with $\chi_a(X_i, z) = \mathbb{1}_{[z > a_i]}$, and

$$K_T^{PNG}(x_1,h_1;x_2,h_2) = \begin{cases} \displaystyle\sum_{e \geq 0} \left(\frac{T-x_2}{T+x_2}\right)^{\frac{h_2+e}{2}} J_{h_2+e}\left(2\sqrt{T^2-x_2^2}\right) \cdot \left(\frac{T+x_1}{T-x_1}\right)^{\frac{h_1+e}{2}} J_{h_1+e}\left(2\sqrt{T^2-x_1^2}\right), & \text{if } -T \leq x_2 \leq x_1 \leq T \\ \displaystyle -\sum_{e<0} (\quad " \quad), & \text{if } T \geq x_2 > x_1 \geq -T. \end{cases}$$

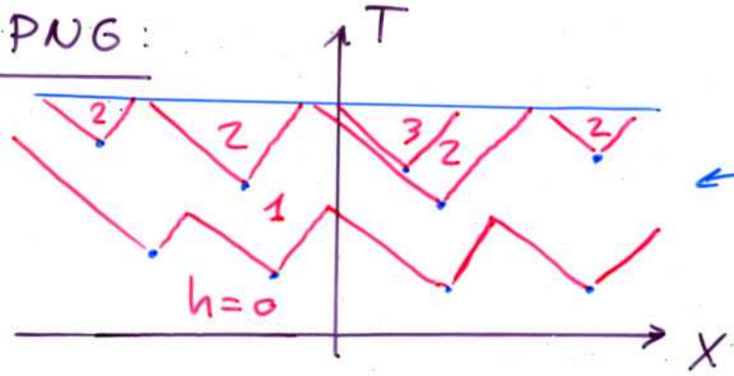
($J_n(x)$: Bessel functions)

② Edge scaling:

$$h_T^{edge}(u) := \frac{h_0(uT^{2/3},T)-(2T-u^2T^{1/3})}{T^{1/3}}$$

$$\boxed{\lim_{T\to\infty} h_T^{edge}(u) = \mathcal{A}_2(u).}$$

- Flat PNG:



← Poisson Points not restricted to $|x|\leq T$ anymore.

- Results: [Borodin-Ferrari-Sasamoto '07]

① $\mathbb{P}\left(\bigcap_{i=1}^{m}\{h(x_i,T)\leq a_i\}\right) = \det(\mathbb{1}-\chi_a K_T^{flat}\chi_a)$

with $K_T^{flat}(x_1,h_1;x_2,h_2) = -I_{|h_1-h_2|}(2(x_2-x_1))\mathbb{1}_{[x_2>x_1]}$

$$+\left(\frac{2T+x_2-x_1}{2T-x_2+x_1}\right)^{\frac{h_1+h_2}{2}}\cdot J_{h_1+h_2}\left(2\sqrt{4T^2-(x_2-x_1)^2}\right)\cdot\mathbb{1}_{[2T\geq|x_2-x_1|]}$$

② Edge scaling:

$$h_T^{edge}(u) := \frac{h(uT^{2/3},T)-2T}{T^{1/3}}$$

$$\boxed{\lim_{T\to\infty} h_T^{edge}(u) = 2^{1/3}\cdot\mathcal{A}_1(2^{-2/3}\cdot u)}$$

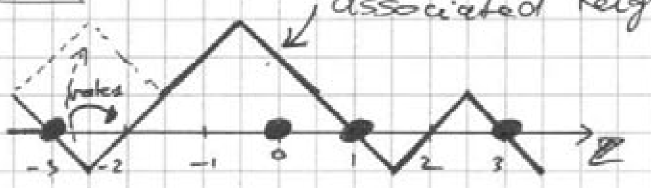
6.b) Application to the Totally Asymmetric Simple Exclusion Process (TASEP).

. The TASEP is a model of interacting particle system.
The configurations consists of particles on $\mathbb{Z}$, with the exclusion constraint that at most one particle per site is allowed.

. The dynamics is (in continuous time) simply the following: each particle try to jump to its right-neighboring site at rate one, and the move occurs only if the site is empty.

. One can also associate an interface to a particles' configuration by replacing a particle by \ and an empty site by /.

. Example:



. From this point of view, the TASEP can be seen as a stochastic growth model. It belongs to the KPZ universality class.

. Two important initial conditions:

(a) Step initial conditions: Particles occupy $\{\ldots,-2,-1,0\}$. Then, the associate limit shape is curved $\Rightarrow$ One expect the fluctuations to be described by the Airy$_2$ process (proven).

(b) Periodic initial conditions: Particles occupy $2\mathbb{Z}$ (for example). Then the limit shape is straight $\Rightarrow$ one expect the fluctuations to be described by another process: the Airy$_1$ process.

6.b.1) Transition probability of a fixed number of particles

• **Proposition {Schütz formula} 26:**

• Consider N particles with initial conditions $x_i(0)=y_i$, $y_1>y_2>\dots>y_N$.

Denote by $G(x_N,\dots,x_1;t)=\mathbb{P}\left(x_i(t)=x_i, 1\le i\le N \,\middle|\, x_i(0)=y_i, 1\le i\le N\right)$.

Then:

$$G(x_N,\dots,x_1;t)= \quad \det\left(F_{i-j}(x_{N+1-i}-y_{N+1-j},t)\right)_{1\le i,j\le N} \tag{6.20}$$

$$\text{with} \quad F_n(x,t)=\frac{(-1)^n}{2\pi i}\oint_{\Gamma_{0,1}}\frac{dw}{w}\cdot\frac{(1-w)^{-n}}{w^{x-n}}\cdot e^{t(1-w)} \tag{6.21}$$

• This formula, obtained by Schütz '97, was remanipulated cleverly by Sasamoto '05.

• The key argument which we explain below, it is based on the following recursion relations:

$$F_{n-1}(x,t)=F_n(x,t)-F_n(x+1,t) \tag{6.22}$$

and its integrated form

$$F_{n+1}(x,t)=\sum_{y\ge x}F_n(y,t). \tag{6.23}$$

Lemma 27: Denote $y_k=: x_1^k$, $k=1,\dots,N$. Then,

$$G(x_N,\dots,x_1;t)=\sum_{\mathcal{D}}\det\left(F_{-j}(x_{i+1}^N-y_{N-j},t)\right)_{0\le i,j\le N-1}, \tag{6.24}$$

where $\mathcal{D}=\{x_i^n, 2\le i\le n\le N \mid x_i^{n+1}<x_i^n\le x_{i+1}^{n+1}\}$

Proof of Lemma 27: The key is to use (6.23) and the antisymmetry of the determinant.

. By Proposition 26, we have:

$$(6.25) \quad G(x_1^N,\dots,x_1^1;t) = \det \begin{bmatrix} F_0(x_1^N - y_N, t) & \dots & F_{-N+1}(x_1^N - y_1, t) \\ \vdots & & \vdots \\ F_{N-1}(x_1^1 - y_N, t) & \dots & F_0(x_1^1 - y_1, t) \end{bmatrix}$$

. Step 1: Rewrite last row as:

$$\sum_{x_2^2 \geq x_1^1} \left[F_{N-2}(x_2^2 - y_N, t) \dots F_{-1}(x_2^2 - y_1, t) \right].$$

. Step 2: Same procedure to the last two rows; which becomes

$$(\text{last row}) : \sum_{x_2^2 \geq x_1^1} \sum_{x_3^3 \geq x_2^2} \left[F_{N-3}(x_3^3 - y_N, t) \dots F_{-2}(x_3^3 - y_1, t) \right]$$

$$(\text{2nd last row}) : \sum_{x_2^3 \geq x_1^2} \left[F_{N-3}(x_2^3 - y_N, t) \dots F_{-2}(x_2^3 - y_1, t) \right].$$

. At this point we have:

$$(6.25) = \sum_{x_2^2 \geq x_1^1} \sum_{x_3^3 \geq x_2^2} \sum_{x_2^3 \geq x_1^2} \det \begin{bmatrix} F_0(x_1^N - y_N, t) & \dots & F_{-N+1}(x_1^N - y_1, t) \\ \vdots & & \vdots \\ F_{N-3}(x_1^3 - y_N, t) & \dots & F_{-2}(x_1^3 - y_1, t) \\ F_{N-3}(x_2^3 - y_N, t) & \dots & F_{-2}(x_2^3 - y_1, t) \\ F_{N-3}(x_3^3 - y_N, t) & \dots & F_{-2}(x_3^3 - y_1, t) \end{bmatrix}$$

. The determinant is antisymmetric in (x_2^3, x_3^3). Thus

$$\sum_{x_3^3 \geq x_2^2} \sum_{x_2^3 \geq x_2^2} (\dots) \equiv 0$$

. It remains: $\displaystyle\sum_{x_2^2 \geq x_1^1} \sum_{x_3^3 \geq x_2^2} \sum_{x_2^2 > x_2^3 \geq x_1^2} (\dots).$

. By repeating the same procedure one gets (6.24). #

. The key next idea of Sasamoto was to write the sum over D as sum "without constraints" times a product of determinants giving 1 when in D and zero otherwise.

Lemma 28: We have a (non-positive in general) measure on $\{x_i^n\}$:

$$W(\{x_i^n, 1\le i\le n\le N\}) := \prod_{n=1}^{N-1} \det\left(\phi_n(x_i^n, x_j^{n+1})\right)_{1\le i,j\le n+1} \cdot \det\left(F_{-j}(x_{i+1}^N - y_{N-j}, t)\right)_{0\le i,j\le N-1} \quad (6.26)$$

with $\phi_n(x,y) = \mathbb{1}[y \geq x]$ and $\phi_n(x_{n+1}^n, x) \doteq 1$ (x_{n+1}^n are fictitious variables).

Such that: $G(x_N^1, \dots, x_1^1; t) = \sum_{x_i^n, 2\le i\le n\le N} W(\{x_i^n, 1\le i\le n\le N\})$. (6.27)

. To see this, one just need to check that the determinants of ϕ_n's enforce the interlacing constraint $\mathcal{D}$.

6.b.2) A determinantal measure.

. Lemma 3.4 of math-ph/0806056:

. Assume that we have a (signed) measure on $\{x_i^n, 1\le i\le n\le N\}$ given in the form:

(6.28) $$\frac{1}{Z_N} \cdot \prod_{n=1}^{N-1} \det\left(\phi_n(x_i^n, x_j^{n+1})\right)_{1\le i,j\le n+1} \cdot \det\left(\Psi_{N-i}^N(x_j^N)\right)_{1\le i,j\le N}$$

where x_{n+1}^n are "fictitious" variables and Z_N a normalization constant.

. Then, if $Z_N \neq 0$, the correlation functions are determinantal.

. Define: $\phi^{(n,m)}(x,y) = \begin{cases} (\phi_n * \dots * \phi_{m-1})(x,y), & \text{if } n < m, \\ 0 & , \text{if } n \geq m. \end{cases}$

$$\boxed{(a*b)(x,y) = \sum_{z\in\mathbb{Z}} a(x,z)b(z,y)}$$

and $\Psi_{n-j}^n(x) := (\phi^{(n,N)} * \Psi_{N-j}^N)(x)$, $j = 1, \dots, N$.

. Set $\phi_0(x_1^0, x) = 1$. Then, the functions

(6.29a) $$\{(\phi_0 * \phi^{(1,n)})(x_1^0, x), \dots, (\phi_{n-2} * \phi^{(n-1,n)})(x_{n-1}^{n-2}, x), \phi_{n-1}(x_n^{n-1}, x)\}$$

are linearly independent and generate the n-dim. space V_n.

Define a set of functions $\{\Phi_j^n, j=0,\dots,n-1\}$ spanning V_n and obtained by the orthonormal relations:

(6.29b) $$\sum_x \Phi_i^n(x)\Psi_j^n(x) = \delta_{i,j}, \quad 0\le i,j\le n-1.$$

. Under assumption (A): $\phi_n(x^n_{n+1}, x) = c_n \cdot \Phi_0^{n+1}(x)$ for some $c_n \neq 0$, $n=1,\dots,N-1$, the kernel takes a simple form:

(6.30) $$K(n_1 x_1; n_2 x_2) = -\phi^{(n_1,n_2)}(x_1,x_2) + \sum_{k=1}^{n_2} \Psi^{n_1}_{n_1-k}(x_1)\,\Phi^{n_2}_{n_2-k}(x_2).$$

We do not prove it here, but several ingredients are similar to the proof of Proposition 19. Notice that here we have

. Application to the measure of Lemma 28 gives:

<u>Proposition 29:</u> Let $V_n = \mathrm{span}\{1, x, \dots, x^{n-1}\}$.

Define $\phi^{(n_1,n_2)}(x_1,x_2) = \binom{x_1-x_2-1}{n_2-n_1-1}$,

$$\Psi_i^n(x) = \frac{1}{2\pi i}\oint_{\Gamma_0} dw \frac{(1-w)^i e^{t(w-1)}}{w^{i+1}\cdot w^{x-y_{n-i}}}$$ and

$\Phi_i^n(x)$ polynomials of degree i, $i=0,\dots,n-1$, satisfying

$$\sum_{x\in\mathbb{Z}} \Psi_i^n(x)\Phi_j^n(x) = \delta_{ij}, \quad 1\le i,j\le n-1.$$

Then, the joint distributions of m particles with indices $\sigma(1)<\sigma(2)<\dots<\sigma(m)$ at time t is given by:

(6.31) $$\mathbb{P}\left(\bigcap_{k=1}^{m} \{x_{\sigma(k)}(t) \geq s_k\}\right) = \det\left(\mathbb{1} - \chi_s K_t \chi_s\right)_{\ell^2(\{\sigma(1),\dots,\sigma(m)\}\times\mathbb{Z})},$$

where $\chi_s(\sigma(k), x) = \mathbb{1}[x < s_k]$, and the extended kernel K_t is given by

(6.32) $$K_t(n_1 x_1; n_2 x_2) = -\phi^{(n_1,n_2)}(x_1,x_2) + \sum_{k=1}^{n_2} \Psi^{n_1}_{n_1-k}(x_1)\,\Phi^{n_2}_{n_2-k}(x_2).$$

6.b.3) Particularization for step and alternating initial conditions:

a) Step I.C.: let $y_i = -i,\ i \geq 1$. Then,

$$\Psi^n_k(x) = \frac{1}{2\pi i} \oint_{\Gamma_0} dw \, \frac{e^{(w-1)t} \cdot (1-w)^k}{w^{x+n+1}} \tag{6.33}$$

and

$$\Phi^n_k(x) = \frac{-1}{2\pi i} \oint_{\Gamma_1} dz \, \frac{e^{-tz} \cdot z^{x+n}}{(1-z)^{k+1}}$$

Consequence: (6.34)

$$\begin{cases} K_t(n_1, x_1; n_2, x_2) = -\dfrac{1}{2\pi i} \displaystyle\oint_{\Gamma_0} dw \left(\frac{w}{1-w}\right)^{n_2-n_1} \cdot \frac{1}{w^{x_1-x_2+1}} \cdot \mathbb{1}_{[n_1<n_2]} \\ \quad + \dfrac{1}{(2\pi i)^2} \displaystyle\oint_{\Gamma_0} dw \oint_{\Gamma_1} dz \, \frac{e^{tw}}{e^{tz}} \cdot \frac{(1-w)^{n_1}}{w^{x_1+n_1+1}} \cdot \frac{z^{x_2+n_2}}{(1-z)^{n_2}} \cdot \frac{1}{w-z} \end{cases}$$

b) Flat I.C.: let $y_i = -2i,\ i \geq 1$. Then,

$$\Psi^n_k(x) = \frac{1}{2\pi i} \oint_{\Gamma_0} dw \, \frac{(w(1-w))^k \cdot e^{tw}}{w^{x+2n+1}}, \tag{6.35}$$

$$\Phi^n_k(x) = \frac{-1}{2\pi i} \oint_{\Gamma_1} dz \, \frac{(2z-1) \cdot z^{x+2n} \cdot e^{-tz}}{(z(1-z))^{k+1}}$$

Consequence: The kernel for flat I.C., i.e., $y_i = -2i$, $i \in \mathbb{Z}$, is obtained by looking around particle N, i.e., $n_i \to n_i + N$ and, of course $x_i \to x_i - 2N$.

After $N \to \infty$ limit, one gets the flat IC kernel:

(6.36)

$$\begin{cases} K_t(n_1 x_1; n_2 x_2) = -\dfrac{1}{2\pi i} \displaystyle\oint_{\Gamma_0} dw \left(\frac{w}{1-w}\right)^{n_2-n_1} \cdot \frac{1}{w^{x_1-x_2+1}} \cdot \mathbb{1}_{[n_1<n_2]} \\ \quad + \dfrac{-1}{2\pi i} \displaystyle\oint_{\Gamma_1} dz \, e^{t(1-2z)} \cdot \frac{z^{n_1+n_2+x_2}}{(1-z)^{n_1+n_2+x_1+1}}. \end{cases}$$

Edge scaling and asymptotics.

- For step I.C. one in the appropriate edge scaling, gets the convergence of K_t to the Airy kernel $\Rightarrow$ Airy$_2$ process.

For example:

(6.37)
$$\lim_{t\to\infty} \frac{x_{[t/4+u(t/2)^{2/3}]}(t) - \left(-2u(t/2)^{2/3} + u^2 (t/2)^{1/3}\right)}{-(t/2)^{1/3}} = \mathcal{A}_2(u).$$

For flat I.C.: let $X_t^{\text{resc}}(u) := \dfrac{x_{[t/4+ut^{2/3}]}(t) + 2ut^{2/3}}{-t^{1/3}}$.

(6.38) Then, $\lim\limits_{t\to\infty} X_t^{\text{resc}}(u) = \mathcal{A}_1(u)$: Airy$_1$ process.

Definition 30: The Airy$_1$ Process is defined by the m-point joint distributions at $u_1 < u_2 < \dots < u_m$ given by

(6.39)
$$\mathbb{P}\left(\bigcap_{k=1}^{m} \{\mathcal{A}_1(u_k) \le s_k\}\right) = \det(\mathbb{1} - \chi_s K_{\mathcal{A}_1} \chi_s)_{L^2(\{u_1,\dots,u_m\}\times\mathbb{R})}$$

where $\chi_s(u_k,x) = \mathbb{1}[x > s_k]$ and

(6.40)
$$K_{\mathcal{A}_1}(u_1 s_1; u_2 s_2) = -\frac{1}{\sqrt{4\pi(u_2-u_1)}} \cdot \exp\left(-\frac{(s_2-s_1)^2}{4(u_2-u_1)}\right) \cdot \mathbb{1}_{[u_2>u_1]}$$
$$+ Ai(s_1+s_2+(u_2-u_1)^2) \cdot \exp\left((u_2-u_1)(s_1+s_2) + \tfrac{2}{3}(u_2-u_1)^3\right).$$

A few properties:
- $\mathcal{A}_1$ is stationary
- $\mathbb{P}(\mathcal{A}_1(0) \le s) = F_1(2s)$
  $$\Rightarrow F_1(s) = \det(\mathbb{1} - B_s)_{L^2(\mathbb{R}_+,dx)},$$
  $$B_s(x,y) = Ai(x+y+s).$$
- $\mathrm{Var}(\mathcal{A}_1(u) - \mathcal{A}_1(0)) \simeq 2|u|$ for small u.

Compact formula: (compare page 40)

$$K_{\mathcal{A}_1}(u,s;u',s') = -(e^{-(s'-s)H_1})(u,u')\,\mathbb{1}_{[s'>s]} + (e^{sH_1} B e^{-s'H_1})(u,u'),$$
$$B(s,s') = Ai(s+s'),\quad H_1 = -\Delta \equiv -\frac{d^2}{dx^2}.$$