5) Extended determinantal point processes.

5.1) Karlin-McGregor theorem.

. The original work of Karlin and McGregor was in continuous time but discrete space. Here we present the analogue for Brownian Motions.

. Consider N Brownian Motions starting from $X_i(0)=Y_i$ and arriving at $X_i(t)=x_i$, $1\leq i\leq N$.

. Let $x_1>x_2>\dots>x_N$ and $Y_1>Y_2>\dots>Y_N$. Denote by $\mathbb{P}_{non-int}(A)$ the probability of A and that the Brownian Motions do not intersect.

Theorem 17:

$$(5.1)\qquad \mathbb{P}_{non-int}\left(X_1(t)=x_1,\dots,X_N(t)=x_N \mid X_1(0)=Y_1,\dots,X_N(0)=Y_N\right)=$$

$$=\det\left[\mathbb{P}(X(t)=x_i \mid X(0)=Y_j)\right]_{1\leq i,j\leq N}.$$

where $\mathbb{P}(X(t)=x_i \mid X(0)=Y_j)$ is the transition density for a single Brownian Motion.

Proof: . Ingredients: (a) Continuity of paths

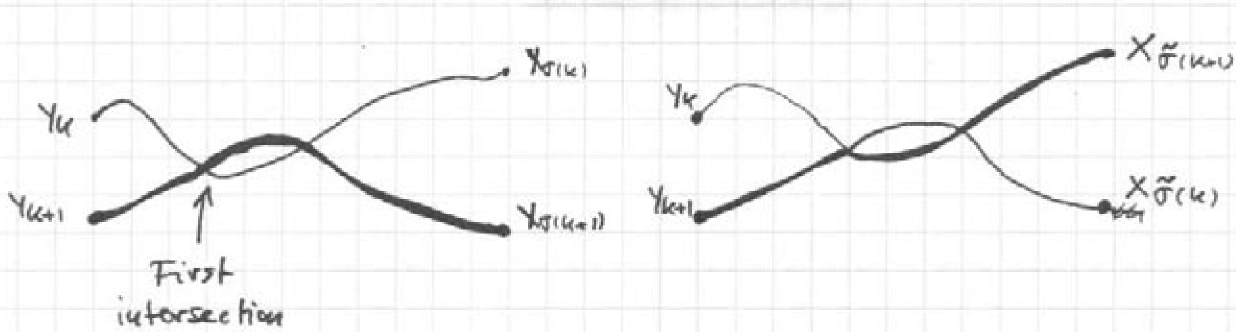
(b) Reflection principle.

. Consider the R.H.S. of (5.1): let $p(x_i,Y_j) \doteq \mathbb{P}(X(t)=x_i \mid X(0)=Y_j)$.

$$\Rightarrow \det\left(p(x_i,Y_j)\right)_{1\leq i,j\leq N}=\sum_{\sigma\in S_N}(-1)^{|\sigma|}\cdot\underbrace{\prod_{k=1}^{N}p(x_k,Y_{\sigma(k)})}_{\text{Transition density from } x_1,\dots,x_N \text{ to } Y_{\sigma(1)},\dots,Y_{\sigma(N)} \text{ with intersections.}}.$$

. We have to see that the contributions of the paths with intersections is exactly cancelled.

. Consider paths with intersections. Let us focus at the first intersection.



Prefactor: $(-1)^{|\sigma|}$ $\qquad$ $(-1)^{|\tilde\sigma|} = (-1)^{|\sigma|}\cdot(-1)$.

The contributions cancels, since have the same weight. $\qquad$ #

. A first generalisation is known on Graphs. It is called the Lindström-Gessel-Viennot theorem (LGV).

5.2) LGV theorem.

. Consider a directed graph (V,E) of vertices V and edges E.

. A path π is a sequence of consecutive vertices joined by directed edges. We denote by $\mathcal{P}(u,v)$ = set of all paths from u to v, $u,v\in V$.

. Two paths π and π' intersects if they have a common vertex.

. Weights: . To every edge $e\in E$ assign a weight $w(e)$.

. The weight of a path π is then $w(\pi) = \prod_{e\in\pi} w(e)$.

. The total weights of paths from u to v is

$$h(u,v) = \sum_{\pi\in\mathcal{P}(u,v)} w(\pi).$$

. Assumption: Given points $(u_1,\dots,u_m)$ and $(v_1,\dots,v_m)$, $\exists$ at most one $\sigma\in S_m$ s.t. one can connect u_i to $v_{\sigma(i)}$, $i=1,\dots,m$, without intersections.

$\oplus$ No loops in the graph.

. If such σ exists, $(u_1,\dots,u_m)$ and $(v_1,\dots,v_m)$ are compatible, and we can relabel them to have $\sigma=\mathbb{1}$.

Theorem (LGV) 18: Denote by $\mathcal{P}_{non\text{-}int}(\vec{u},\vec{v})$ the set of non-intersecting n-tuples of paths from $\vec{u}=(u_1,\dots,u_n)$ to $\vec{v}=(v_1,\dots,v_n)$. Then,

$$w\left(\mathcal{P}_{non\text{-}int}(\vec{u},\vec{v})\right)=\sum_{\substack{(\pi_1,\dots,\pi_n)\\ \in\mathcal{P}_{non\text{-}int}(\vec{u},\vec{v})}} w(\pi_1)\dots w(\pi_n)=\det\left(h(u_i,v_j)\right)_{1\le i,j\le N}. \tag{5.2}$$

Remark: The proof uses the same ingredients of Karlin-McGregor Theorem, namely weight = product of local weights (Markov) and the fact that to exchange their position they first have to intersect.

5.3) Non-intersecting Brownian Motions.

- Consider N Brownian motions with fixed initial and final positions. Focus at m intermediate times: $\underset{\underset{\tau_0}{\shortparallel}}{0}<\tau_1<\tau_2<\dots<\tau_m<\underset{\underset{\tau_{m+1}}{\shortparallel}}{T}$

- Denote $x_k^n := x_k(\tau_n)$, $\begin{array}{l}0\le n\le m+1\\ 1\le k\le N\end{array}$

- Let $\phi_{n,n+1}(x,y)=\mathbb{P}(X.(\tau_{n+1})=y\,|\,X.(\tau_n)=x)$, the transition density from time τ_n to time τ_{n+1}.

Then, by Theorem 17, the measure on $\{x_k^n,\ 1\le k\le N,\ 1\le n\le m\}$ is given by

$$\frac{1}{Z_{N,m}}\cdot\det\left(\phi_{0,1}(x_i^0,x_j^1)\right)_{1\le i,j\le N}\cdot\left(\prod_{n=1}^{m-1}\det\left(\phi_{n,n+1}(x_i^n,x_j^{n+1})\right)_{1\le i,j\le N}\right)\cdot\det\left(\phi_{m,m+1}(x_i^m,x_j^{m+1})\right)_{1\le i,j\le N}, \tag{5.3}$$

where $Z_{N,m}$ is the normalisation constant.

- Assume that we have a measure of the form (5.3) with $\underline{\underline{Z_{N,m}\neq 0}}$.

Then, the space-time correlation functions have determinantal form.

Rem.: x_i^0's and x_j^{m+1}'s are fixed.

Proposition 19: The space-time correlation functions of (5.3) are given by:

(5.4) $$S^{(n)}(x_1,t_1;\dots;x_n,t_n) = \det\left(K(x_i,t_i;x_j,t_j)\right)_{1\le i,j\le n}, \text{ where}$$

$x_i \in \mathbb{R}$, $t_i \in \{\tau_1,\dots,\tau_m\}$ and the Kernel is given by

(5.5) $$K(x,\tau_r;y,\tau_s) = -\phi_{r,s}(x,y) + \sum_{i,j=1}^{N} \phi_{r,m+1}(x,x_i^{m+1})\cdot[A^{-1}]_{ij}\,\phi_{0,s}(x_j^0,y)$$

with $\phi_{r,s}(x,y) = \begin{cases} (\phi_{r,r+1} * \cdots * \phi_{s-1,s})(x,y), & \text{if } r<s, \\ 0 & \text{, if } r\ge s. \end{cases}$

and $A_{ij} = \phi_{0,m+1}(x_i^0, x_j^{m+1})$.

$(\phi_{r,r+1} * \phi_{r+1,r+2})(x,y) \equiv \int dz\, \phi_{r,r+1}(x,z)\,\phi_{r+1,r+2}(z,y)$ Notation $*$

Remark: Applying Cauchy-Binet m times one obtains $Z_{N,m} = \det(A)$, so since by assumption $Z_{N,m}\neq 0$, A is invertible.

Proof of Proposition 19: We prove it for $m=2$. The proof for any m can be made on the same line.

First a small Lemma; Lemma 20: Let $(x_1,\dots,x_N;y_1,\dots,y_N) \equiv (X,Y)$ and consider the matrix M with blocs $\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22}\end{pmatrix}$ and entries.

$$\langle x,y| M(x,y\rangle \equiv \begin{pmatrix} \langle X, M_{11} X\rangle & \langle X, M_{12} Y\rangle \\ \langle Y, M_{21} X\rangle & \langle Y, M_{22} Y\rangle \end{pmatrix}. \quad \text{Then,}$$

(5.6) $$\underline{\det\left(\langle x,y| M |x y\rangle\right) = \det\left(\langle x,y| M\cdot Q\, |x y\rangle\right)} \text{ for any } Q$$ of the form $\begin{pmatrix} \mathbb{1} & 0 \\ A & \mathbb{1}\end{pmatrix}$.

Proof:
$$\det(\langle x,y| M |x,y\rangle) = \det\begin{pmatrix} \langle x, M_{11} x\rangle & \langle x, M_{12} Y\rangle \\ \langle Y, M_{21} x\rangle & \langle Y, M_{22} Y\rangle \end{pmatrix} \cdot \int dz\,dw\, \langle z, \mathbb{1} x\rangle \langle w, \mathbb{1} Y\rangle$$
$$= \int dz\,dw\, \det\begin{pmatrix} \langle x, M_{11} z\rangle & \langle x, M_{12} w\rangle \\ \langle Y, M_{21} z\rangle & \langle Y, M_{22} w\rangle \end{pmatrix} \cdot \det\begin{pmatrix} \langle z, \mathbb{1} x\rangle & 0 \\ \langle z, A\, x\rangle & \langle w, \mathbb{1} Y\rangle \end{pmatrix}$$
$$= \det\left(\langle x,y|\ M\cdot Q\, |x,y\rangle\right) \quad \#$$

. For $m=2$, (5.3) becomes:

$$(5.7)\quad \frac{1}{Z}\cdot \det\left(\phi_{0,1}(x_i^0,x_j^1)\right) \det\left(\phi_{12}(x_i^1,x_j^2)\right)\cdot \det\left(\phi_{2,3}(x_i^2,x_j^3)\right).$$

. Let $A_{ij} = (\phi_{0,1} * \phi_{12} * \phi_{2,3})(x_i^0, x_j^3)$. Then, $\underline{Z = \det A} \neq 0$.

. Notations:

. $\Psi_j^2(x) \equiv \phi_{2,3}(x, x_j^3)$

. $\Psi_j^1(x) := (\phi_{12} * \Psi_j^2)(x)$

Define : . $\Phi_j^1(x) = \sum_{k=1}^{N} A_{jk}^{-1}\cdot \phi_{0,1}(x_k^0, x)$ and set $\Phi_j^2(x) := (\Phi_j^1 * \phi_{12})(x)$.

Then, (5.7) becomes:

$$(5.7) = \frac{1}{\cancel{\det A}}\,\cancel{\det A}\cdot \det\left(\Phi_i^1(x_j^1)\right)\cdot \det\left(\Psi_i^2(x_j^2)\right)\cdot \det\left(\phi_{12}(x_i^1,x_j^2)\right).$$

$$(5.8)\qquad = \det\begin{pmatrix} 0 & -\phi_{1,2}(x_i^1,x_j^2) \\ \sum_{k=1}^{N} \Psi_k^2(x_i^2)\,\Phi_k^1(x_j^1) & 0 \end{pmatrix}$$

In operator form: $\det\left(\langle x^1, x^2 | M | x^1, x^2\rangle\right)$ with

$$M = \begin{pmatrix} 0 & -\phi_{12} \\ \sum_{k=1}^{N} |\Psi_k^2\rangle\langle \Phi_k^1| & 0 \end{pmatrix}$$

. Consider $\tilde{M} = \begin{pmatrix} 1 & \phi_{12} \\ 0 & 1 \end{pmatrix}\cdot M \cdot \begin{pmatrix} 1 & \phi_{12} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^{N} |\phi_{12} * \Psi_k^2\rangle\langle \Phi_k^1| & -\phi_{12} + \sum_{k=1}^{N} |\phi_{12} * \Psi_k^2\rangle\langle \Phi_k^1 * \phi_{12}| \\ \sum_{k=1}^{N} |\Psi_k^2\rangle\langle \Phi_k^1| & \sum_{k=1}^{N} |\Psi_k^2\rangle\langle \Phi_k^1 * \phi_{12}| \end{pmatrix}$

By Lemma 20,

$$(5.8) = \det\left(\langle x^1 x^2 | \tilde{M} | x^1 x^2\rangle\right).$$

So, we have obtained:

$$P(x_1^1, \dots, x_N^1 ; x_1^2, \dots, x_N^2) = \det\begin{pmatrix} K_{11}(x_i^1, x_j^1) & K_{12}(x_i^1, x_j^2) \\ K_{21}(x_i^2, x_j^1) & K_{22}(x_i^2, x_j^2) \end{pmatrix}$$

. Replacing the definitions of Ψ and Φ we get:

$$(5.9)\quad \begin{cases} K_{11}(x,y) = \displaystyle\sum_{i,j=1}^{N} \phi_{1,3}(x,x_i^3)\, A^{-1}_{ij}\, \phi_{0,1}(x_j^0,y), \\ K_{12}(x,y) = -\phi_{1,2}(x,y) + \displaystyle\sum_{i,j=1}^{N} \phi_{1,3}(x,x_i^3)\, A^{-1}_{ij}\, \phi_{0,2}(x_j^0,y), \\ K_{21}(x,y) = \displaystyle\sum_{i,j=1}^{N} \phi_{2,3}(x,x_i^3)\, A^{-1}_{ij}\cdot \phi_{0,1}(x_j^0,y), \\ K_{22}(x,y) = \displaystyle\sum_{i,j=1}^{N} \phi_{2,3}(x,x_i^3)\, A^{-1}_{ij}\cdot \phi_{0,2}(x_j^0,y). \end{cases}$$

. What remains to do is to integrate over some of the variable and see that the form for the correlations is maintained.

. <u>An orthogonal relation:</u> $\boxed{\Phi_k^n * \Psi_\ell^n = \delta_{k\ell} \quad, 1 \le k,\ell \le N \quad, n=1,2.}$

Let us verify for $n=1$: $\quad \Phi_k^1 * \Psi_\ell^1 = \displaystyle\sum_{j=1}^{N} A^{-1}_{kj} \cdot \underbrace{\phi_{0,1}(x_j^0,\cdot) * \phi_{1,2} * \phi_{2,3}(\cdot,x_\ell^3)}_{= \phi_{0,3}(x_j^0,x_\ell^3) = A_{j\ell}}$

$$= \sum_{j=1}^{N} A^{-1}_{kj} A_{j\ell} = \delta_{k\ell}. \#$$

. The correlation function for n_1 points at time τ_1 and n_2 points at time τ_2 is then given by:

$$\rho^{(n_1,n_2)}(x_1^1,\dots,x_{n_1}^1;\ x_1^2,\dots,x_{n_2}^2) = \frac{N}{(N-n_1)!}\cdot\frac{N}{(N-n_2)!} \int dx^1_{n_1+1}\cdots dx^1_N\, dx^2_{n_2+1}\cdots dx^2_N \cdot \det\begin{pmatrix} K_{11}(x_i^1,x_j^1) & K_{12}(x_i^1,x_j^2) \\ K_{21}(x_i^2,x_j^1) & K_{22}(x_i^2,x_j^2) \end{pmatrix}$$

. Using the orthogonal relation it is very easy to verify that:

$$\left.\begin{array}{lll} \cdot\ K_{22} * K_{21} = K_{21} & ; & K_{22} * K_{22} = K_{22} \\ \cdot\ K_{12} * K_{21} = 0 & ; & K_{12} * K_{22} = 0 \\ \cdot\ K_{11} * K_{11} = K_{11} & ; & K_{11} * K_{12} = 0 \\ \cdot\ K_{21} * K_{12} = 0 & ; & K_{21} * K_{11} = K_{21}. \end{array}\right\}$$

The analogous of (2.9), see page 12.

and: $\int dx\, K_{ii}(x,x) = N \quad$: Analogous of (2.8).

Remark:

. The generalisation to any m is made using the following matrix multiplications: (ex. for $m=4$)

$$M = \begin{pmatrix} 0 & -\phi_{12} & 0 & 0 \\ 0 & 0 & -\phi_{23} & 0 \\ 0 & 0 & 0 & -\phi_{34} \\ \sum_k |\Psi_k^4\rangle\langle\Phi_k^1| & 0 & 0 & 0 \end{pmatrix}$$

$$\tilde{M} = \begin{pmatrix} \mathbb{1} & 0 & 0 & \phi_{14} \\ 0 & \mathbb{1} & 0 & \phi_{24} \\ 0 & 0 & \mathbb{1} & \phi_{34} \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot M \cdot \begin{pmatrix} \mathbb{1} & \phi_{12} & \phi_{13} & \phi_{14} \\ 0 & \mathbb{1} & \phi_{23} & \phi_{24} \\ 0 & 0 & \mathbb{1} & \phi_{34} \\ 0 & 0 & 0 & \mathbb{1} \end{pmatrix}.$$

. The final step is to use these relations and develop the determinant along the column with the variable to integrate out, exactly like in the proof of Proposition 6 (see page 13) and one gets the result. #

5.4) Application to Brownian Bridges.

. From Brownian Motions to Brownian Bridges one does the usual limit procedure: $x_i^0 = x_i^{m+1} = -\varepsilon \cdot i$ and send $\varepsilon \to 0$.

. First term: $\det\left(\phi_{0,1}(x_i^0, x_j^1)\right)$:

$$\phi_{0,1}(-\varepsilon k, x) = \text{const} \times e^{\frac{-x^2}{2\tau_1}} \cdot \underbrace{e^{-\frac{\varepsilon^2 k^2}{2\tau_1}}}_{\xrightarrow[\varepsilon\to 0]{} 1} \cdot \underbrace{e^{-\frac{\varepsilon k x}{\tau_1}}}_{= 1 - \frac{k\varepsilon x}{\tau_1} + \dots + \frac{(-k\varepsilon x)^{N-1}}{(N-1)!\,\tau_1^{N-1}} + \sigma(\varepsilon^N)}$$

Linear combinations

$$\Rightarrow \det\left(\phi_{0,1}(x_i^0, x_j^1)\right) = \text{const} \cdot \det\begin{pmatrix} 1 + \sigma(\varepsilon^N) \\ \varepsilon x_j^1 + \sigma(\varepsilon^N) \\ \vdots \\ (\varepsilon x_j^1)^{N-1} + \sigma(\varepsilon^N) \end{pmatrix}$$

$$= \varepsilon^{\frac{N(N-1)}{2}} \cdot \text{cte} \cdot \underbrace{\det\begin{pmatrix} 1 + \sigma(\varepsilon^N) \\ x_j^1 + \sigma(\varepsilon^{N-1}) \\ \vdots \\ (x_j^1)^{N-1} + \sigma(\varepsilon) \end{pmatrix}}_{\xrightarrow[\varepsilon\to 0]{} \Delta_N(x_1^1, \dots, x_N^1)}.$$

. The last term is analogous and the factors $\varepsilon^{\frac{N(N-1)}{2}}$ enters in the normalization.

. Also, we can freely choose two sets of polynomials $\{\Phi_{i-1}^1(x), i=1,\dots,N\}$ and $\{\Psi_{i-1}^m(x), i=1,\dots,N\}$ with $\Phi_{i-1}^1(x)$, $\Psi_{i-1}^m(x)$ of degree $i-1$, instead of the monomials in the Vandermonde determinant.

Notations:

(5.10)
$$\begin{cases} \cdot\ \phi_{n,m}(x,y) = \dfrac{\exp\left(-\frac{(y-x)^2}{2(\tau_m-\tau_n)}\right)}{\sqrt{2\pi(\tau_m-\tau_n)}} \cdot \mathbb{1}_{[\tau_m>\tau_n]} \\ \cdot\ \Phi_i^r(x) = \dfrac{\sqrt[4]{2\pi T}}{\sqrt{i!\,2^i}} \cdot \left(\dfrac{T-\tau_r}{\tau_r}\right)^{i/2} \cdot H_i\left(\dfrac{x}{\sqrt{2\tau_r(T-\tau_r)/T}}\right) \cdot \dfrac{e^{-\frac{x^2}{2\tau_r}}}{\sqrt{2\pi\tau_r}} \\ \cdot\ \Psi_j^s(x) = \dfrac{\sqrt[4]{2\pi T}}{\sqrt{j!\,2^j}} \cdot \left(\dfrac{\tau_s}{T-\tau_s}\right)^{j/2} \cdot H_j\left(\dfrac{x}{\sqrt{2\tau_s(T-\tau_s)/T}}\right) \cdot \dfrac{e^{-\frac{x^2}{2(T-\tau_s)}}}{\sqrt{2\pi(T-\tau_s)}} \end{cases}$$

with $H_i(x)$ the standard Hermite polynomial of degree i.

. These functions satisfy:

(5.11)
$$\begin{cases} \displaystyle\int_{\mathbb{R}} dx\ \Phi_i^r(x)\phi_{r,s}(x,y) = \Phi_i^s(y), \text{ for } s>r. \\ \displaystyle\int_{\mathbb{R}} dx\ \phi_{r,s}(x,y)\Psi_j^s(y) = \Psi_j^r(x), \text{ for } r<s. \end{cases}$$

. Moreover,
$$\begin{cases} \mathrm{vect}\{\Phi_i^1(x), i=0,..,N-1\} = \mathrm{vect}\{e^{-\frac{x^2}{2\tau_1}}\cdot x^i, i=0,..,N-1\} \\ \mathrm{vect}\{\Psi_j^m(x), i=0,..,N-1\} = \mathrm{vect}\{e^{-\frac{x^2}{2(T-\tau_m)}}\cdot x^i, i=0,..,N-1\}. \end{cases}$$

. Therefore, as $\varepsilon \to 0$, the measure on $\{x_k^n, 1\le k\le N, n=1,..,m\}$ for Brownian Bridges from 0 at time 0 to 0 at time T, is given by:

(5.12)
$$\frac{1}{Z}\cdot \det\left(\Phi_i^1(x_j^1)\right)\left(\prod_{k=1}^{m-1}\det\left(\phi_{k,k+1}(x_i^k,x_j^{k+1})\right)\right)\cdot\det\left(\Psi_i^m(x_j^m)\right).$$

This measure is exactly of the form (5.3). Therefore, the space-time correlations are <u>determinantal</u> <u>(extended determinantal point process)</u>, with Kernel:

(5.13)
$$K_N(x,\tau_r;y,\tau_s) = -\phi_{r,s}(x,y) + \sum_{k=0}^{N-1}\Psi_k^r(x)\,\Phi_k^s(y).$$

<u>Remark:</u> This for $\tau_r=\tau_s$ is also the GUE kernel!

5.5) Edge scaling and Airy$_2$ process.

- We consider the following edge scaling:

  - $T = 4N \implies$ at $t \approx 2N$, the top B.B is around $2N$.

(5.14) $$\begin{cases} \tau_i = 2N + 2u_i N^{2/3} & , u_1, u_2, \dots, u_m \text{ fixed.} \\ x_i = \underbrace{2N - u_i^2 \cdot N^{1/3}}_{\text{position from the limit shape}} + \underbrace{s_i N^{1/3}}_{\text{Fluctuations: } \frac{1}{3} \text{ exponent as in R.M.!}} & \left(\equiv H_N(\tau_i) + s_i N^{1/3} + O(1)\right). \end{cases}$$

$\boxed{\text{Limit shape}: \; x_1(\tau) \cong \sqrt{\tau(4N-\tau)}} =: H_N(\tau)$

- Rescaled Kernel: $K_N^{edge}(u_1 s_1; u_2 s_2) \doteq N^{1/3} \cdot K_N(x_1, \tau_1; x_2, \tau_2)$.

- One uses the asymptotics for Ψ and Φ, setting $k = N - \lambda N^{1/3}$, $\lambda \in \frac{\mathbb{N}}{N^{1/3}}$.

(5.15) $$\begin{cases} \Psi^{\tau_i}_{N-\lambda N^{1/3}}(x_i) \cong N^{-1/3} \cdot Ai(s_i + \lambda) \cdot e^{\lambda u_i} \cdot \underbrace{f(s_i, u_i)}_{= \exp\left(-\frac{u_i^3}{3} + s_i u_i\right)} \\ \Phi^{\tau_i}_{N-\lambda N^{1/3}}(x_i) \cong N^{-1/3} Ai(s_i + \lambda) \cdot e^{-\lambda u_i} \cdot f(s_i u_i)^{-1} \end{cases}$$

For $\tau_j > \tau_i$: $$\phi_{\tau_i, \tau_j}(x, y) \cong N^{-1/3} \cdot \frac{f(s_j u_j)}{f(s_i u_i)} \cdot \frac{1}{\sqrt{4\pi(u_j - u_i)}} \cdot \exp\left(-\frac{(s_j - s_i)^2}{4(u_j - u_i)} + \frac{1}{12}(u_j - u_i)^3 - \frac{(u_j - u_i)(s_i + s_j)}{2}\right)$$

$$\sum_{k=0}^{N-1} \Longrightarrow \frac{1}{N^{1/3}} \sum_{\lambda \in \frac{\mathbb{N}}{N^{1/3}}} \to \int_0^\infty d\lambda.$$

- Final result: **Theorem 21:** Let N Brownian Bridges from $t=0$ to $t = 4N$ be conditioned on non-intersecting (excepts at $t=0$, $t=4N$).

  Let $$\begin{cases} \tau = 2N + 2u N^{2/3}, \\ x = 2N - u^2 N^{1/3} + s N^{1/3} \end{cases}$$

  Then, in the $N \to \infty$ limit, the extended determinantal point process has kernel given by:

$$(5.16)\quad K_{\mathcal{A}_2}(u,s;u',s') = \begin{cases} \int_0^\infty d\lambda\, Ai(s+\lambda) Ai(s'+\lambda)\cdot e^{(u'-u)\lambda}, & \text{if } u'\leq u, \\ -\int_{-\infty}^0 d\lambda\, Ai(s+\lambda) Ai(s'+\lambda)\cdot e^{(u'-u)\lambda}, & \text{if } u'>u. \end{cases}$$

. This is called the extended Airy kernel.

Remark: . $\int_{\mathbb{R}} d\lambda\, Ai(s+\lambda) Ai(s'+\lambda)\, e^{\lambda u} \overset{u>0}{=} \frac{1}{\sqrt{4\pi u}}\cdot \exp\left(-\frac{(s'-s)^2}{4u} + \frac{1}{12}u^3 - u\frac{(s'+s)}{2}\right)$.

Corollary 22: Let $x_1(t)$ be trajectory of the top Brownian Bridge. Define the rescaled process:

$$(5.17)\quad Y_N(u) := \frac{x_1(2N+2uN^{2/3}) - (2N-u^2N^{1/3})}{N^{1/3}}.$$

Then, $\lim_{N\to\infty} Y_N(u) = \mathcal{A}_2(u)$

in the sense of finite-dimensional distributions, where $\mathcal{A}_2(u)$ is called the Airy$_2$ process.

Definition 23: The Airy$_2$ process is defined via the finite-dimensional distributions given by:

$$(5.18)\quad \mathbb{P}\left(\bigcap_{k=1}^{m} \mathcal{A}_2(u_k)\leq s_k\right) = \det(\mathbb{1}-\chi_s K_{\mathcal{A}_2}\chi_s)_{L^2(\{u_1,\dots,u_m\}\times\mathbb{R})}$$

with $\chi_s(u_k,x) = \mathbb{1}[x>s_k]$ and $K_{\mathcal{A}_2}$ is the extended Airy kernel given above.

. Compact formula:

$$K_{\mathcal{A}_2}(u,s;u',s') = -\left(e^{-(s'-s)H_{Ai}}\right)(u,u')\cdot \mathbb{1}_{[s'>s]} + \left(e^{sH_{Ai}}\, \overline{K}_{\mathcal{A}_2}\cdot e^{-s'H_{Ai}}\right)(u,u'), \text{ with}$$

$\overline{K}_{\mathcal{A}_2}(s,s') = \int_0^\infty d\lambda\, Ai(\lambda+s) Ai(\lambda+s')$, and H_{Ai} is the Airy operator: $(H_{Ai}\psi)(x) = -\frac{d^2}{dx^2}\psi(x) + x\psi(x)$.

Eigenfunctions: $H_{Ai}\, Ai(x-\lambda) = \lambda\, Ai(x-\lambda)$.

5.6) Dyson's Brownian Motion.

. It is a Brownian Motion on space of Matrices.

. let us denote by H_μ, $\mu = 1,\dots,\frac{N(N+1)}{2} =: p$, for GOE, the
$\mu = 1,\dots, N^2 =: p$, for GUE

independent entries of the symmetric/hermitian $N \times N$ matrices.

• let these H_μ perform independent Brownian Motion in a quadratic potential: $dH_\mu = -\gamma \cdot H_\mu dt + \sigma_\mu \underset{\substack{\uparrow \\ \text{standard B.M.} \\ (\mathrm{Var}(B_\mu(t)) = t)}}{dB_\mu}$, with $\sigma_\mu = \begin{cases} 1, & \mu = (i,i), \\ \frac{1}{2}, & \text{otherwise.} \end{cases}$

. Then, $P(H_1,\dots,H_p;t)$ be the probability density of H at time t, it satisfies the Smoluchowski equation:

$$\frac{\partial P}{\partial t} = \sum_{\mu=1}^{p} \left[\frac{1}{2}\sigma_\mu \frac{\partial^2}{\partial H_\mu^2} P + \gamma \cdot \frac{\partial}{\partial H_\mu}(H_\mu P) \right]. \tag{5.19}$$

. Its stationary solution is the GOE/GUE measure:

$$P^{stat}(H) = \frac{1}{Z} \cdot e^{-\gamma \mathrm{Tr}(H^2)}. \tag{5.20}$$

. The <u>stationary process</u> is called <u>Dyson's Brownian Motion</u>, and has transition probability density

$$P(H(t) \mid H(0)) = \frac{cte}{(1-q_t^2)^{p/2}} \cdot \exp\left(- \frac{\gamma \cdot \mathrm{Tr}\,(H(t) - q_t \cdot H(0))^2}{1-q_t^2} \right) \tag{5.21}$$

where $q_t = \exp(-\gamma \cdot t)$.

• The evolution of the eigenvalues satisfy then (one need some computations):
$$d\lambda_i = \Big[-\gamma \lambda_i + \frac{\beta}{2} \sum_{j \neq i} \frac{1}{\lambda_j - \lambda_i} \Big] dt + db_i, \quad i = 1,\dots,N$$
where db_i are standard independent B.M.

5.6.1) $\beta=2$ Dyson's Brownian Motion.

. Consider GUE ($\beta=2$) case and set $\gamma = \frac{1}{2N}$.
Let $\mathbb{P}(H(t=0)) = P^{stat}(H(t=0))$.

. Consider m times $\underset{t_0}{\overset{\shortparallel}{0}} < t_1 < t_2 < \dots < t_m$.

Then, the multi-time measure is, with $H_i \doteq H(t_i)$, given by

$$(5.22) \qquad \frac{1}{Z} \cdot e^{-\frac{Tr(H_0^2)}{2N}} \cdot \prod_{j=0}^{m-1} e^{-\frac{Tr(H_{j+1}-q_j H_0)^2}{2N(1-q_j^2)}} dH_0 \dots dH_m$$

where $q_j \doteq \exp\left(-\frac{t_{j+1}-t_j}{2N}\right)$.

. Denote by $\alpha_k \doteq \frac{1}{2N(1-q_k^2)}$, $\gamma_k = \frac{1-q_{k-1}^2 q_k^2}{2N(1-q_{k-1}^2)(1-q_k^2)}$, $\beta_k = \frac{q_k}{N(1-q_k^2)}$.

Then, one verifies that (5.22) can be rewritten as

$$(5.23) \qquad \frac{1}{Z} \cdot e^{-\alpha_0 \cdot Tr(H_0^2)} \left(\prod_{k=1}^{m-1} e^{-\gamma_k \cdot Tr(H_k^2)}\right) e^{-\alpha_{m-1} Tr(H_m^2)} \cdot$$

$$\cdot \prod_{k=0}^{m-1} e^{\beta_k \cdot Tr(H_k \cdot H_{k+1})} dH_0 \dots dH_m .$$

. What about the measure on their eigenvalues?
For GUE we have seen that $dH_k = \Delta_N^2(\lambda^k) d\lambda^k \cdot \underset{\hookrightarrow \text{ Haar measure on } U(N).}{dU_k}$
with $\lambda^k = (\lambda_1^k, \dots, \lambda_N^k)$, and $H_k = U_k \cdot \Lambda_k^{-1} U_k^{-1}$, with

$$\Lambda_k = \begin{pmatrix} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \lambda_N^k \end{pmatrix}.$$

. Thus, $\prod_{k=0}^{m-1} e^{\beta_k Tr(H_k H_{k+1})} dH_0 \dots dH_m = \prod_{k=0}^{m-1} e^{\beta_k Tr(U_k \Lambda_k U_k^{-1} U_{k+1} \Lambda_{k+1} U_{k+1}^{-1})} \cdot$

$$\cdot \Delta_N^2(\lambda^1) \dots \Delta_N^2(\lambda^m) d\lambda^0 \dots d\lambda^m \, dU_0 \dots dU_m$$

. Defining $U_k^{-1} U_{k+1} \doteq V_k$, we get the terms of type

$$(5.24) \qquad \int_{U(N)^m} \prod_{k=0}^{m-1} e^{\beta_k Tr(\Lambda_k V_k \Lambda_{k+1} V_k^{-1})} dV_k$$

to evaluate.

Lemma 24 [Harish-Chandra / Itzykson-Zuber formula]:

$$(5.25)\qquad \int_{U(N)} dU\, e^{\gamma \operatorname{Tr}(\Lambda_1 U \Lambda_2 U^{-1})} = \frac{1}{\gamma^{N(N-1)/2}} \cdot \frac{\left(\prod_{p=1}^{N-1} p!\right) \cdot \det\left(e^{\gamma \lambda_i^1 \lambda_j^2}\right)_{1\le i,j\le N}}{\Delta_N(\lambda_1)\,\Delta_N(\lambda_2).}$$

. Using this, the measure on the eigenvalues $\{\lambda_i^k, i=1,\dots,N, k=0,\dots,m\}$ is given by

$$(5.26)\qquad \frac{1}{Z}\cdot\left(\prod_{i=1}^{N} e^{-\alpha_0(\lambda_i^0)^2}\right)\cdot\left(\prod_{k=1}^{m-1}\prod_{i=1}^{N} e^{-\gamma_k(\lambda_i^k)^2}\right)\cdot\prod_{i=1}^{N} e^{-\alpha_{m-1}(\lambda_i^m)^2}\, d\lambda^0\dots d\lambda^m$$
$$\cdot\,\Delta_N(\lambda^0)\left(\prod_{k=0}^{m-1}\det\left(e^{\beta_k \lambda_i^k \lambda_j^{k+1}}\right)_{1\le i,j\le N}\right)\cdot\Delta_N(\lambda^m).$$

. This measure has the form (5.3), so one can apply Proposition 19, and finds:

- The $\beta=2$ Dyson's Brownian Motion's eigenvalues form an extended determinantal point process with kernel (Extended Hermite):

$$(5.27)\qquad \boxed{K_N(x_1,t_1;x_2,t_2) = \begin{cases} \sum_{k=0}^{N-1} e^{-\frac{k(t_2-t_1)}{2N}}\cdot P_k(x)\,P_k(y)\, e^{-\frac{x^2+y^2}{4N}}, & \text{for } t_1\le t_2,\\ -\sum_{k=N}^{\infty} e^{-\frac{k(t_2-t_1)}{2N}}\cdot P_k(x)\,P_k(y)\cdot e^{-\frac{x^2+y^2}{4N}}, & \text{for } t_1>t_2, \end{cases} \\ \text{where } P_k(x) = \frac{1}{\sqrt[4]{2\pi N}\sqrt{2^k k!}}\cdot H_k\left(\frac{x}{\sqrt{2N}}\right)}$$

. Edge scaling: $\lambda_i(t)$ be the i^{th} largest e.v.

$$\lambda_{i,N}^{edge}(s) := \frac{\lambda_i(2\cdot s\cdot N^{2/3}) - 2N}{N^{1/3}}$$

. The rescaled kernel converges to the Airy_2 kernel as $N\to\infty$.

$$\Rightarrow \lim_{N\to\infty} \lambda_{N,N}^{edge}(s) = \mathcal{A}_2(s) : \text{ the Airy}_2 \text{ process.}$$