3) Determinantal point processes.

3.1) Point processes.

Definition 9: A point process η on Λ is a random point measure, where point measures are measures which are locally finite sum of Dirac measures.

Definition 10: A point process is simple if

$$\mathbb{P}\left(\eta(\{x\}) \leq 1, \forall x \in \Lambda\right) = 1,$$

i.e., no double points.

Remark: A simple point process can be identified with the support of the random point measure.

. Let us do two examples:

(1) Poisson point process on $\mathbb{R}^d$ with intensity ρ:

. Take $\Lambda = \mathbb{R}^d$ and $\mathbb{P}$ the probability measure s.t., $\forall B_1, B_2 \subset \Lambda$, bounded and $\forall n_1, n_2 \geq 0$:

$$\mathbb{P}(\# B_1 = n_1) = \frac{(\rho\,|B_1|)^n}{n!} \cdot e^{-\rho|B|}$$

and if $B_1 \cap B_2 = \phi \Rightarrow \mathbb{P}(\# B_1 = n_1, \# B_2 = n_2) = \mathbb{P}(\# B_1 = n_1)\,\mathbb{P}(\# B_2 = n_2)$.

(2) GUE eigenvalues:

. In this case, $\Lambda = \mathbb{R}$ and $\mathbb{P}$ is the probability of the GUE ensemble. Then, the point process η associated with the GUE eigenvalues is

$$\eta(x) = \sum_{i=1}^{N} \delta(x - \lambda_i). \tag{3.1}$$

3.2) Correlation functions.

. For a point process η on Λ, the total mass (points with multiplicity, in the support of η) in a set $A \subset \Lambda$ is given by

(3.2) $$\eta(\mathbb{1}_A), \quad \mathbb{1}_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}.$$

. For a general function, we denote:

(3.3) $$\eta(f) = \int_\Lambda f(x)\,\eta(x)\,dx$$

. <u>Moments:</u> Let $\rho^{(n)}(x_1,\dots,x_n)$ be the n-point correlation functions of the point process η with respect to a measure μ.

<u>Lemma 11:</u> Let $A \subset \Lambda$ a subset. Then,

(3.4) $$\int_{A^n} d\mu(x_1)\dots d\mu(x_n)\,\rho^{(n)}(x_1,\dots,x_n) = \mathbb{E}\left(\frac{\eta(\mathbb{1}_A)!}{(\eta(\mathbb{1}_A)-n)!}\right)$$

<u>Proof.:</u> . For $n=1$: $\eta(\mathbb{1}_A) = \sum_i \mathbb{1}_{[x_i \in A]} \Rightarrow \mathbb{E}(\eta(\mathbb{1}_A)) = \mathbb{E}(\# x_i \in A) = \int_A d\mu(x)\rho^{(1)}(x)$

. For $n=2$: notice that $\eta(\mathbb{1}_A)(\eta(\mathbb{1}_A)-1) = \sum_i \mathbb{1}_{[x_i \in A]} \sum_{j \neq i} \mathbb{1}_{[x_j \in A]}$, because if $i \notin A \Rightarrow \sum_{j\neq i}(\dots)$ is irrelevant, while if $i \in A \Rightarrow \sum_{j \neq i}(\dots) = (\eta(\mathbb{1}_A)-1)$

. For general n:

$$\eta(\mathbb{1}_A)(\eta(\mathbb{1}_A)-1)\cdots(\eta(\mathbb{1}_A)-n+1) =$$

$$= \sum_{i_1} \sum_{i_2 \neq i_1} \cdots \sum_{i_n \neq \{i_1,\dots,i_{n-1}\}} \prod_{k=1}^n \left(\mathbb{1}_{[x_{i_k} \in A]}\right)$$

$$\Rightarrow \mathbb{E}(\quad) = \int_{A^n} d\mu(x_1)\dots d\mu(x_n)\,\rho^{(n)}(x_1,\dots,x_n). \quad \#$$

In particular: (a) $\mathbb{E}(\eta(\mathbb{1}_A)) = \int_A \rho^{(1)}(x)\,d\mu(x)$

$$(3.5)\qquad \text{(b)}\quad Var(\eta(\mathbb{1}_A)) = \int_{A^2} \rho^{(2)}(x,y)\,d\mu(x)\,d\mu(y) + \int_A \rho^{(1)}(x)\,d\mu(x) - \left(\int_A \rho^{(1)}(x)\,d\mu(x)\right)^2.$$

3.3) Determinantal class of point processes.

Definition 12: A point process is determinantal if the n-point correlation functions are given by

$$(3.6)\qquad \rho^{(n)}(x_1,\dots,x_n) = \det\left(K(x_i,x_j)\right)_{1\le i,j\le n}$$

where $K(x,y)$ is a kernel (of an integral operator: $K: L^2(\Lambda, d\mu) \to L^2(\Lambda, d\mu)$, non-negative, locally trace-class).

Theorem 13: (Macchi; Soshnikov). In the case of Hermitian K, K defines a determinantal point process

iff $\quad 0 \le K \le 1$.

. If the corresponding point process exists, then it is unique.

Remarks: . A determinantal point process is simple.

. The # of points is n with proba. 1 iff K is an orthogonal projection with rank$(K)=n$.

. Example: . GUE eigenvalues' point process!

3.4) Hole probability.

- One of the quantity interest is the hole probability of a set A, i.e., the probability that no points is in A.

- $$\mathbb{P}(\eta(\mathbb{1}_A)=0) = \mathbb{E}\Big(\prod_i (1-\mathbb{1}_A(x_i))\Big) \quad \text{, because } \prod_i (1-\mathbb{1}_A(x_i)) = \begin{cases} 0, & \exists i \text{ st. } x_i \in A, \\ 1, & A \text{ is empty.} \end{cases}$$

$$= \sum_{n=0}^{\infty} (-1)^n \, \mathbb{E}\Big(\sum_{i_1<\dots<i_n} \prod_{k=1}^{n} \mathbb{1}_A(x_{i_k}) \Big)$$

(3.7)
$$\overset{\text{symmetry}}{=} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \, \mathbb{E}\Big(\sum_{\substack{i_1,\dots,i_n \\ \text{all different}}} \prod_{k=1}^{n} \mathbb{1}_A(x_{i_k}) \Big)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{A^n} d\mu(x_1)\dots d\mu(x_n)\, \rho^{(n)}(x_1,\dots,x_n).$$

- For determinantal point processes:

(3.8)
$$\underline{\mathbb{P}(\eta(\mathbb{1}_A)=0) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{A^n} d\mu(x_1)\dots d\mu(x_n) \cdot \det\big(K(x_i,x_j)\big)_{1\le i,j\le n}}$$

$$\underline{\equiv \det(\mathbb{1}-K)_{L^2(A,d\mu)}} \quad \Big(= \det(\mathbb{1} - \mathbb{1}_A K \mathbb{1}_A)_{L^2(\Lambda,d\mu)}\Big)$$

- This is called <u>Fredholm determinant.</u>

<u>Remark:</u> There is a whole theory on Fredholm determinants of operators, but here we do not need it. As soon as the series is well defined it is fine.

Example: $\mu(A)$ finite, and $|K(x,y)| \le C_1$ in A. Then,

$$\Big| \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{A^n} d\mu(x_1)\dots d\mu(x_n) \det(K(x_i,x_j)) \Big| \le \underbrace{\sum_{n=0}^{\infty} \frac{1}{n!} C_1^n \mu(A)^n n^{n/2}}_{<\infty}$$

Hadamard bound!

. Application: Distribution of the largest eigenvalue for GUE.

$$\mathbb{P}(\lambda_{max}^{GUE,N} \leq s) = \mathbb{P}(\eta^{GUE,N}(\mathbb{1}_{(s,\infty)})=0) = \det(\mathbb{1} - \mathbb{1}_{(s,\infty)} K_N^{GUE} \mathbb{1}_{(s,\infty)}) \quad (3.9)$$

where K_N^{GUE} is the Kernel for GUE eigenvalues of $N\times N$ matrices.

. 3.5) When a measure defines a determinantal point process?

. We have seen that for GUE eigenvalues, the measure

$$\frac{1}{Z}\left(\det(\lambda_i^{j-1})\right)^2 \prod_{i=1}^{N}\left(e^{-\frac{\lambda_i^2}{2N}} d\lambda_i\right)$$ induces a determinantal

point process. This is a particular case of the following theorem.

Theorem 14: (Borodin; Tracy-Widom for GUE).

Consider a measure of the form

$$\frac{1}{Z_N} \det_{1\leq i,j\leq N}(\phi_i(x_j)) \cdot \det_{1\leq i,j\leq N}(\psi_i(x_j))\, d\mu(x_1)\cdots d\mu(x_N) \quad (3.10)$$

with $Z_N \neq 0$. Then, (3.10) defines a determinantal point process with Kernel

$$K_N(x,y) = \sum_{i=1}^{N} \psi_i(x)[A^{-1}]_{ij}\ \phi_j(y) \quad (3.11)$$

where: $A_{ij} = \int_\Lambda \phi_i(s)\psi_j(s)\,d\mu(s)$.

Proof.: Notation: $\langle a|b\rangle \doteq \int_\Lambda a(x)b(x)\,d\mu(x)$, $\langle x|b\rangle \doteq b(x)$, $\langle a|x\rangle \doteq a(x)$

. Suppose that we can find functions $\tilde{\phi}_i, \tilde{\psi}_i$, $i=1,\dots,N$ such that:

(a) $\mathrm{vect}(\{\phi_i\}) = \mathrm{vect}(\{\tilde{\phi}_i\})$

(b) $\mathrm{vect}(\{\psi_i\}) = \mathrm{vect}(\{\tilde{\psi}_i\})$

and (c) $\langle\tilde{\phi}_k|\tilde{\psi}_\ell\rangle = \delta_{k,\ell}$.

[(a),(b) possible by Gram-Schmidt; (c) is possible since $Z_N\neq 0$].

. Then: $(3.10) = \text{const.} \det\left(\tilde{\Phi}_i(x_j)\right)_{1\le i,j\le N} \cdot \det\left(\tilde{\Psi}_i(x_j)\right)_{1\le i,j\le N} d^N\mu(x)$

as for GUE

$= \text{const} \cdot \det\left(K_N(x_i,x_j)\right)_{1\le i,j\le N} d^N\mu(x)$

with $K_N(x,y) = \sum_{k=1}^N \tilde{\Psi}_k(x)\tilde{\Phi}_k(y)$.

. Using $\langle\tilde{\Phi}_k,\tilde{\Psi}_l\rangle = \delta_{kl}$, one verifies:

$$\int_\Lambda d\mu(x)\, K_N(x,y) = N$$

$$\int_\Lambda d\mu(z)\, K_N(x,z)K_N(z,y) = K_N(x,y).$$

. These two properties are the only two used in the GUE case.

Thus: $\rho^{(n)}(x_1,\ldots,x_n) = \det\left(K_N(x_i,x_j)\right)_{1\le i,j\le n}$.

. $K_N = \sum_{k=1}^N |\tilde{\Psi}_k\rangle\langle\tilde{\Phi}_k|$.

Let S and T the change of basis matrices:

$$\phi_i = \sum_{j=1}^N S_{ij}\tilde{\Phi}_j \quad ; \quad \Psi_i = \sum_{j=1}^N T_{ij}\tilde{\Psi}_j$$

$$\Rightarrow K_N = \sum_{k=1}^N |\tilde{\Psi}_k\rangle\langle\tilde{\Phi}_k| = \sum_{k=1}^N \sum_{i,j=1}^N (T^{-1})_{ki}|\Psi_i\rangle (S^{-1})_{kj}\langle\phi_j|$$

$$= \sum_{i,j=1}^N |\Psi_i\rangle \underbrace{\left((T^t)^{-1}\cdot S^{-1}\right)_{ij}}_{=(S\cdot T^t)^{-1}_{ij}} \langle\phi_j|$$

. Define $A = S\cdot T^t$ and compute

$$\langle\phi_i|\Psi_j\rangle = \sum_{kl} S_{ik}T_{jl}\cdot\underbrace{\langle\tilde{\Phi}_k,\tilde{\Psi}_l\rangle}_{=\delta_{kl}} = \sum_k S_{ik}\cdot T_{jk} = A_{ij}\ \#.$$

<u>Remark:</u> . A det. p.p. with Kernel $K(x,y)$ is the same as the one with Kernel $\tilde{K}(x,y) = \frac{f(x)}{f(y)}K(x,y)$ for any $f(x)$ with $f(x)\neq 0, x\in\Lambda$.

. One say that <u>K and $\tilde{K}$ are conjugate Kernels.</u>