6.b) Application to the Totally Asymmetric Simple Exclusion Process (TASEP).

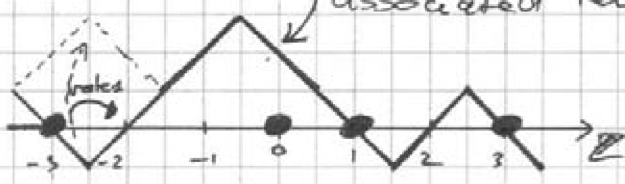
- The TASEP is a model of interacting particle system.

The configurations consists of particles on $\mathbb{Z}$, with the exclusion constraint that at most one particle per site is allowed.

- The dynamics is (in continuous time) simply the following: each particle try to jump to its right-neighboring site at rate one, and the move occurs only if the site is empty.

- One can also associate an interface to a particles' configuration by replacing a particle by \ and an empty site by /.

- Example:



- From this point of view, the TASEP can be seen as a stochastic growth model. It belongs to the KPZ universality class.

- Two important initial conditions:

(a) Step initial conditions: Particles occupy $\{\ldots,-2,-1,0\}$. Then, the associate limit shape is curved $\Rightarrow$ One expect the fluctuations to be described by the Airy_2 process (proven).

(b) Periodic initial conditions: Particles occupy $2\mathbb{Z}$ (for example). Then the limit shape is straight $\Rightarrow$ one expect the fluctuations to be described by another process: the Airy_1 process.

6.b.1) Transition probability of a fixed number of particles

Proposition {Schütz formula} 26:

- Consider N particles with initial conditions $x_i(0)=y_i$, $y_1>y_2>\dots>y_N$.

Denote by $G(x_N,\dots,x_1;t)=\mathbb{P}(x_i(t)=x_i, 1\leq i\leq N \mid x_i(0)=y_i, 1\leq i\leq N)$.

Then:

(6.20) $$G(x_N,\dots,x_1;t)= \det\left(F_{i-j}(x_{N+1-i}-y_{N+1-j},t)\right)_{1\leq i,j\leq N}$$

(6.21) with $$F_n(x,t)=\frac{(-1)^n}{2\pi i}\oint_{\Gamma_{0,1}}\frac{dw}{w}\cdot\frac{(1-w)^{-n}}{w^{x-n}}\cdot e^{-t(1-w)}$$

- This formula, obtained by Schütz '97, was remanipulated cleverly by Sasamoto '05.

- The key argument which we explain below, it is based on the following recursion relations:

(6.22) $$F_{n-1}(x,t)=F_n(x,t)-F_n(x+1,t)$$

and its integrated form

(6.23) $$F_{n+1}(x,t)=\sum_{y\geq x}F_n(y,t).$$

Lemma 27: Denote $x_k=:x_1^k$, $k=1,\dots,N$. Then,

(6.24) $$G(x_N,\dots,x_1;t)=\sum_{\mathcal{D}}\det\left(F_{-j}(x_{i+1}^N-y_{N-j},t)\right)_{0\leq i,j\leq N-1},$$

where $\mathcal{D}=\{x_i^n, 2\leq i\leq n\leq N \mid x_i^{n+1}<x_i^n\leq x_{i+1}^{n+1}\}$

Proof of Lemma 27: The key is to use (6.23) and the antisymmetry of the determinant.

. By Proposition 26, we have:

$$(6.25)\quad G(x_1^N,\dots,x_1^1;t)=\det\begin{bmatrix} F_0(x_1^N-y_N,t) & \cdots & F_{-N+1}(x_1^N-y_1,t)\\ \vdots & & \vdots\\ F_{N-1}(x_1^1-y_N,t) & \cdots & F_0(x_1^1-y_1,t)\end{bmatrix}$$

. Step 1: Rewrite last row as:

$$\sum_{x_2^2\geq x_1^1}\left[F_{N-2}(x_2^2-y_N,t)\ \cdots\ F_{-1}(x_2^2-y_1,t)\right].$$

. Step 2: Same procedure to the last two rows; which becomes

$$(\text{last row}):\ \sum_{x_2^2\geq x_1^1}\ \sum_{x_3^3\geq x_2^2}\left[F_{N-3}(x_3^3-y_N,t)\ \cdots\ F_{-2}(x_3^3-y_1,t)\right]$$

$$(\text{2nd last row}):\ \sum_{x_2^3\geq x_1^2}\left[F_{N-3}(x_2^3-y_N,t)\ \cdots\ F_{-2}(x_2^3-y_1,t)\right].$$

. At this point we have:

$$(6.25)=\sum_{x_2^2\geq x_1^1}\ \sum_{x_3^3\geq x_2^2}\ \sum_{x_2^3\geq x_1^2}\det\begin{bmatrix} F_0(x_1^N-y_N,t) & \cdots & F_{-N+1}(x_1^N-y_1,t)\\ \vdots & & \vdots\\ F_{N-3}(x_1^3-y_N,t) & \cdots & F_{-2}(x_1^3-y_1,t)\\ F_{N-3}(x_2^3-y_N,t) & \cdots & F_{-2}(x_2^3-y_1,t)\\ F_{N-3}(x_3^3-y_N,t) & \cdots & F_{-2}(x_3^3-y_1,t)\end{bmatrix}$$

. The determinant is antisymmetric in (x_2^3,x_3^3). Thus

$$\sum_{x_3^3\geq x_2^2}\ \sum_{x_2^3\geq x_2^2}(\dots)\equiv 0$$

. It remains: $\displaystyle\sum_{x_2^2\geq x_1^1}\ \sum_{x_3^3\geq x_2^2}\ \sum_{x_2^2> x_2^3\geq x_1^2}(\dots).$

. By repeating the same procedure one gets (6.24). #

. The key next idea of Sasamoto was to write the sum over D as sum "without constraints" times a product of determinants giving 1 when in D and zero otherwise.

Lemma 28: We have a (non-positive in general) measure on $\{x_i^n\}$:

$$W(\{x_i^n, 1\le i\le n\le N\}) := \prod_{n=1}^{N-1} \det\left(\phi_n(x_i^n, x_j^{n+1})\right)_{1\le i,j\le n+1} \cdot \det\left(F_{-j}(x_{i+1}^N - y_{N-j}, t)\right)_{0\le i,j\le N-1} \quad (6.26)$$

with $\phi_n(x,y) = \mathbb{1}[y\geq x]$ and $\phi_n(x_{n+1}^n, x) \doteq 1$ (x_{n+1}^n are fictitious variables).

Such that: $$G(x_N^1, \dots, x_1^1; t) = \sum_{x_i^n, 2\le i\le n\le N} W(\{x_i^n, 1\le i\le n\le N\}) \quad (6.27)$$

. To see this, one just need to check that the determinants of ϕ_n's enforce the interlacing constraint $\mathcal{D}$.

6.b.2) A determinantal measure.

. Lemma 3.4 of math-ph/0806056:

. Assume that we have a (signed) measure on $\{x_i^n, 1\le i\le n\le N\}$ given in the form:

(6.28) $$\frac{1}{Z_N}\cdot\prod_{n=1}^{N-1} \det\left(\phi_n(x_i^n, x_j^{n+1})\right)_{1\le i,j\le n+1} \cdot \det\left(\Psi_{N-i}^N(x_j^N)\right)_{1\le i,j\le N}$$

where x_{n+1}^n are "fictitious" variables and Z_N a normalization constant.

. Then, if $Z_N \neq 0$, the correlation functions are determinantal.

. Define: $$\phi^{(n,m)}(x,y) = \begin{cases} (\phi_n * \dots * \phi_{m-1})(x,y), & \text{if } n < m, \\ 0 & \text{, if } n \geq m. \end{cases}$$

$(a*b)(x,y) = \sum_{z\in\mathbb{Z}} a(x,z)b(z,y)$

and $\Psi_{n-j}^n(x) := (\phi^{(n,N)} * \Psi_{N-j}^N)(x)$, $j=1,\dots,N$.

. Set $\phi_0(x_1^0, x) = 1$. Then, the functions

(6.29a) $$\{(\phi_0 * \phi^{(1,n)})(x_1^0, x), \dots, (\phi_{n-2} * \phi^{(n-1,n)})(x_{n-1}^{n-2}, x), \phi_{n-1}(x_n^{n-1}, x)\}$$

are linearly independent and generate the n-dim. space V_n.

Define a set of functions $\{\Phi_j^n, j=0,\dots,n-1\}$ spanning V_n and obtained by the orthonormal relations:

(6.29b) $$\sum_x \Phi_i^n(x)\Psi_j^n(x)=\delta_{i,j}, \quad 0\le i,j\le n-1.$$

. Under assumption (A): $\phi_n(x_{n+1}^n, x) = c_n \cdot \Phi_0^{n+1}(x)$ for some $c_n\neq 0$, $n=1,\dots,N-1$, the kernel takes a simple form:

(6.30) $$K(n_1x_1;n_2x_2) = -\phi^{(n_1,n_2)}(x_1,x_2) + \sum_{k=1}^{n_2} \Psi_{n_1-k}^{n_1}(x_1)\,\Phi_{n_2-k}^{n_2}(x_2).$$

We do not prove it here, but several ingredients are similar to the proof of Proposition 19. Notice that here we have

. Application to the measure of Lemma 28 gives:

<u>Proposition 29</u>: Let $V_n = \operatorname{span}\{1,x,\dots,x^{n-1}\}$.

Define $\phi^{(n_1,n_2)}(x_1,x_2) = \binom{x_1-x_2-1}{n_2-n_1-1}$,

$\Psi_i^n(x) = \frac{1}{2\pi i}\oint_{\Gamma_0} dw \frac{(1-w)^t e^{t(w-1)}}{w^{i+1}\, w^{x-x_{n-i}}}$ and

$\Phi_i^n(x)$ polynomials of degree i, $i=0,\dots,n-1$, satisfying
$$\sum_{x\in\mathbb{Z}} \Psi_i^n(x)\Phi_j^n(x) = \delta_{ij}, \quad 1\le i,j\le n-1.$$

Then, the joint distributions of m particles with indices $\sigma(1)<\sigma(2)<\dots<\sigma(m)$ at time t is given by:

(6.31) $$\mathbb{P}\left(\bigcap_{k=1}^m \{x_{\sigma(k)}(t)\ge s_k\}\right) = \det(\mathbb{1}-\chi_s K_t \chi_s)_{\ell^2(\{\sigma(1),\dots,\sigma(m)\}\times\mathbb{Z})},$$

where $\chi_s(\sigma(k),x) = \mathbb{1}[x<s_k]$, and the extended kernel K_t is given by

(6.32) $$K_t(n_1x_1;n_2x_2) = -\phi^{(n_1,n_2)}(x_1,x_2) + \sum_{k=1}^{n_2}\Psi_{n_1-k}^{n_1}(x_1)\,\Phi_{n_2-k}^{n_2}(x_2).$$

6.b.3) Particularization for step and alternating initial conditions:

. a) Step I.C.: let $y_i = -i,\ i \geq 1$. Then,

$$\Psi^n_k(x) = \frac{1}{2\pi i} \oint_{\Gamma_0} dw\, \frac{e^{(w-1)t} \cdot (1-w)^k}{w^{x+n+1}} \qquad (6.33)$$

and

$$\Phi^n_k(x) = \frac{-1}{2\pi i} \oint_{\Gamma_1} dz\, \frac{e^{-tz} \cdot z^{x+n}}{(1-z)^{k+1}}$$

. Consequence: (6.34)

$$K_t(n_1, x_1; n_2, x_2) = -\frac{1}{2\pi i} \oint_{\Gamma_0} dw \left(\frac{w}{1-w}\right)^{n_2-n_1} \cdot \frac{1}{w^{x_1-x_2+1}} \cdot \mathbb{1}_{[n_1<n_2]}$$

$$+ \frac{1}{(2\pi i)^2} \oint_{\Gamma_0} dw \oint_{\Gamma_1} dz\, \frac{e^{tw}}{e^{tz}} \cdot \frac{(1-w)^{n_1}}{w^{x_1+n_1+1}} \cdot \frac{z^{x_2+n_2}}{(1-z)^{n_2}} \cdot \frac{1}{w-z}$$

. b) Flat I.C.: let $y_i = -2i,\ i \geq 1$. Then,

$$\Psi^n_k(x) = \frac{1}{2\pi i} \oint_{\Gamma_0} dw\, \frac{(w(1-w))^k \cdot e^{tw}}{w^{x+2n+1}}, \qquad (6.35)$$

$$\Phi^n_k(x) = \frac{-1}{2\pi i} \oint_{\Gamma_1} dz\, \frac{(2z-1) \cdot z^{x+2n} \cdot e^{-tz}}{(z(1-z))^{k+1}}$$

. Consequence: The kernel for flat I.C., i.e., $y_i = -2i$, $i \in \mathbb{Z}$, is obtained by looking around particle N, i.e.,
$n_i \to n_i + N$ and, of course $x_i \to x_i - 2N$.

After $N \to \infty$ limit, one gets the flat IC kernel:

(6.36)

$$K_t(n_1 x_1; n_2 x_2) = -\frac{1}{2\pi i} \oint_{\Gamma_0} dw \left(\frac{w}{1-w}\right)^{n_2-n_1} \cdot \frac{1}{w^{x_1-x_2+1}} \cdot \mathbb{1}_{[n_1<n_2]}$$

$$+ \frac{-1}{2\pi i} \oint_{\Gamma_1} dz\, e^{t(1-2z)} \cdot \frac{z^{n_1+n_2+x_2}}{(1-z)^{n_1+n_2+x_1+1}}.$$

Edge scaling and asymptotics.

- For step I.C. one in the appropriate edge scaling, gets the convergence of K_t to the Airy kernel $\Rightarrow$ Airy$_2$ process.

For example:

(6.37) $$\lim_{t\to\infty} \frac{x_{[t/4+u(t/2)^{2/3}]}(t) - \left(-2u(t/2)^{2/3} + u^2(t/2)^{1/3}\right)}{-(t/2)^{1/3}} = \mathcal{A}_2(u).$$

For flat I.C.: let $$X_t^{resc}(u) := \frac{x_{[t/4+ut^{2/3}]}(t) + 2ut^{2/3}}{-t^{1/3}}.$$

(6.38) Then, $\lim_{t\to\infty} X_t^{resc}(u) = \mathcal{A}_1(u)$: Airy$_1$ process.

Definition 30: The Airy$_1$ Process is defined by the m-point joint distributions at $u_1 < u_2 < \dots < u_m$ given by

(6.39) $$\mathbb{P}\left(\bigcap_{k=1}^{m} \{\mathcal{A}_1(u_k) \le s_k\}\right) = \det(\mathbb{1} - \chi_s K_{\mathcal{A}_1} \chi_s)_{L^2(\{u_1,\dots,u_m\}\times\mathbb{R})}$$

where $\chi_s(u_k, x) = \mathbb{1}[x > s_k]$ and

(6.40) $$K_{\mathcal{A}_1}(u_1 s_1; u_2 s_2) = -\frac{1}{\sqrt{4\pi(u_2-u_1)}} \cdot \exp\left(-\frac{(s_2-s_1)^2}{4(u_2-u_1)}\right) \cdot \mathbb{1}_{[u_2>u_1]}$$
$$+ Ai(s_1+s_2+(u_2-u_1)^2) \cdot \exp\left((u_2-u_1)(s_1+s_2) + \frac{2}{3}(u_2-u_1)^3\right).$$

A few properties:
- $\mathcal{A}_1$ is stationary
- $\mathbb{P}(\mathcal{A}_1(0) \le s) = F_1(2s)$
  $$\Rightarrow F_1(s) = \det(\mathbb{1} - B_s)_{L^2(\mathbb{R}_+, dx)},$$
  $$B_s(x,y) = Ai(x+y+s).$$
- $Var(\mathcal{A}_1(u) - \mathcal{A}_1(0)) \approx 2|u|$ for small u.

Compact formula: (compare page 40)
$$K_{\mathcal{A}_1}(u,s;u',s') = -(e^{-(s'-s)H_1})(u,u')\mathbb{1}_{[s'>s]} + (e^{sH_1} B e^{-s'H_1})(u,u'),$$
$$B(s,s') = Ai(s+s'),\quad H_1 = -\Delta \equiv -\frac{d^2}{dx^2}.$$