Random Matrices and Related Problems

Lecture's notes for Beg Rohu Summer School 2008.

by

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Outline:

1) Introduction : universality and examples.

2) Gaussian Unitary / Orthogonal Ensembles (GUE / GOE):

. An example to introduce the mathematical structure.

3) Determinantal point processes:

. From point processes to Gap probability given by Fredholm det.

4) Edge scaling limit and Tracy-Widom distributions:

. The universal limit distributions.

5) Extended determinantal point processes:

. From Karlin-McGregor and LGV theorem to Airy_2 Process.

6.a) Application to the polynuclear growth in droplet geometry.

6.b) Application to the Totally Asymmetric Simple Exclusion Process

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References used in the preparation of the lecture notes

1. Lecture notes on the same framework [19, 10]
2. My PhD thesis [6]
3. The standard book on Random Matrices [12]
4. Booklet on random matrices [11]
5. Universality in Mathematics and Physics [5]
6. Point processes [14] and determinantal class [17, 18, 10, 13, 1, 4]
7. Airy processes [15, 9, 2] (papers) and my review [7]
8. Tracy-Widom distributions [20, 8, 21]
9. Application to the PNG [15, 9, 3]
10. Application to the TASEP [2, 16]

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1) Introduction.

1.1) From micro to macro: universality.

. On a macroscopic scale, there are physical laws which are shared by different systems.

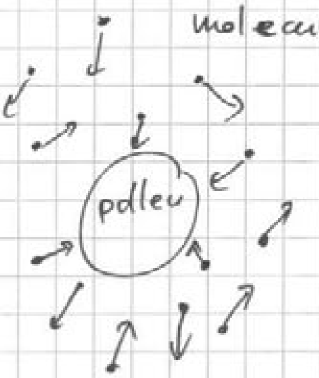
For example, consider the diffusion equation in the space-homogeneous case:

$$(1.1) \qquad \frac{\partial \phi}{\partial t} = D \cdot \nabla^2 \phi \quad \text{, for some function } \phi(x,t),\ x \in \mathbb{R}^d \text{ the space, } t \in \mathbb{R} \text{ the time.}$$

. "D" is called the diffusion coefficient.

. This equations appears in several situations, just two examples:

(a) ϕ represents the probability density of finding a grain of pollen in suspension in water, at position x and time t. The evolution of the grain of pollen being determined by the shocks with the water molecules: it looks random (→ Brownian Motion).



. Changing the dimension of the pollen or the water temperature, equation (1.1) still holds.
The only difference will be the diffusion coefficient D:

$$(1.2) \qquad D = \frac{k_B \cdot T}{m \cdot \gamma} \quad \text{, where: } k_B = \text{Boltzmann constant, } T = \text{(absolute) temperature, } m = \text{mass of pollen, } \gamma = \text{friction coefficient of the liquid}$$

(b) ϕ represents the temperature profile in a metal. Again, equation (1.1) holds and the only material dependence is in D.

. The key observation is that the same macroscopic laws emerges "no matters" of the details of the microscopic interactions.

. In the above example, the details of the atomic interactions emerge only in the diffusion constant D (which is material/system dependent), but the diffusion equation is universal.

Remark: It is the emergence of such universal behaviors for macroscopic systems, which allow the existence of physical law. If the form of (11) would change for every mass, temperature, ..., then one would not have the law of diffusion.

1.2) A simple mathematical example for universality.

. The simplest example is the Central Limit Theorem (CLT).
Let $\{X_n\}_{n\geq 1}$ be iid random variables. Assume that $\mu = \mathbb{E}(X_i)$, $\sigma^2 = \mathrm{Var}(X_i)$ are both finite (and $\mathbb{E}(|X_i|^3) < \infty$ too). Then,

$$(13) \quad \lim_{N\to\infty} \mathbb{P}\left(\frac{\sum_{i=1}^{N} X_i - \mu \cdot N}{\sigma\sqrt{N}} \leq S \right) = \int_{-\infty}^{S} du \, \frac{e^{-u^2/2}}{\sqrt{2\pi}} .$$

. The r.h.s. of (13) is universal: it does not depend on the details of the distribution of the x_i's. The details of it enters only via the centering (μ) and rescaling (σ).

$S_N := \sum_{i=1}^{N} x_i$ is the macroscopic observable.

. Two universal quantities: $\begin{cases} \text{(a) Fluctuation exponent is } \tfrac{1}{2}: \ S_N - \mu N \simeq N^{1/2} \\ \text{(b) Limit law is Gaussian : r.h.s. of (3).} \end{cases}$

1.3) Random Matrices.

. There are systems with limit laws non-Gaussian. One class of such models share the same limit laws (on a mesoscopic or macroscopic) as Random Matrices.

. Consider the following example: $N \times N$ real symmetric matrices with independent entries :
$$\begin{cases} H_{i,i} \sim \mathcal{N}(0,2N) & , 1 \le i \le N \\ H_{i,j} = H_{j,i} \sim \mathcal{N}(0,N), & 1 \le i < j \le N \end{cases}$$

. Often the quantity of interest are the eigenvalues, and not the specific entries, since they are independent of the basis used to describe the system.

. In this case, we have N real eigenvalues : $\lambda_1, \lambda_2, \ldots, \lambda_N$ distributed as
$$p(\lambda_1, \ldots, \lambda_N)\, d\lambda_1 \ldots d\lambda_N = \text{const} \times \underbrace{\prod_{1 \le i < j \le N} |\lambda_i - \lambda_j|}_{\doteq \Delta(\lambda) \text{ the Vandermonde determinant.}} \cdot \prod_{i=1}^{N} \left(e^{-\frac{\lambda_i^2}{2N}} d\lambda_i\right).$$

. Key feature of random matrices : eigenvalue's repulsion, due to the $\Delta(\lambda)$ term.

. What can one analyze? $\rightarrow$ Statistial properties.

(a) Macroscopic behavior : For $N \gg 1$, the density of eigenvalues around λ is given by the Wigner semi-circle law:

$$\rho(\lambda) \cong \begin{cases} \frac{1}{\pi} \cdot \sqrt{1 - \left(\frac{\lambda}{2N}\right)^2}, & \lambda \in [-2N, 2N], \quad \text{(Figure 1.)} \\ 0 & , \lambda \notin [-2N, 2N]. \end{cases} \tag{1.4}$$

This is not a universal quantity, but depends on the details (like D).

. Some universal quantities are the following:

(b) Nearest-neighbor spacing (in the bulk): see below.

(c) Fluctuations of the largest eigenvalue (see later part of the lecture).

. Question: Is the system behaving like a random matrix?

. Roughly speaking, we say that a system is modeled by a random matrix theory if it behaves statistically as the eigenvalues of large matrices.

. Suppose that a scientist makes an experiment and get some data as output (e.g., the neutron resonance spectrum of heavy nuclei). (Figure 3.)

Step 1: Centering and rescaling

Exp. Data		Eigenvalues
$\{a_k\}$ around A		$\{\lambda_k\}$ around E
$\downarrow$ centering		$\downarrow$ centering
$\{a_k - A\}$		$\{\lambda_k - E\}$
$\downarrow$ rescale to density one		$\downarrow$
$\left\{\frac{a_k - A}{\gamma_A}\right\}$		$\left\{\frac{\lambda_k - E}{\gamma_E}\right\}$

. Now we have two comparable sets of data.

Step 2: Comparison with R.M.

. The scientist can compare the statistical properties of the two set of data and if the fit is good, he concludes that the system is well-modeled by a R.M. Theory.

. Examples: Nearest-neighbor spacing statistics.

(a) R.M. considered above : "GOE" : Analytic
(b) Nuclei resonances : Experimental.
(c) Spectrum of a free particle in a stadium: Numerics.
(d) Zeros of Riemann-zeta function : Deterministic! (This is GUE-like)

A List of Figures:

(a) GOE = nearest-neighbor spacing for eigenvalues: Figures 1, 2.

(b) Nuclei resonances: Figures 3, 4.2.

(c) Stadium spectrum : Figure 4.3.
($\text{Solve } -\Delta \psi = E\psi,\ \psi(\partial\text{stadium}) = 0$).

(d) Riemann-ζ function: Figures 5.1, 5.2, 5.3.

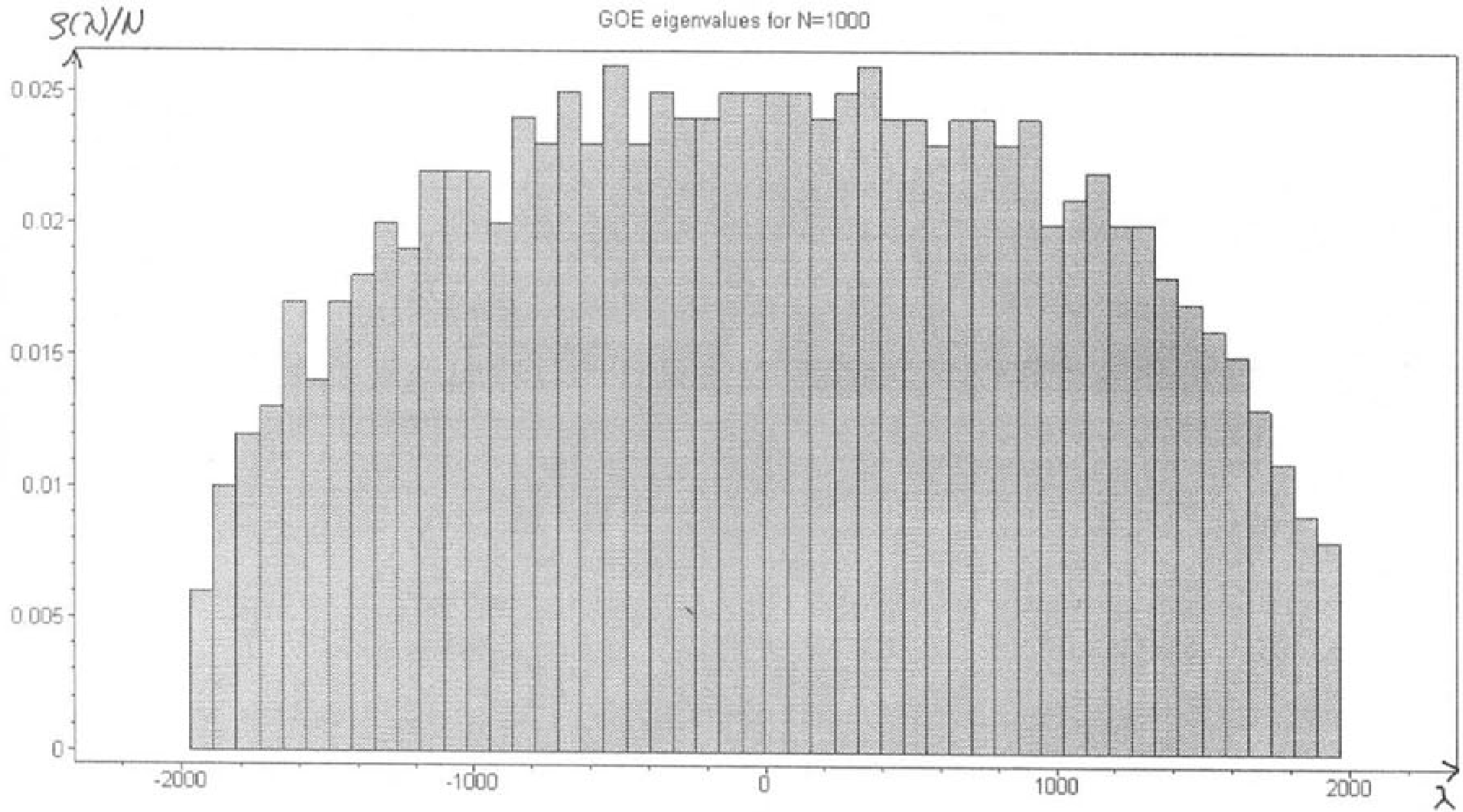
$$\zeta(z) \doteq \sum_{n=1}^{\infty} \frac{1}{n^z} = \prod_{\text{primes } p} \frac{1}{1-\frac{1}{p^z}}, \ \operatorname{Re} z > 1$$

$\oplus$ Analytic continuation for $\operatorname{Re} z \leq 1$.

Look at $\{\gamma_n\}_{n\geq 1}$ s.t. $\zeta(\frac{1}{2} \pm i\gamma_n) = 0$. (non-trivial zeros)

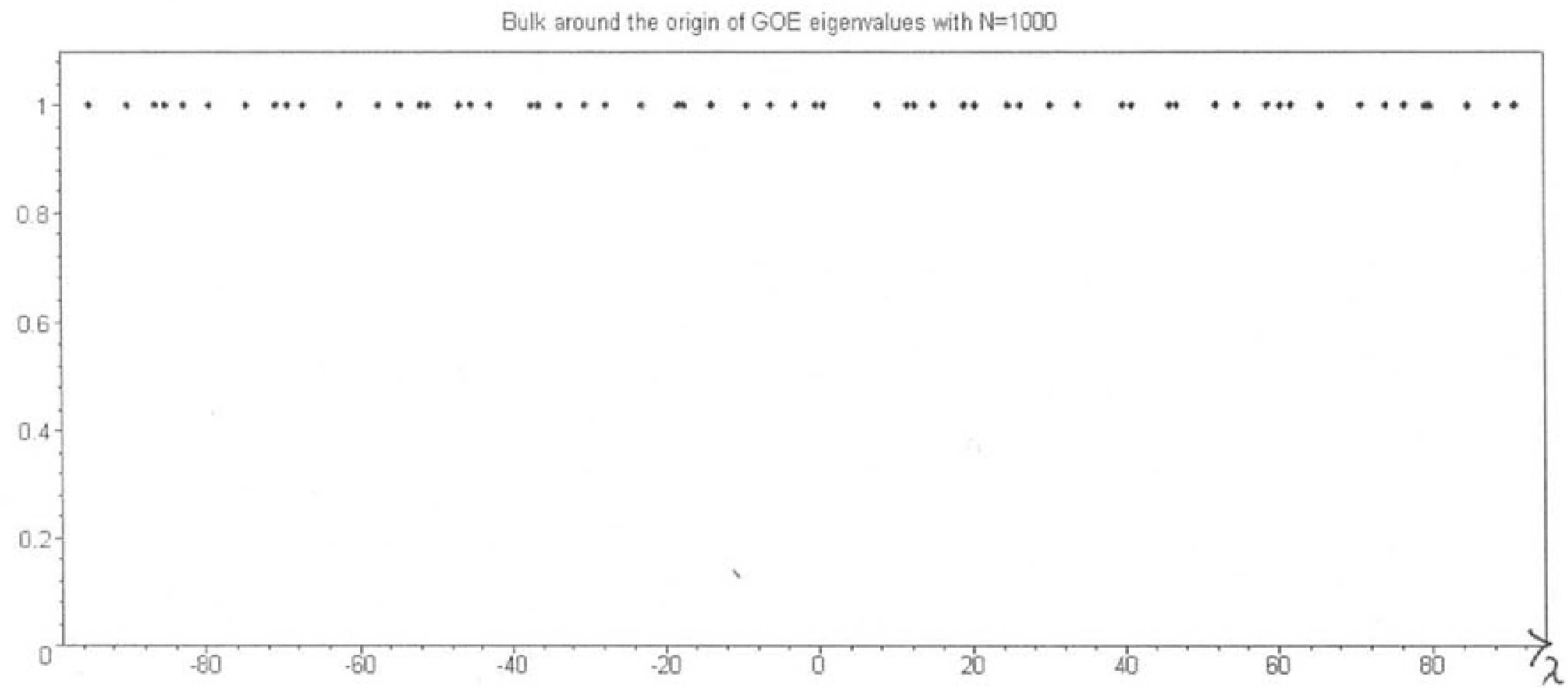
$\cong$ GUE instead of GOE

Figure 1



Eigenvalue density of a $N \times N$ matrix in the GOE ensemble, $N=1000$.

Figure 2



Eigenvalues in the bulk of the $N \times N$ GOE matrix, $N = 1000$,

Figure 3

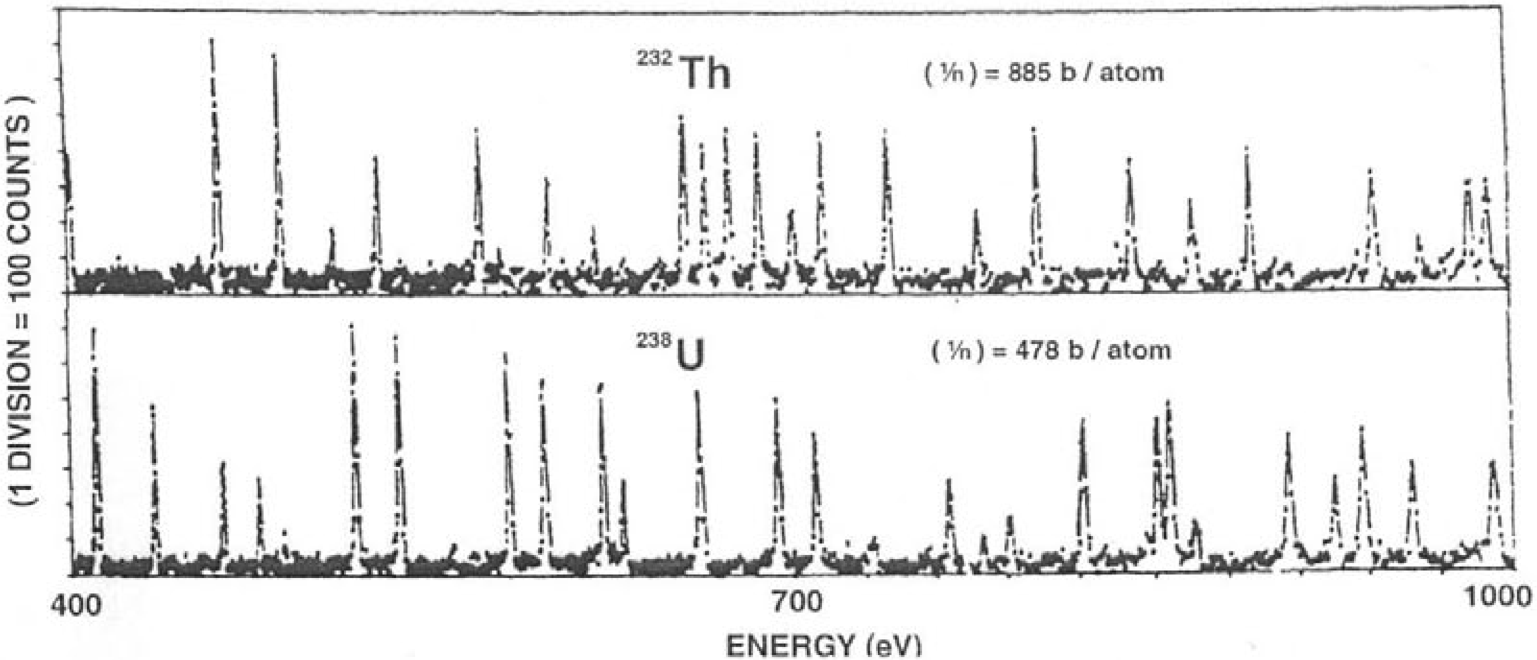


Figure 1.1. Slow neutron resonance cross-sections on thorium 232 and uranium 238 nuclei. Reprinted with permission from The American Physical Society, Rahn et al., Neutron resonance spectroscopy, X, *Phys. Rev. C* 6, 1854–1869 (1972).

. Taken from Mehta book "Random Matrices", page 2.

Figure 4.3

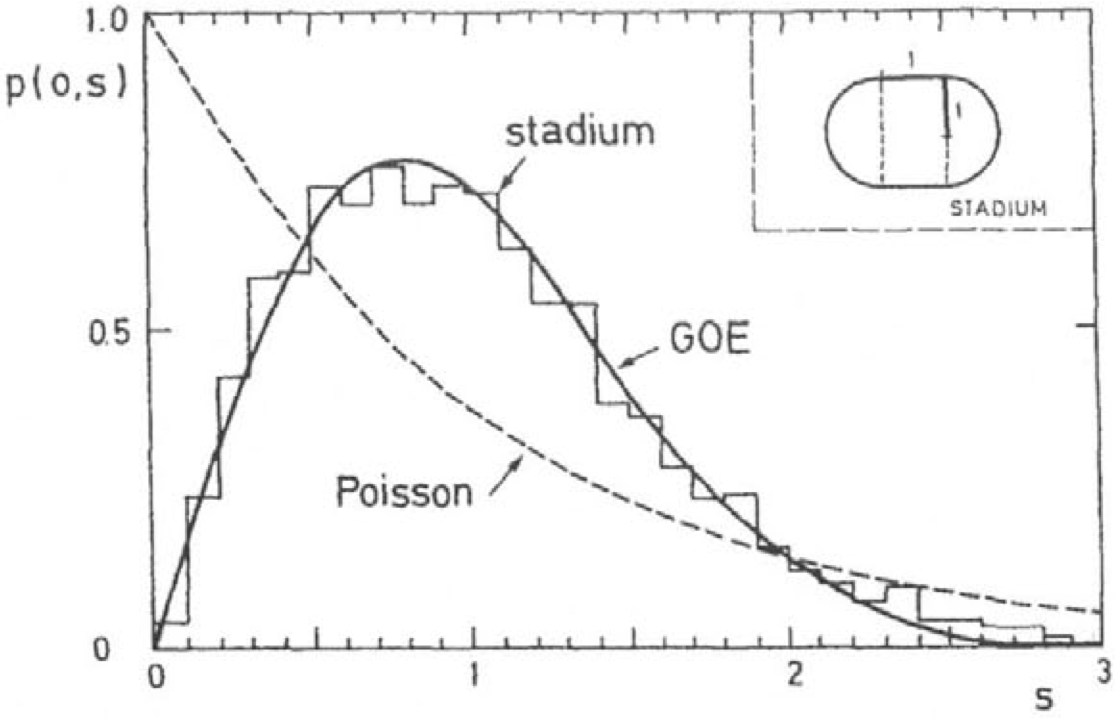


Figure 7.7. Empirical probability density of the nearest neighbor spacings of the possible energies of a particle free to move on the stadium consisting of a rectangle of size 1×2 with semi-circular caps of radius 1, depicted in the right upper corner. The stadium can be defined by the inequalities $|y| \leqslant 1$, and either $|x| \leqslant 1/2$ or $(x \pm 1/2)^2 + y^2 \leqslant 1$. The solid curve represents Eq. (7.3.19) corresponding to the Gaussian orthogonal ensemble (GOE), while the dashed curve is for the Poisson process corresponding to no correlations. Supplied by O. Bohigas, from Bohigas et al. (1984a).

. Mehta book, page 172.

Figure 4.2

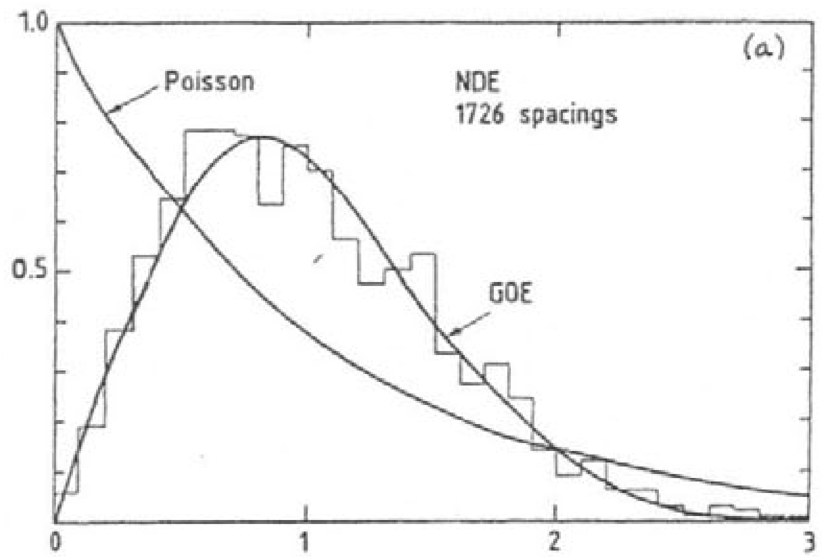


Figure 1.4. Level spacing histogram for a large set of nuclear levels, often referred to as nuclear data ensemble. The data considered consists of 1407 resonance levels belonging to 30 sequences of 27 different nuclei: (i) slow neutron resonances of Cd(110, 112, 114), Sm(152, 154), Gd(154, 156, 158, 160), Dy(160, 162, 164), Er(166, 168, 170), Yb(172, 174, 176), W(182, 184, 186), Th(232) and U(238); (1146 levels); (ii) proton resonances of Ca(44) ($J = 1/2+$), Ca(44) ($J = 1/2-$), and Ti(48) ($J = 1/2+$); (157 levels); and (iii) (n, γ)-reaction data on Hf(177) ($J = 3$), Hf(177) ($J = 4$), Hf(179) ($J = 4$), and Hf(179) ($J = 5$); (104 levels). The data chosen in each sequence is believed to be complete (no missing levels) and pure (the same angular momentum and parity). For each of the 30 sequences the average quantities (e.g. the mean spacing, spacing/mean spacing, number variance μ_2, etc., see Chapter 16) are computed separately and their aggregate is taken weighted according to the size of each sequence. The solid curves correspond to the Poisson distribution, i.e. no correlations at all, and that for the eigenvalues of a real symmetric random matrix taken from the Gaussian orthogonal ensemble (GOE). Reprinted with permission from Kluwer Academic Publishers, Bohigas O., Haq R.U. and Pandey A., Fluctuation properties of nuclear energy levels and widths, comparison of theory with experiment, in: *Nuclear Data for Science and Technology*, Bökhoff K.H. (Ed.), 809–814 (1983).

. Mehta book, page 13.

Figuro 5.1

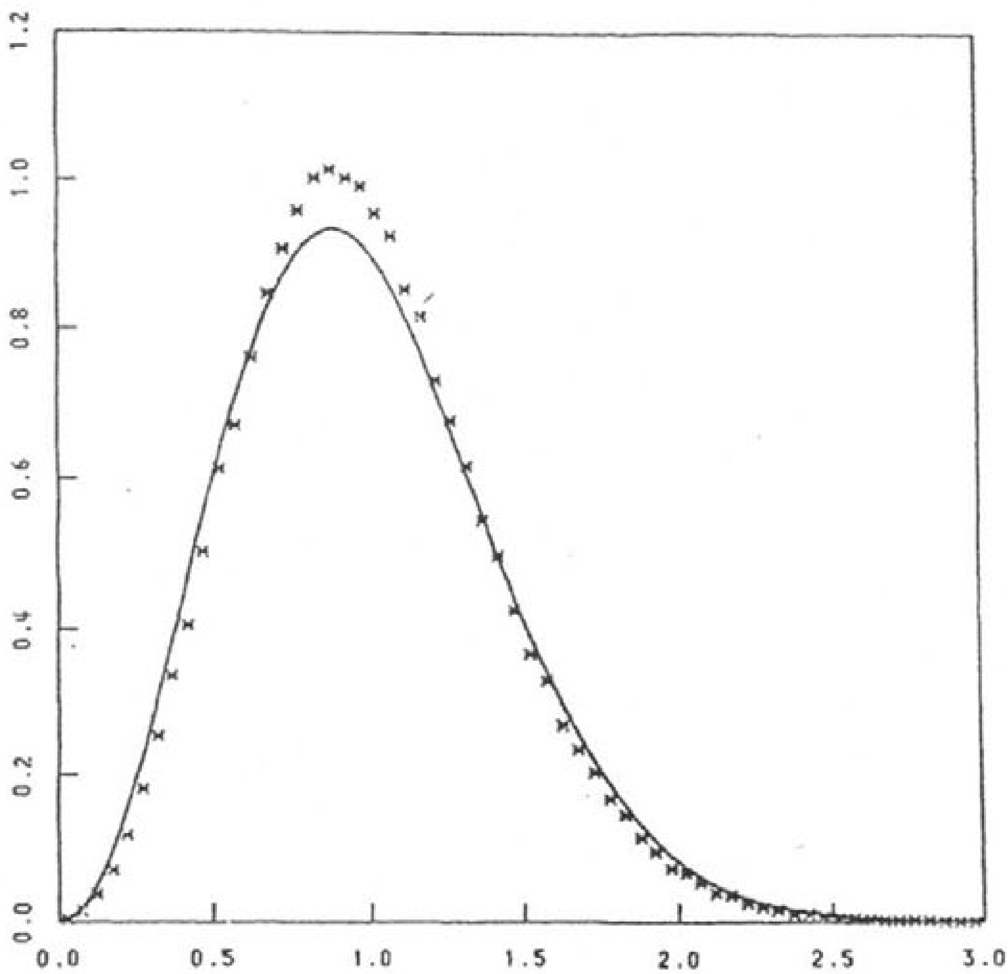


Figure 1.12. Plot of the density of normalized spacings for the zeros $0.5 \pm i\gamma_n$, γ_n real, of the Riemann zeta function on the critical line. $1 < n < 10^5$. The solid curve is the spacing probability density for the Gaussian unitary ensemble, Eq. (6.4.32). From Odlyzko (1987). Reprinted from "On the distribution of spacings between zeros of the zeta function," *Mathematics of Computation* (1987), pages 273–308, by permission of The American Mathematical Society.

. Mehta book, page 24.

Figure 5.2

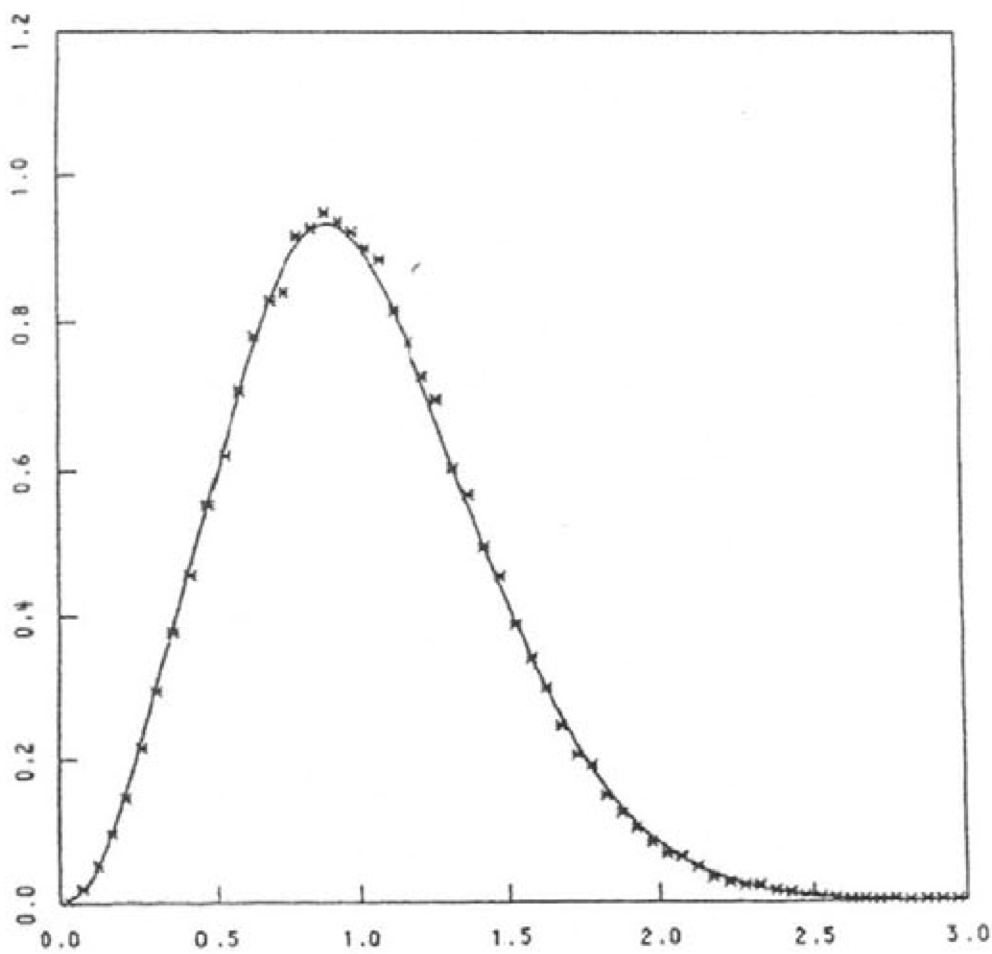


Figure 1.13. The same as Figure 1.12 with $10^{12} < n < 10^{12} + 10^5$. Note the improvement in the fit. From Odlyzko (1987). Reprinted from "On the distribution of spacings between zeros of the zeta function," *Mathematics of Computation* (1987), pages 273–308, by permission of The American Mathematical Society.

. Mehta book, page 25.

Figura 5.3

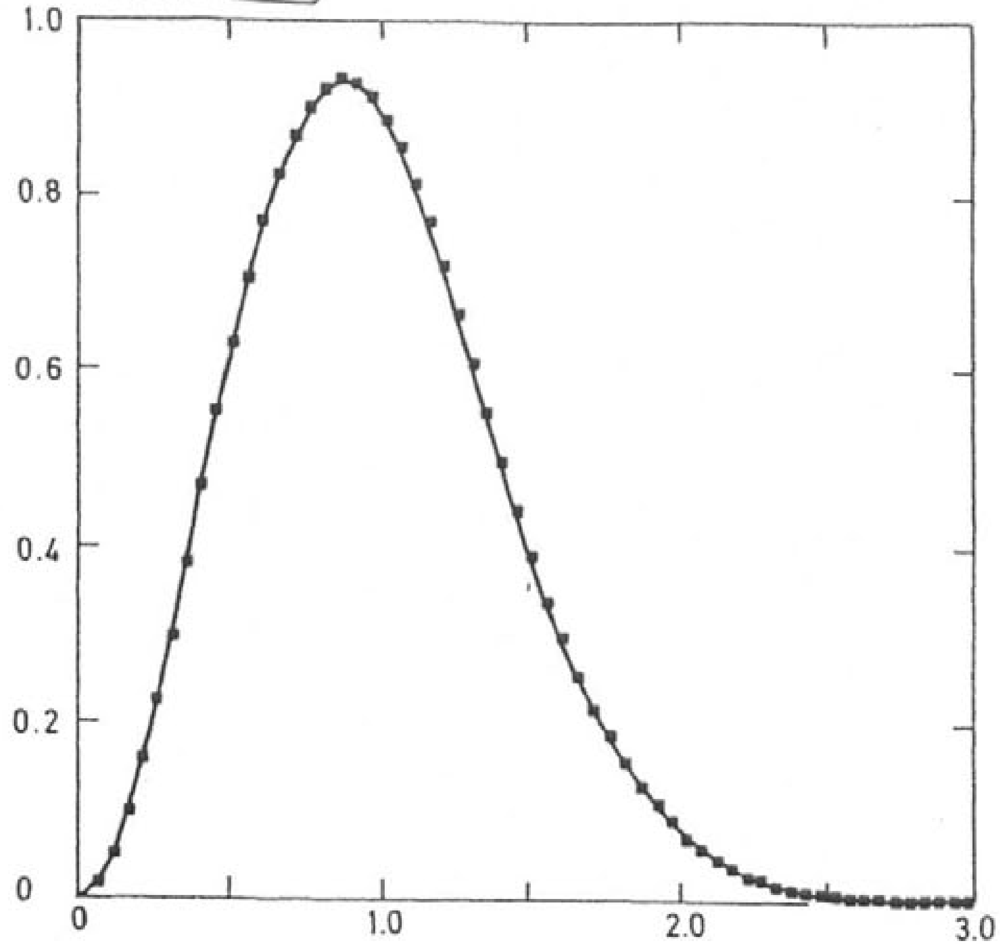


Figure 1.14. The same as Figure 1.12 but for the 79 million zeros around the 10^{20}th zero. From Odlyzko (1989).

. Mehta book, page 26.