2) Gaussian Unitary / Orthogonal Ensemble.

2.1) Definitions.

. The Gaussian Ensembles of random matrices have been introduced by physicists (Dyson, Wigner...) in the 60's to model statistical properties of heavy nuclei resonance spectrum. The different ensembles are related with intrinsic symmetries of the system.

. A real symmetric matrix can a priori describe a system with : $\to$ time reversal and (rotation invariant or integer magnetic momentum).

[. A real quaternionic matrix (basis $\equiv$ Pauli matrices) : time reversal and half-integer magnetic momentum.]

. A complex hermitian matrix : not time-reversal (e.g. with external magnetic field).

Definition 1 : (GOE random matrices).

. The Gaussian Orthogonal Ensemble of random matrices is a measure P on the set of $N \times N$ real symmetric matrices given by :

$$(2.1) \qquad P(H)\,dH = \text{const} \times \exp\left(-\frac{\mathrm{Tr}(H^2)}{4N}\right) dH,$$

where dH is the flat reference measure: $dH = \prod_{1 \le i \le j \le N} dH_{ij}$

Remark: . "Orthogonal" because (2.1) is invariant under orthogonal transformation.

Definition 2 : (GUE). The Gaussian Unitary Ensemble of random matrices is a measure P on the set of $N \times N$ complex hermitian matrices given by :

$$(2.2) \qquad P(H)\,dH = \text{const} \times \exp\left(-\frac{\mathrm{Tr}(H^2)}{2N}\right) dH,$$

where $dH = \prod_{i=1}^{N} dH_{ii} \prod_{1 \le i < j \le N} d\,\mathrm{Re}H_{ij}\, d\,\mathrm{Im}H_{ij}$.

Remark: . The original definition is different: $p(H)$ is defined to be : (1) invariant under the change of basis (the group of symmetry: $\mathcal{O}(N)$, $\mathcal{U}(N)$).

$\hookrightarrow p(H) = \mathrm{fct}(\mathrm{Tr}(H^k), k=1,\dots,N)$.

(2) the entries of the matrices are independent random variables (up to the imposed symmetry).

$\hookrightarrow p(H) \propto \exp\left(-a \cdot \mathrm{Tr}(H^2) + b\,\mathrm{Tr}(H) + c\right)$, $a>0$, $b, c \in \mathbb{R}$.

→ Shifting the zero of the energy, one can get rid of b, while c is just a normalisation constant.

2.2) Eigenvalues' distributions.

. Often one is interested in the eigenvalues of the matrix, since they are independent of the choice of basis.

Proposition 3: Let $\lambda_1, \lambda_2, \dots, \lambda_N$ be the eigenvalues of a GOE/GUE. Then, the joint distribution of eigenvalues is given by

$$(2.3) \qquad p(\lambda)\, \underset{(d\lambda_1 \cdots d\lambda_N)}{\underset{\shortparallel}{d\lambda}} = \mathrm{const} \times \Big[\underbrace{\prod_{1 \le i < j \le N} (\lambda_j - \lambda_i)}_{\equiv \Delta_N(\lambda)}\Big]^{\beta} \cdot \prod_{i=1}^{N} \left(e^{-\frac{\lambda_i^2 \beta}{4N}} d\lambda_i\right)$$

where: $\begin{cases} \beta = 1 \text{ for GOE}, \\ \beta = 2 \text{ for GUE}. \end{cases}$

Remark: $\prod_{1 \le i < j \le N} (\lambda_j - \lambda_i) = \det\left(\lambda_i^{j-1}\right)_{1 \le i,j \le N}$, so it is a determinant, the Vandermonde determinant.

. One can prove by induction:

$$\det(\lambda_i^{j-1})_{1 \le i,j \le N} = \begin{vmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{N-1} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{N-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \lambda_N & \lambda_N^2 & \cdots & \lambda_N^{N-1} \end{vmatrix} = \begin{vmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{N-1} \\ 0 & \lambda_2 - \lambda_1 & \lambda_2^2 - \lambda_1^2 & \cdots & \lambda_2^{N-1} - \lambda_1^{N-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & \lambda_N - \lambda_1 & \lambda_N^2 - \lambda_1^2 & \cdots & \lambda_N^{N-1} - \lambda_1^{N-1} \end{vmatrix}$$

$$= \prod_{i=2}^{N} (\lambda_i - \lambda_1) \cdot \underbrace{\begin{bmatrix} 1 & \lambda_2+\lambda_1 & \cdots & \lambda_2^{N-2}+\dots+\lambda_1^{N-2} \\ \vdots & & & \\ 1 & \lambda_N+\lambda_1 & \cdots & \lambda_N^{N-2}+\dots+\lambda_1^{N-2} \end{bmatrix}}_{\overset{\text{lin. comb.}}{=} \Delta_{N-1}(\lambda_2,\dots,\lambda_N)} \quad \#$$

<u>Proof of Proposition 3</u>: Let us prove it for GOE. The GUE case is proven similarly.

. Given a matrix H, real symmetric, $\exists\, g \in O(N)$ st.

$$H = g \Lambda g^{-1}, \quad \text{with } \Lambda_{ij} = \lambda_i \cdot \delta_{ij}, \quad 1 \le i,j \le N.$$

. Since we have $\frac{N(N+1)}{2}$ independent entries of H and only N eigenvalues, it remains $\frac{N(N-1)}{2}$ "angular variables" in g's.

(a) . An infinitesimal transformation of H gives:

$$dH = dg \cdot \Lambda \cdot g^{-1} + g \cdot \Lambda \cdot dg^{-1} + g\, d\Lambda \cdot g^{-1}$$

Since: $\mathbb{1} = g \cdot g^{-1} \Rightarrow dg \cdot g^{-1} = -g \cdot dg^{-1} \Rightarrow dg^{-1} = -g^{-1} dg\, g^{-1}$

$$\Rightarrow dH = g \cdot [g^{-1} dg \Lambda - \Lambda g^{-1} dg + d\Lambda] \cdot g^{-1}$$

$$= g \cdot d\tilde{H} \cdot g^{-1} \quad \text{where } d\tilde{H} = d\Lambda + [\underbrace{g^{-1}dg}_{\doteq\, d\Omega \text{ : angular variables}}, \Lambda]$$

$\Rightarrow$ Jacobian $H \to \tilde{H}$ is one.

(b) Jacobian $\tilde{H} \to (\Lambda, \Omega)$:

In components: $d\tilde{H}_{ij} = d\lambda_i \cdot \delta_{ij} + \sum_{i=1}^{N} (d\Omega_{ik}\, \delta_{kj} \lambda_j - \lambda_i \delta_{ik} d\Omega_{kj})$

$$= d\lambda_i \cdot \delta_{ij} + d\Omega_{ij}\, (\lambda_j - \lambda_i).$$

$\Rightarrow$ Jacobian $\tilde{H} \to (\Lambda, \Omega)$ equal to

$$\left| \det \left[\frac{\partial(\tilde{H}_{11}, \dots, \tilde{H}_{NN}; \tilde{H}_{12}, \dots, \tilde{H}_{1N}; \dots; \tilde{H}_{N-1,N})}{\partial(\lambda_1, \dots, \lambda_N; \Omega_{12}, \dots, \Omega_{1N}; \dots; \Omega_{N-1,N})} \right] \right| = \ /.$$

$$= \left| \det \begin{pmatrix} \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} & & & & \\ & \lambda_1 - \lambda_2 & & & \bigcirc \\ & & \ddots & & \\ & & & \lambda_1 - \lambda_N & \\ & \bigcirc & & & \ddots \\ & & & & \lambda_{N-1} - \lambda_N \end{pmatrix} \right| = |\Delta_N(\lambda)|.$$

. Therefore: $dH = |\Delta_N(\lambda)| d\lambda \cdot \underbrace{d\Omega}_{\text{Haar measure on } SO(N).}$

$$(2.1) \Rightarrow p(H)\, dH = \text{const} \times \exp\left(-\frac{\beta}{4N} \cdot \mathrm{Tr}(H^2)\right) \cdot \Delta_N(\lambda)\, d\lambda\, d\Omega$$
$$= \text{const} \cdot d\Omega \cdot |\Delta_N(\lambda)|^{\beta} \prod_{i=1}^{N} \left(e^{-\frac{\beta}{4N}\lambda_i^2} d\lambda_i\right).$$

. (c) Integrate out the angular variables, leads to (2.3).

. For the GUE ensemble, the only difference is that for the upper-diagonal components we have $\mathrm{Re}\, H_{ij}$ and $\mathrm{Im}\, H_{ij}$, and in the angular variables too: $\mathrm{Re}\,\Omega_{ij}$, $\mathrm{Im}\,\Omega_{ij}$. This gives each factor $\lambda_i - \lambda_j$ twice. #

2.3) Correlation functions for GUE.

2.3.1) Generalities.

. Consider a measure like (2.3) and take any bounded <u>disjoints</u> Borel sets $A_1, \dots, A_n$ of $\mathbb{R}$. Then, denote

$$M_n(A_1, \dots, A_n) = \mathbb{E}\left(\prod_{i=1}^{n} (\#\text{eigenvalues in } A_i)\right), \qquad (2.4)$$

where $\mathbb{E}$ is the expectation under the measure.

<u>Definition 4:</u> (Correlation functions). If M_n is absolutely continuous with respect to a <u>reference measure μ^n</u> on $\mathbb{R}^n$, i.e., if $M_n(A_1, \dots, A_n) = \int_{A_1 \times \dots \times A_n} d\mu(x_1) \dots d\mu(x_n)\, \rho^{(n)}(x_1, \dots, x_n)$, $\forall$ Borel sets A_i in $\mathbb{R}$, then we call <u>$\rho^{(n)}$ the n-point correlation function.</u>

Remarks: • $\rho^{(n)}(x_1,\dots,x_n) = \rho^{(n)}(x_{\sigma_1},\dots,x_{\sigma_n})$, $\forall \sigma \in S_n$ (symmetry).

• It makes sense to speak about $\rho^{(n)}$ only specifying the reference measure.

If $\mu(dx) = dx$ • Probabilistic interpretation: In the case where a.s. no double points occurs (i.e., for simple point processes), we have the following probabilistic interpretation:

. Let $[x_i, x_i+\Delta x_i]$, $i=1,\dots,n$ be disjoint infinitesimally small sets. Then, we will have at most one point in each $[x_i, x_i+\Delta x_i]$ and

$$\rho^{(n)}(x_1,\dots,x_n) = \lim_{\substack{\Delta x_i \to 0 \\ i=1,\dots,n}} \frac{\mathbb{P}(\text{one point in each } [x_i, x_i+\Delta x_i], 1\leq i\leq n)}{\Delta x_1 \cdots \Delta x_n}$$

[To see it: Take $A_i = [x_i, x_i+dx_i]$ in (2.4).]

• $\rho^{(1)}(x)$ is the density of points at x.

Lemma 5: Consider the case (like GUE) where the probability density $P_N(x_1,\dots,x_N)$ is symmetric on $\mathbb{R}^N$.

Then,

$$\rho^{(n)}(x_1,\dots,x_n) = \frac{N!}{(N-n)!} \int_{\mathbb{R}^{N-n}} dx_{n+1}\cdots dx_N \cdot P_N(x_1,\dots,x_N). \tag{2.6}$$

Proof: $\rho^{(n)}(x_1,\dots,x_n)$ is the probability density of finding a particle at $x_1, \dots$, a particle at x_n, but it does not say which of the N particle is at which x_i's.

By the symmetry of P_N, each choice gives a contribution

$$\int_{\mathbb{R}^{N-n}} dx_{n+1}\cdots dx_N P_N(x_1,\dots,x_N),$$

and there are $n! \binom{N}{n} = \frac{N!}{(N-n)!}$ possible choices. #

2.3.2) $P_N(x_1, \dots, x_N)$ and orthogonal polynomials for GUE.

. Consider the weight $w(x) = \exp\left(-\frac{x^2}{2N}\right)$ and define the orthogonal polynomials

$\{q_k(x), k = 0, \dots, N-1\}$ by the following conditions:

(1) $q_k(x)$ is of degree k with $q_k(x) = u_k x^k + \dots$, $u_k > 0$.

(2) they are orthonormal:

$$\int_{\mathbb{R}} dx\, w(x)\, q_k(x)\, q_\ell(x) = \delta_{k\ell}.$$

. Notice that: $\det\left(\lambda_i^{j-1}\right)_{1\le i,j\le N} = \text{const} \times \det\left(q_{j-1}(\lambda_i)\right)_{1\le i,j\le N}$.

Thus, the eigenvalues' measure for GUE ((2.3) for $\beta = 2$) is

$$P_N(\lambda_1, \dots \lambda_N) = \frac{1}{Z_N} \times \left(\prod_{i=1}^N w(\lambda_i)\right) \cdot \left(\det\left(q_{i-1}(\lambda_j)\right)_{1\le i,j\le N}\right)^2$$

$$\underset{\text{multiplications}}{\overset{\text{matrix}}{=\!=}} \frac{1}{Z_N}\left(\prod_{i=1}^N w(\lambda_i)\right) \cdot \det\left(\sum_{k=1}^N q_{k-1}(\lambda_i)\, q_{k-1}(\lambda_j)\right)_{1\le i,j\le N}.$$

. <u>Define the kernel</u>: $\boxed{K_N(x,y) = w(x)^{1/2} w(y)^{1/2} \cdot \sum_{k=1}^N q_{k-1}(x)\, q_{k-1}(y).}$

(2.7) Then, $P_N(\lambda_1, \dots, \lambda_N) = \frac{1}{Z_N} \cdot \det\left[K_N(\lambda_i, \lambda_j)\right]_{1\le i,j\le N}$.

. <u>Two properties of K_N</u>: (proof follows from orthonormality property of q_k's).

(2.8) . $\displaystyle\int_{\mathbb{R}} dx\, K_N(x,x) = N.$

(2.9) . $\displaystyle\int_{\mathbb{R}} dz\, K_N(x,z) K_N(z,y) = K_N(x,y).$

⚠ Orthogonal polynomials plays a central role here!

<u>Proposition 6</u>: The n-point correlation functions of GUE are given by:

(2.10) $$\rho^{(n)}(\lambda_1, \dots, \lambda_n) = \det\left(K_N(\lambda_i, \lambda_j)\right)_{1\le i,j\le n},$$ with respect

to the flat measure $d\lambda$.

Proof of Proposition 6:

(a) Determination of Z_N in (2.7).

. For $n=N$, by Lemma 5, $\rho^{(N)}(\lambda_1,\dots,\lambda_N)=P_N(\lambda_1,\dots,\lambda_N)\,N!$

$$\Rightarrow \int_{\mathbb{R}^N} d\lambda_1\dots d\lambda_N\, \rho^{(N)}(\lambda_1,\dots,\lambda_N)=N! \quad ,\text{i.e.},$$

$$\int_{\mathbb{R}^N} d\lambda_1\dots d\lambda_N\, P_N(\lambda_1\dots\lambda_N)=1$$

$$|||$$

$$\frac{1}{Z_N}\int_{\mathbb{R}^N} d\lambda_1\dots d\lambda_N\, w(\lambda_1)\dots w(\lambda_N)\, \det(q_{i-1}(\lambda_j))\det(q_{i-1}(\lambda_j))$$

$$=\frac{1}{Z_N}\cdot N!\cdot\det\Big(\underbrace{\int_{\mathbb{R}^N} d\lambda\, w(\lambda)\, q_{i-1}(\lambda)\, q_{j-1}(\lambda)}_{=\delta_{i,j}}\Big)=\frac{N!}{Z_N}\Rightarrow Z_N=N!$$

Used Cauchy-Binet (Heine) identity:

$$\int_{\Lambda^N} d\lambda(x_1)\dots d\lambda(x_N)\,\det(\phi_i(x_j))\det(\psi_i(x_j))=N!\det\Big(\int_\Lambda d\lambda(x)\,\phi_i(x)\,\psi_j(x)\Big)$$

(b) By Lemma 5:

$$\rho^{(n)}(\lambda_1,\dots,\lambda_n)=\frac{1}{(N-n)!}\cdot\int_{\mathbb{R}^{N-n}} d\lambda_{n+1}\dots d\lambda_N\,\det\big(K_N(\lambda_i,\lambda_j)\big)_{1\le i,j\le N}.$$

. We integrate $N-n$ variables. Consider the case when we have a $m\times m$ matrix

$$\int_{\mathbb{R}} dx_m\,\det(K_N(x_i,x_j))_{1\le i,j\le m}=\int_{\mathbb{R}} K_N(x_m,x_m)\cdot\det(K_N(x_i,x_j))_{1\le i,j\le m-1}\,dx_m$$

$$+\int\sum_{k=1}^{m-1}(-1)^{m-k}\cdot K_N(x_k,x_m)\det\begin{bmatrix}K_N(x_i,x_j)\\ \hline K_N(x_m,x_j)\end{bmatrix}_{\substack{1\le i,j\le m-1\\ i\ne k}}dx_m$$

$\boxed{\text{Develop on last column.}}$

$$\overset{\text{linearity}}{=}\int_{\mathbb{R}} dx_m\,K_N(x_m,x_m)\cdot\det(K_N(x_i,x_j))_{1\le i,j\le m-1}$$

$$+\sum_{k=1}^{m-1}(-1)^{m-k}\cdot\det\begin{bmatrix}K_N(x_i,x_j)\\ \hline \int K_N(x_k,x_m)\cdot K_N(x_m,x_j)\,dx_m\end{bmatrix}$$

$$\overset{(2.8),(2.9)}{=}(N-(m-1))\cdot\det(K_N(x_i,x_j))_{1\le i,j\le m-1}.$$

. By iteration we get: $\rho^{(n)}(\lambda_1,\dots,\lambda_n)=\frac{\cancel{1\cdot 2\cdots(N-n)}}{\cancel{(N-n)!}}\cdot\det(K_N(\lambda_i,\lambda_j))_{1\le i,j\le n}$. #

2.3.3) Different representations of the Kernel.

(A) $$K_N(x,y) = w(x)^{1/2}\, w(y)^{1/2} \cdot \sum_{k=1}^{N} q_{k-1}(x)\, q_{k-1}(y).$$

with $w(x) = \exp\left(-\frac{x^2}{2N}\right)$ and q_k are the orthogonal polynomials, degree k, satisfying $\int_{\mathbb{R}} dx\, w(x)\, q_k(x)\, q_\ell(x) = \delta_{k,\ell}$.

. q_k's can be written in terms of the standard Hermite polynomials:

(2.11) $$H_k(x) := e^{x^2} \frac{d^k}{dx^k} e^{-x^2} \text{ satisfying } \int_{\mathbb{R}} H_k(x)\, H_\ell(x)\, e^{-x^2} dx = \sqrt{\pi} \cdot 2^k k!\, \delta_{k,\ell}.$$

with $H_k(x) = 2^k x^k + \dots$

. A simple calculation gives:

(2.12) $$q_k(x) = \frac{1}{\sqrt[4]{2\pi N}} \cdot \frac{1}{\sqrt{2^k k!}} \cdot H_k\left(\frac{x}{\sqrt{2N}}\right) \Rightarrow u_k = \frac{1}{\sqrt[4]{2\pi N}} \cdot \frac{1}{\sqrt{2^k k!}} \cdot \left(\frac{2}{N}\right)^{k/2}.$$

(B) Using Christoffel-Darboux formula:

(2.13) $$\sum_{k=0}^{N-1} q_k(x)\, q_k(y) = \begin{cases} \frac{u_{N-1}}{u_N} \cdot \frac{q_N(x)\, q_{N-1}(y) - q_{N-1}(x)\, q_N(y)}{x-y}, & \text{for } x \neq y, \\ \frac{u_{N-1}}{u_N} \cdot \left[q'_N(x)\, q_{N-1}(x) - q'_{N-1}(x)\, q_N(x)\right], & \text{for } x = y. \end{cases}$$

. Then, (A) can be rewritten as:

(2.14) $$K_N(x,y) = \begin{cases} N \cdot e^{-\frac{x^2+y^2}{4N}} \cdot \frac{q_N(x)\, q_{N-1}(y) - q_{N-1}(x)\, q_N(y)}{x-y}, & \text{for } x \neq y, \\ N \cdot e^{-\frac{x^2}{2N}} \cdot \left[q'_N(x)\, q_{N-1}(x) - q'_{N-1}(x)\, q_N(x)\right], & \text{for } x = y. \end{cases}$$

2.4) Bulk and edge scaling limits.

2.4.1) Wigner's semicircle law.

. With the chosen rescaling: $p(H) = e^{-\frac{\mathrm{Tr}(H^2)}{2N}}$, the largest eigenvalue is around $2N$ and the smallest eigenvalue around $-2N$. Since there are exactly N eigenvalues, the density will be of order one, without extra rescaling.

. Let $\rho_N(\lambda)$ be the density of eigenvalues around $2N.\lambda$.
Then,

$$\rho_\infty(\lambda) := \lim_{N\to\infty} \rho_N(\lambda) = \lim_{N\to\infty} K_N(2N\lambda, 2N\lambda) = \begin{cases} \frac{1}{\pi}\cdot\sqrt{1-\lambda^2}, & \lambda \in [-1,1], \\ 0, & \text{otherwise.} \end{cases} \tag{2.16}$$

. This is called Wigner's semicircle law.

2.4.2) Bulk scaling limit: Sine kernel.

. Let us consider the eigenvalues in the bulk around $2N\lambda$ for some $\lambda \in (-1,1)$ and <u>rescale</u> space <u>to</u> have <u>density one</u>:

$$\begin{cases} X = 2N\lambda + \frac{\xi_1}{\rho_\infty(\lambda)} \\ Y = 2N\lambda + \frac{\xi_2}{\rho_\infty(\lambda)} \end{cases} \tag{2.17}$$

Since the kernel is related to correlation functions ($\sim$ densities), we also have to rescale it accordingly:

$$K_N^{\mathrm{resc}}(\xi_1,\xi_2) := \frac{1}{\rho_\infty(\lambda)} K_N\left(2N\lambda + \frac{\xi_1}{\rho_\infty(\lambda)}, 2N\lambda + \frac{\xi_2}{\rho_\infty(\lambda)}\right). \tag{2.18}$$

<u>Proposition 7:</u>

$$\lim_{N\to\infty} K_N^{\mathrm{resc}}(\xi_1,\xi_2) = \underline{\frac{\sin(\pi\cdot(\xi_1-\xi_2))}{\pi(\xi_1-\xi_2)}}. \tag{2.19}$$

<u>This is called the Sine Kernel.</u>

Proof of Proposition 7: Recall the form of the Kernel:

$$K_N^{\text{resc}}(\xi_1,\xi_2) = \frac{N\cdot}{\rho_0(\lambda)} \frac{q_N(x)q_{N-1}(y) - q_{N-1}(x)q_N(y)}{(\xi_1-\xi_2)\frac{1}{\rho_0(\lambda)}} \cdot e^{-\frac{x^2}{4N}} \cdot e^{-\frac{y^2}{4N}}$$

. The asymptotics of the Hermite polynomials are known, from which:

$$\sqrt{N}\cdot q_{N-m}(x)\cdot e^{-\frac{x^2}{4N}} \underset{\uparrow}{\cong} \frac{1}{\pi\cdot\sqrt{\rho_0(\lambda)}} \cdot \sin\left[\alpha_0 N + \pi\cdot\xi + m\cdot\varphi\right]$$

$$x = 2N\lambda + \frac{\xi}{\rho_0(\lambda)} \qquad \text{with } \varphi = \operatorname{acos}(\lambda).$$

$$\Rightarrow K_N^{\text{resc}}(\xi_1,\xi_2) \underset{N\to\infty}{\cong} \frac{1}{\pi^2\cdot\rho_0(\lambda)(\xi_1-\xi_2)} \cdot \Big\{\sin(\alpha_0 N + \pi\xi_1)\sin(\alpha_0 N + \pi\xi_2+\varphi) - \sin(\alpha_0 N+\pi\xi_1+\varphi)\cdot\sin(\alpha_0 N + \pi\xi_2)\Big\}$$

. Use the identity: $\sin(a)\cdot\sin(b+\varphi) - \sin(a+\varphi)\sin(b) = \sin(\varphi)\cdot\sin(a-b)$

and obtain:

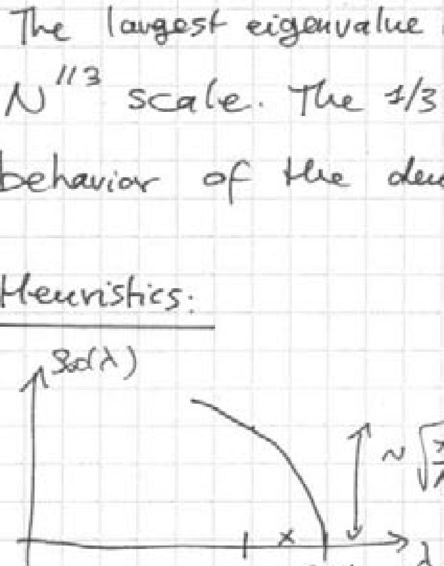
$$\lim_{N\to\infty} K_N^{\text{resc}}(\xi_1,\xi_2) = \frac{1}{\pi^2(\xi_1-\xi_2)} \cdot \frac{1}{\frac{1}{\pi}\sqrt{1-\lambda^2}} \cdot \sin(\pi(\xi_1-\xi_2)) \cdot \underbrace{\sin(\operatorname{acos}(\lambda))}_{=\sqrt{1-\lambda^2}}$$

$$= \frac{\sin(\pi(\xi_1-\xi_2))}{\pi(\xi_1-\xi_2)}. \quad \#$$

2.4.3) Edge scaling limit:

. The largest eigenvalue is around $2N$ and its fluctuations are on a $N^{1/3}$ scale. The 1/3 exponent is connected with the square root behavior of the density at the edge of the spectrum.

. Heuristics:



$\#$ e.v. $\geq 2N-x \approx N\cdot\left(\frac{x}{N}\right)^{3/2} = \frac{x^{3/2}}{\sqrt{N}}$ over distance x.

$\Rightarrow$ $\#$ e.v. $\geq 2N-x$ is of order one for $x\sim N^{1/3}$, i.e., the top eigenvalues fluctuates over distances $\mathcal{O}(N^{1/3})$.

. Therefore, the scaling limit is as follows:

(2.20) $\begin{cases} x = 2N + \xi_1 N^{1/3} \\ y = 2N + \xi_2 N^{1/3} \end{cases}$ and the rescaled kernel is

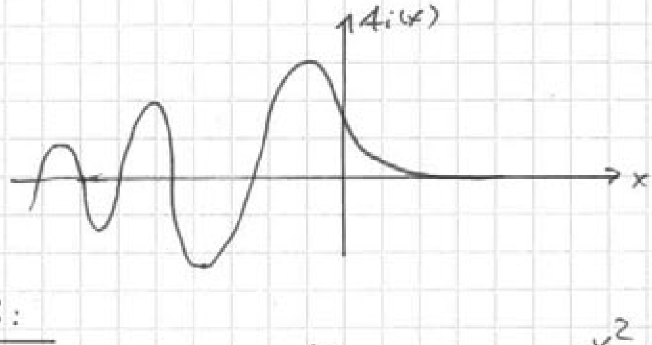
(2.21) $K_N^{resc}(\xi_1, \xi_2) = N^{1/3} K_N(x, y)$.

. Proposition 8:

(2.22) $$\lim_{N\to\infty} K_N^{resc}(\xi_1,\xi_2) = \frac{Ai(\xi_1)Ai'(\xi_2) - Ai'(\xi_1)Ai(\xi_2)}{\xi_1 - \xi_2}$$

. This is called the Airy kernel: $K_{Ai_2}(\xi_1,\xi_2)$

. Remark: . $Ai(x)$ is the Airy function, solution of $y''(x) = x \cdot y(x)$ and with asymptotic behavior: $y(x) \sim \frac{e^{-\frac{2}{3}x^{3/2}}}{2\sqrt{\pi}\, x^{1/4}}$ as $x \to +\infty$.



Proof of Proposition 8:

. One uses the asymptotics: $N^{1/3} \cdot q_{N-m}(x) \cdot e^{-\frac{x^2}{4N}} \underset{x=2N+\xi N^{1/3}}{\simeq} Ai(\xi + m \cdot N^{-1/3})$.

$$\Rightarrow K_N^{resc}(\xi_1,\xi_2) \underset{N\to\infty}{\simeq} \frac{N^{1/3} \cdot N \cdot N^{-2/3}}{N^{1/3} \cdot (\xi_1 - \xi_2)} \cdot \left(Ai(\xi_1) Ai(\xi_2 + N^{-1/3}) - Ai(\xi_1 + N^{-1/3}) Ai(\xi_2) \right)$$

$$\simeq \frac{N^{1/3}}{(\xi_1 - \xi_2)} \left(Ai(\xi_1)(Ai(\xi_2) + N^{-1/3} Ai'(\xi_2)) - (\xi_1 \leftrightarrow \xi_2) \right)$$

$$= \frac{Ai(\xi_1)Ai'(\xi_2) - Ai'(\xi_1)Ai(\xi_2)}{\xi_1 - \xi_2} . \#$$

. Properties of the Airy Kernel, $K_{\mathcal{A}_2}$:

① $K_{\mathcal{A}_2}(u,v) = \int_0^\infty d\lambda \, Ai(u+\lambda) Ai(v+\lambda)$.

Proof.: $(u-v) \cdot \int_0^\infty d\lambda \, Ai(u+\lambda) Ai(v+\lambda) = \int_0^\infty d\lambda \, (u+\lambda) Ai(u+\lambda) Ai(v+\lambda)$
$- \int_0^\infty d\lambda \, Ai(u+\lambda)(v+\lambda) Ai(v+\lambda)$

$Ai''(x) = x Ai(x)$

$$= \int_0^\infty d\lambda \, Ai''(u+\lambda) Ai(v+\lambda) - (u \leftrightarrow v)$$

$$= Ai(v+\lambda) Ai'(u+\lambda)\Big|_0^\infty - \int_0^\infty d\lambda \, Ai'(u+\lambda) Ai'(v+\lambda)$$

$$- (u \leftrightarrow v)$$

$$= Ai(u) Ai'(v) - Ai'(u) Ai(v). \quad \#$$

② $K_{\mathcal{A}_2}^2 = K_{\mathcal{A}_2}$ $\quad \int_0^\infty d\lambda \int_0^\infty d\mu \, Ai(u+\lambda) \Big(\underbrace{\int_{\mathbb{R}} dz \, Ai(\lambda+z) Ai(\mu+z)}_{= \delta(\mu-\lambda)} \Big) Ai(v+\mu) = A_2(u,v)$.

③ let $H = -\frac{d^2}{dx^2} + x$ the Airy operator.

then: $H \psi_\lambda = \lambda \psi_\lambda$ with $\psi_\lambda(x) = Ai(x-\lambda)$.

$\Rightarrow K_{\mathcal{A}_2}$ is the spectral projection of H on $\{H \leq 0\}$.

④ $K_{\mathcal{A}_2}$ is locally trace-class and $\| \mathbb{1}_{[s,\infty)} K_{\mathcal{A}_2} \mathbb{1}_{[s,\infty)} \| < 1, \ \forall s > -\infty$.

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