4) Edge scaling and Tracy-Widom distributions.

4.1) TW distribution.

. Reminder: . Consider the GUE ensembles of $N \times N$ matrices.

. At the edge:

$$\lambda = 2N + \xi N^{1/3} \tag{4.1}$$

the rescaled kernel converges to the Airy kernel:

$$\lim_{N\to\infty} K_N^{resc}(\xi_1,\xi_2) = K_{Ai_2}(\xi_1,\xi_2) \tag{4.2}$$

$$= \int_0^\infty d\lambda \, Ai(\xi_1+\lambda) Ai(\xi_2+\lambda) .$$

. Eigenvalues' rescaling:

$$. \lambda_i^{resc,N} \doteq \frac{\lambda_i - 2N}{N^{1/3}} \tag{4.3}$$

Gap probability.

$$\Rightarrow \mathbb{P}\left(\lambda_{max}^{resc,N} \le s\right) = \det\left(\mathbb{1} - P_s \cdot K_N^{resc} \cdot P_s\right)_{L^2(\mathbb{R},dx)} . \tag{4.4}$$

where $P_s(x) = \begin{cases} 1, & x > s, \\ 0, & x \le s. \end{cases}$

. The convergence of the kernel K_N^{resc} to the Airy kernel (plus some bounds for large values of ξ s.t. the $\lim_{N\to\infty}$ can be taken inside the Fredholm series expansion by using dominated convergence) implies that:

$$\underline{F_2(s)} := \lim_{N\to\infty} \mathbb{P}\left(\lambda_{max}^{resc,N} \le s\right) = \underline{\det\left(\mathbb{1} - P_s \cdot K_{Ai_2} \cdot P_s\right)_{L^2(\mathbb{R},dx)}} . \tag{4.5}$$

. This is called the (GUE) Tracy-Widom distribution.

- A rewriting on a fixed Hilbert space.

. Let $B_s(u,v) := Ai(u+v+s)$, then

$$(4.6) \qquad F_2(s) = \det(\mathbb{1} - B_s^2)_{L^2(\mathbb{R}_+, dx)}.$$

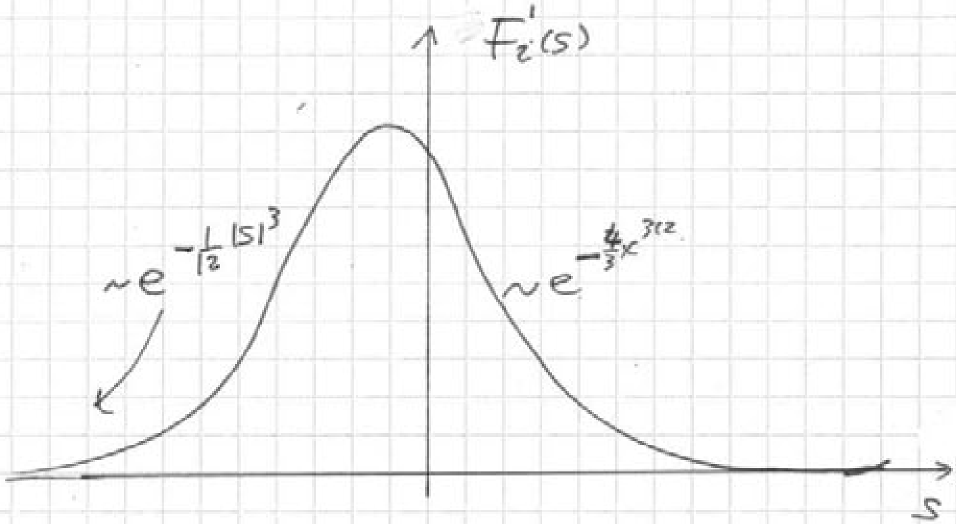
Indeed, $B_s^2(u,v) = \int_0^\infty d\lambda\, B_s(u,\lambda) B_s(\lambda,v) = \int_0^\infty d\lambda\, Ai(u+\lambda+s) Ai(v+\lambda+s)$

$$\Rightarrow \det(\mathbb{1} - B_s^2)_{L^2(\mathbb{R}_+,dx)} \underset{\substack{u+s\to u\\ v+s\to v}}{=} \det(\mathbb{1} - K_{A_2})_{L^2((s,\infty),dx)} = F_2(s).$$

Remarks: ① B_s is not a positive operator.

② B_s is HS on $L^2(\mathbb{R}_+, dx)$, $\forall s > -\infty$

③ B_s^2 is trace-class on $L^2(\mathbb{R}_+,dx)$ $\forall s > -\infty$.



- GOE Tracy-Widom distribution: a det. formula (Ferrari-Spohn).

. With the same rescaling as (4.3) but for GOE eigenvalues, the limit distribution function is called (GOE) Tracy-Widom distribution, denoted by $F_1(s)$.

$$(4.7). \quad F_1(s) = \det(\mathbb{1} - B_s)_{L^2(\mathbb{R}_+, dx)}.$$

4.2) F_2 and Painlevé-II equations.

Theorem 15: (Tracy-Widom)

(4.8) $$F_2(s) = \exp\left(-\int_s^\infty dx\,(x-s)\,q^2(x)\right),$$

where $q(x)$ is the unique solution of the Painlevé-II equation: $q''(x) = x\,q(x) + 2q^3(x)$ satisfying the asymptotic condition: $q(s) \cong \mathrm{Ai}(s)$ for $s \to +\infty$.

Remarks: ① Also $F_1(s)$ can be written in terms of q.

(4.9) $$F_1(s) = \exp\left(-\frac{1}{2}\int_s^\infty dx\,q(x)\right) \cdot F_2(s)^{1/2}.$$

② The importance of the relation to the P.-II equation was also that it was providing a way to numerically obtain explicit quantities, like moments, for F_1, F_2.
. A recent work of Bornemann(*) shows however that there exists efficient numerical evaluation of the Fredholm determinant for Analytic Kernels, which is the case for F_2 and F_1 (see (4.5) and (4.7)).
. Thus the importance of the P-II connection is relative and will be presented only if time permits.

(*) arXiv: 0804.2543 "On the Numerical Evaluation of Fredholm Determinants" by F. Bornemann.

Proof of Theorem 15:

. We use the expression: $F_2(s) = \det(\mathbb{1} - B_s^2)$ and the space will be always $L^2(\mathbb{R}_+, dx)$. $\boxed{K_s \doteq B_s^2.}$

. Let $A_s(x) \doteq Ai(x+s)$. Then,

$$\frac{\partial}{\partial s} K_s(x,y) = \frac{\partial}{\partial s}\left(\int_0^\infty d\lambda\, Ai(x+\lambda+s)\, Ai(y+\lambda+s)\right)$$

$$= \int_0^\infty d\lambda\, Ai'(x+\lambda+s) Ai(y+\lambda+s) + \int_0^\infty d\lambda\, Ai(x+\lambda+s) Ai'(y+\lambda+s)$$

$$= Ai(x+\lambda+s) Ai(y+\lambda+s)\Big|_0^\infty - \cancel{(\ )} + \cancel{(\ )}$$

$$= - A_s(x) A_s(y).$$

. In bra-ket notations:

$$\frac{\partial}{\partial s} K_s = -|A_s\rangle\langle A_s| \qquad (4.10)$$

. let $u(s) := \frac{\partial}{\partial s} \ln\left(\det(\mathbb{1} - K_s)\right)$ (4.11)

Using the identity: $\det(\mathbb{1}+A) = \exp(\mathrm{Tr}(\ln(\mathbb{1}+A)))$, we get

$$\underline{u(s)} = \frac{\partial}{\partial s} \mathrm{Tr}\left(\ln(\mathbb{1} - K_s)\right)$$

$$= -\mathrm{Tr}\left((\mathbb{1}-K_s)^{-1} \frac{\partial}{\partial s} K_s\right) \quad \text{: used cyclicity of trace}$$

$$= \mathrm{Tr}\left((\mathbb{1}-K_s)^{-1} |A_s\rangle\langle A_s|\right) \quad \text{: rank-one operator}$$

$$= \underline{\langle A_s | (\mathbb{1}-K_s)^{-1} A_s\rangle} \qquad (4.12)$$

. Another expression for $u(s)$ is:

$$\underline{u(s)} = \frac{\partial}{\partial s} \sum_{n\geq 1} \frac{-1}{n} \mathrm{Tr}(K_s^n) = -\sum_{n\geq 1} \frac{1}{\cancel{n}} \cdot \cancel{n} \cdot \mathrm{Tr}\left(K_s^{n-1} \frac{\partial}{\partial s} K_s\right)$$

$$= \sum_{n\geq 1} \mathrm{Tr}(K_s^{n-1} |A_s\rangle\langle A_s|) = \sum_{n\geq 1} \langle A_s | K_s^{n-1} A_s\rangle$$

$$|A_s\rangle = B_s |\delta_0\rangle \;\; \overset{\downarrow}{=} \sum_{n\geq 1} \langle \delta_0 | K_s^n \delta_0\rangle = \underline{\langle \delta_0 | K_s (\mathbb{1}-K_s)^{-1} \delta_0\rangle}. \qquad (4.13).$$

Define:
$$\begin{cases} q(s) = \langle \delta_0 | (\mathbb{1}-K_s)^{-1} A_s \rangle \\ p(s) = \langle \delta_0 | (\mathbb{1}-K_s)^{-1} A'_s \rangle \\ v(s) = \langle A_s | (\mathbb{1}-K_s)^{-1} A'_s \rangle \end{cases}$$

Lemma 16:

(a) $\dfrac{\partial u(s)}{\partial s} = -q^2(s)$

(b) $q^2(s) = u^2(s) - 2v(s)$

(c) $\dfrac{\partial q(s)}{\partial s} = p(s) - q(s)u(s)$

(d) $\dfrac{\partial p(s)}{\partial s} = s\,q(s) - 2q(s)v(s) + p(s)u(s).$

. By Lemma 16, we have:

$$\begin{aligned} \frac{\partial^2 q(s)}{\partial s^2} &= p'(s) - q'(s)u(s) - q(s)u'(s) \\ &= s\cdot q(s) - 2q(s)v(s) + \cancel{p(s)u(s)} - \cancel{p(s)u(s)} + q(s)u^2(s) + q^3(s) \\ &= s\,q(s) + q(s)\big(\underbrace{q^2(s) + u^2(s) - 2v(s)}_{=2q^2(s)}\big). \end{aligned}$$

. Moreover, as $s \to \infty$, $(\mathbb{1}-K_s)^{-1} \to \mathbb{1} \Rightarrow q(s) \to \mathrm{Ai}(s)$.

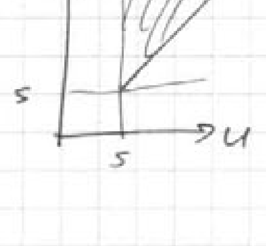
. To get the final formula we need to integrate twice:

$$\frac{\partial u(s)}{\partial s} = \frac{\partial^2}{\partial s^2} \ln(F_2(s)) = -q^2(s)$$

$$\Rightarrow -\int_s^\infty dt\, q^2(t) = \int_s^\infty dt \frac{d^2}{dt^2}(\ln F_2(s)) = \frac{d}{dt} \ln F_2(t)\Big|_s^\infty = -\frac{d}{ds}\ln F_2(s)$$

$$\Rightarrow -\int_s^\infty du \int_u^\infty dt\, q^2(t) = -\int_s^\infty du \frac{d}{du} \ln F_2(u) = -\ln F_2(u)\Big|_s^\infty = \underline{\ln F_2(s)}.$$

$$\hookrightarrow -\int_s^\infty dt\, q^2(t) \int_s^t du = \underline{-\int_s^\infty dt\,(t-s)\,q^2(t).} \qquad \#$$



Proof of Lemma 16: (a) $\frac{\partial u(s)}{\partial s} = \frac{\partial}{\partial s}\langle \delta_0 | (\mathbb{1}-K_s)^{-1}\delta_0\rangle$

$\frac{d}{d\alpha}(\mathbb{1}-K)^{-1} \overset{(*)}{=} (\mathbb{1}-K)^{-1}\frac{dK}{d\alpha}(\mathbb{1}-K)^{-1}$

$$= \langle \delta_0 | (\mathbb{1}-K_s)^{-1}\frac{\partial K_s}{\partial s}(\mathbb{1}-K_s)^{-1}\delta_0\rangle \overset{(4.10)}{=} -(\langle\delta_0|(\mathbb{1}-K_s)^{-1}A_s\rangle)^2 = -q^2(s).$$

(b) From (4.12): $\frac{\partial u(s)}{\partial s} = \frac{\partial}{\partial s}\langle A_s|(\mathbb{1}-K_s)^{-1}A_s\rangle$

$$\overset{(*)}{=} 2\cdot\langle A_s|(\mathbb{1}-K_s)^{-1}A_s'\rangle - \langle A_s|(\mathbb{1}-K_s)^{-1}A_s\rangle\langle A_s|(\mathbb{1}-K_s)^{-1}A_s\rangle$$

$$= 2v(s) - u(s)^2 \overset{(a)}{=} -q^2(s).$$

(c) $\frac{\partial q(s)}{\partial s} \underset{(4.12)}{\overset{(*)}{=}} \langle\delta_0|(\mathbb{1}-K_s)^{-1}A_s'\rangle - \langle\delta_0|(\mathbb{1}-K_s)^{-1}A_s\rangle\langle A_s|(\mathbb{1}-K_s)^{-1}A_s\rangle$

$$= p(s) - q(s)\cdot u(s).$$

(d) We also need: $[L,(\mathbb{1}-K)^{-1}] = (\mathbb{1}-K)^{-1}[L,K](\mathbb{1}-K)^{-1}$ (4.14)

and $[Q,K_s] = |A_s\rangle\langle A_s'| - |A_s'\rangle\langle A_s|$ (4.15)

where Q is the operator multiplication by the position.

$$\Rightarrow \frac{\partial p(s)}{\partial s} \underset{(4.12)}{\overset{(*)}{=}} -\langle\delta_0|(\mathbb{1}-K_s)^{-1}A_s\rangle\langle A_s|(\mathbb{1}-K_s)^{-1}A_s'\rangle + \langle\delta_0|(\mathbb{1}-K_s)^{-1}A_s''\rangle$$

Using: $Ai''(x+s) = (x+s)Ai(x+s)$: $A_s'' = (Q+s)A_s$

$$\Rightarrow \langle\delta_0|(\mathbb{1}-K_s)^{-1}A_s''\rangle = s\cdot\langle\delta_0|(\mathbb{1}-K_s)^{-1}A_s\rangle + \langle\delta_0|(\mathbb{1}-K_s)^{-1}QA_s\rangle$$

$$= s\cdot q(s) + \underbrace{\langle\delta_0|Q(\mathbb{1}-K_s)^{-1}A_s\rangle}_{=0} - \langle\delta_0|[Q,(\mathbb{1}-K_s)^{-1}]A_s\rangle$$

$$\underset{(4.15)}{\overset{(4.14)}{=}} s\cdot q(s) - \langle\delta_0|(\mathbb{1}-K_s)^{-1}A_s\rangle\langle A_s'|(\mathbb{1}-K_s)^{-1}A_s\rangle + \langle\delta_0|(\mathbb{1}-K_s)^{-1}A_s'\rangle\langle A_s|(\mathbb{1}-K_s)^{-1}A_s\rangle$$

$$= s\cdot q(s) - q(s)v(s) + p(s)u(s)$$

$$\Rightarrow \frac{\partial p(s)}{\partial s} = sq(s) - 2q(s)v(s) + p(s)u(s) .$$ #