CRC701

What is the Relation between Eigenvalues & Singular Values?

Mario Kieburg

Fakultät für Physik; Universität Bielefeld
Fakultät für Physik; Universität Duisburg-Essen

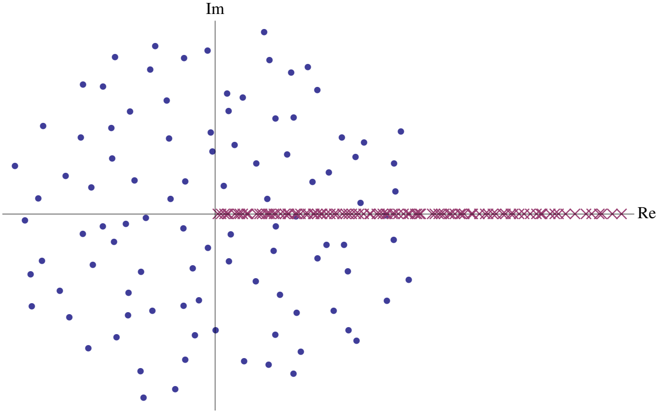
Hausdorff Center for Mathematics (Bonn)
January 12th, 2016

In Collaboration with:

Holger Kösters
Faculty of Mathematics, Bielefeld University

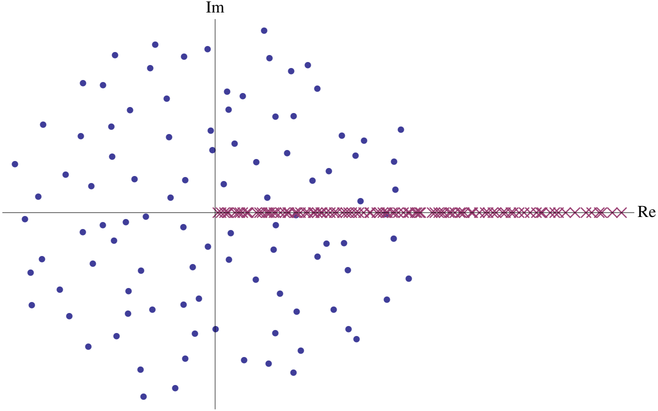
- Kieburg, Kösters: arXiv:1601.02586 [math.CA]

The Main Question in this Talk



- $z \in \mathbb{C}$ eigenvalue of $X \in \mathbb{C}^{n\times n}$ $:\Leftrightarrow \det(X - z\mathbb{1}) = 0$
- $\lambda \geq 0$ singular value of $X \in \mathbb{C}^{n\times n}$ $:\Leftrightarrow \det(X^*X - \lambda^2\mathbb{1}) = 0$

The Main Question in this Talk



When I give you the singular values of a matrix, what are its eigenvalues?

What is the relation vice versa?

Outline of this Talk

- What is known? $\longleftrightarrow$ What isn't known?
- The Answer for Bi-Unitarily Invariant Ensembles!
- Example: Polynomial Ensembles
- Idea & Results

In the most General Case

Assume ordering: $\overbrace{|z_1| \geq \ldots \geq |z_n|}^{\text{eigenvalues}}$ and $\overbrace{a_1 \geq \ldots \geq a_n}^{\text{squared singular values}}$

- determinant, the only equality:

$$\det X^*X = \prod_{j=1}^{n} |z_j|^2 = \prod_{j=1}^{n} a_j$$

- Weyl's inequalities ('49), $k = 1, \ldots, n$:

$$\prod_{j=1}^{k} |z_j|^2 \leq \prod_{j=1}^{k} a_j$$

- Horn's inequalities ('54), $k = 1, \ldots, n$:

$$\sum_{j=1}^{k} |z_j|^2 \leq \sum_{j=1}^{k} a_j$$

We have only inequalities!

Normal Matrices

X is normal $:\Leftrightarrow [X, X^*]_- = 0$

- equalities:

$$|z_j|^2 = a_j, \qquad j = 1, \ldots, n$$

- special case, Hermitian matrices:

$$z_j \in \mathbb{R} \qquad \Rightarrow \qquad a_j = z_j^2$$

- special case, unitary matrices:

$$z_j = e^{\imath \varphi_j} \in \mathbb{S}_1 \subset \mathbb{C} \qquad \Rightarrow \qquad a_j = 1$$

Inequalities become equalities!

Normal Random Matrices

A particular model (e.g. Chau, Zaboronsky ('98); Teodorescu et al. ('05); Bleher, Kuijlaars ('12)):

- $g = k^* z k$ distributed by

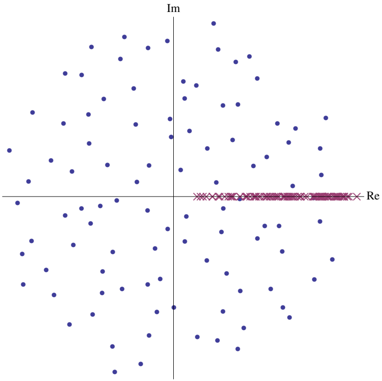
$$f_G(g)dg = f_{\text{ev}}(z)dz d^*k$$

- d^*k: Haar measure of $\mathrm{U}(n) = K$
- joint density of ev's

$$f_{\text{ev}}(z) \propto |\Delta_n(z)|^2 \prod_{j=1}^{n} \omega(|z_j|^2)|\chi(z_j)|^2$$

- χ: analytic function
- $\Delta_n(z) = \prod_{i<j}(z_j - z_i)$
- $\chi(z) = 1$, jpdf of squared sv's

$$f_{\text{sv}}(\lambda) \propto \text{Perm}[a_i^{j-1}\omega(a_i)]$$



No level repulsion for singular values!

Bi-Unitarily Invariant Random Matrices

$f_G(g) = f_G(k_1 g k_2)$, for all $g \in \mathrm{Gl}\,(n) = G$ and $k_1, k_2 \in \mathrm{U}\,(n) = K$

- Schur decomposition: $g = k^* z t k$ with
  - unitary matrix: $k \in \mathrm{U}\,(n) = K$
  - unitriangular matrix: $t \in T$
  - complex diagonal matrix: $z \in [\mathrm{Gl}\,(1)]^n = Z$

$\Rightarrow$ joint density of eigenvalues

$$f_{\mathrm{ev}}(z) \propto |\Delta_n(z)|^2 \left(\prod_{j=1}^{n} |z_j|^{2n-2j} \right) \int_T f_G(zt) dt$$

- singular value decomposition: $g^* g = k^* a k$ with
  - unitary matrix: $k \in \mathrm{U}\,(n) = K$
  - positive diagonal matrix: $a \in \mathbb{R}_+^n = A$

$\Rightarrow$ joint density of squared singular values

$$f_{\mathrm{sv}}(a) \propto |\Delta_n(a)|^2 f_G(\sqrt{a})$$

Ev's and sv's usually exhibit level repulsion!

Bi-Unitarily Invariant Random Matrices

- joint density of eigenvalues

$$f_{\mathrm{ev}}(z) \propto |\Delta_n(z)|^2 \left(\prod_{j=1}^{n} |z_j|^{2n-2j} \right) \int_T f_G(zt)dt$$

- joint density of squared singular values

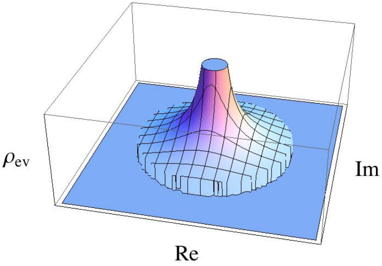
$$f_{\mathrm{sv}}(a) \propto |\Delta_n(a)|^2 f_G(\sqrt{a})$$

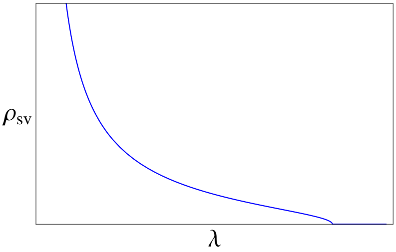
What is the relation between f_{ev} and f_{sv}?

Bi-Unitarily Invariant Random Matrices (at large matrix dimension n)

- single ring theorem (Feinberg, Zee ('97); Guionnet et al. ('11))
  bi-unitary invariance + some conditions $\Rightarrow$ connected support for radii

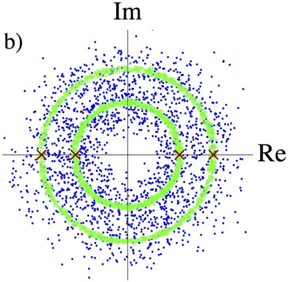
- Haagerup-Larson theorem (Haagerup, Larsen ('00), Haagerup, Schultz ('07))
  bi-unitary invariance $\Rightarrow$ bijection between level densities $\rho_{\rm ev}$ and $\rho_{\rm sv}$





Bi-Unitarily Invariant Random Matrices (Product of infinitely many matrices)

- spectral statistics of $g_1 \cdots g_M$ with g_j Bi-Unitarily invariant random matrix and $M \to \infty$ at finite n
- for particular Meijer G-ensembles: Akemann, Kieburg, Burda ('14); Ipsen ('15)
- singular values and radii of eigenvalues become deterministic at the same positions



Bi-Unitarily Invariant Random Matrices

What is the relation between $f_{\rm ev}$ and $f_{\rm sv}$ at finite n and M?

Is there a bijection between $f_{\rm ev}$ and $f_{\rm sv}$?

Example: Polynomial Ensembles

Definition:

- Let $w_0, \ldots, w_{n-1}$ and ω some functions with suitable integrability and differentiability conditions.

(a) $f_{\rm sv}$ is polynomial ensemble (Kuijlaaars et al. ('14/'15))

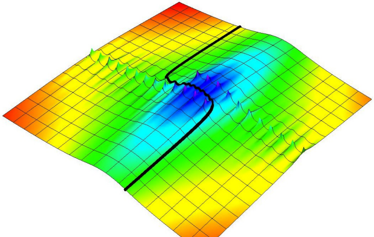
$$:\Leftrightarrow f_{\rm sv}(a) \propto \Delta_n(a) \det[w_{j-1}(a_i)]$$

(b) $f_{\rm sv}$ is polynomial ensemble of derivative type (Kieburg, Kösters ('16))

$$:\Leftrightarrow w_{j-1}(a) = (-a\partial_a)^{j-1}\,\omega(a)$$

(c) $f_{\rm sv}$ is Meijer G-ensemble (Kieburg, Kösters ('16))

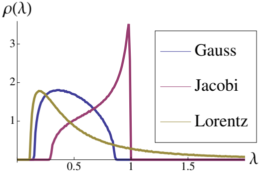
$:\Leftrightarrow \omega$ is Meijer G-function



Example: Polynomial Ensembles

Some polynomial ensembles of derivative type:

- Laguerre (χGaussian, Ginibre, Wishart) ensemble: $\omega(a) = a^{\nu} e^{-a}$
- Jacobi (truncated unitary) ensemble: $\omega(a) = a^{\nu}(1-a)^{\mu-1}\Theta(1-a)$
- Cauchy-Lorentz ensemble: $\omega(a) = a^{\nu}(1+a)^{-\nu-\mu-1}$
- products of random matrices: $\omega(a) =$Meijer G-function



- Muttalib-Borodin of Laguerre-type
  - (a) $\omega(a) = a^{\nu} e^{-\alpha a^{\theta}}$ generating $\Delta_n(a^{\theta})$
  - (b) $\theta \to 0$: $\omega(a) = a^{\nu} e^{-\alpha'(\ln a)^2}$ generating $\Delta_n(\ln a)$
  - (c) $\theta \to \infty$: $\omega(a = 1 + a'/\theta) = e^{\nu a'} e^{-\alpha e^{a'}}$ generating $\Delta_n(e^{a'})$

works also for Jacobi-type or even Cauchy-Lorentz-type

Example: Polynomial Ensembles

- Laguerre:

$$f_{\rm sv}(a) = \Delta_n^2(a)\det{}^\nu a e^{-{\rm tr}\, a} \propto \Delta_n(a)\det[(-a_i\partial_{a_i})^{j-1}\, a_i^\nu e^{-a_i}],$$

$$f_{\rm ev}(z) = |\Delta_n(z)|^2 \prod_{j=1}^{N} |z_i|^{2\nu} e^{-|z_i|^2}$$

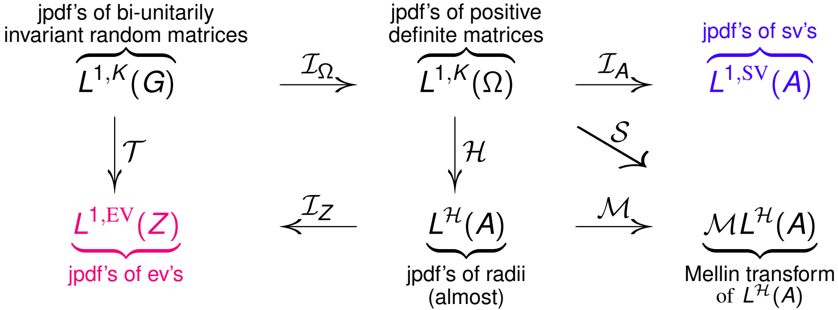
- Jacobi:

$$\begin{aligned} f_{\rm sv}(a) &= \Delta_n^2(a)\det{}^\nu a \det(1\!\!1_n - a)^{\mu-n} \\ &\propto \Delta_n(a)\det[(-a_i\partial_{a_i})^{j-1}\, a_i^\nu (1-a_i)^{\mu-1}], \\ f_{\rm ev}(z) &= |\Delta_n(z)|^2 \prod_{j=1}^{n} |z_i|^{2\nu}(1-|z_i|^2)^{\mu-1} \end{aligned}$$

- similar for other known Meijer G-ensembles

Does this simple relation hold for other ensembles as well?

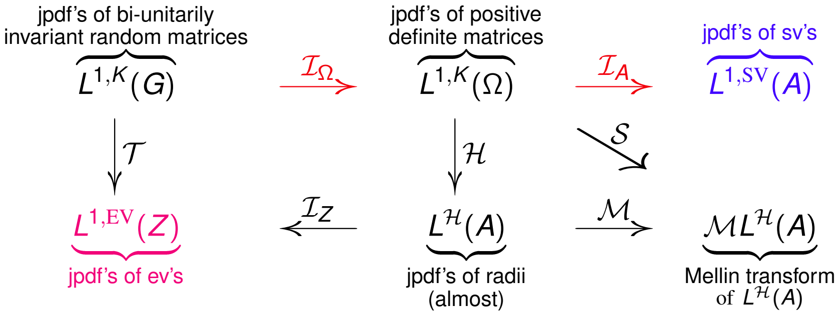
The Idea



This diagram is commutative!

All maps are linear and bijective!

The Idea



- g^*g in positive definite Hermitian matrices $y \in \Omega = \mathrm{Gl}\,(n)/\mathrm{U}\,(n)$:

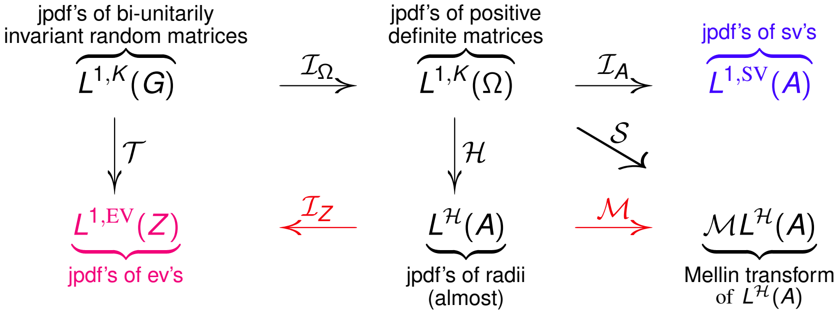
$$\mathcal{I}_\Omega f_G(y) \propto f_G(\sqrt{y}), \qquad \mathcal{I}_\Omega^{-1} f_\Omega(g) \propto f_\Omega(g^*g)$$

- positive definite diagonal matrices A in Ω:

$$\mathcal{I}_A f_\Omega(a) \propto |\Delta_n(a)|^2 f_\Omega(a), \qquad \mathcal{I}_A^{-1} f_{\mathrm{sv}}(y) \propto f_{\mathrm{sv}}(\lambda(y))/|\Delta_n(\lambda(y))|^2$$

($\lambda(y)$: ev's of $y \in \Omega$ and squared sv's of $g \in G$)

The Idea



- positive definite diagonal matrices (squared radii) A in Z:

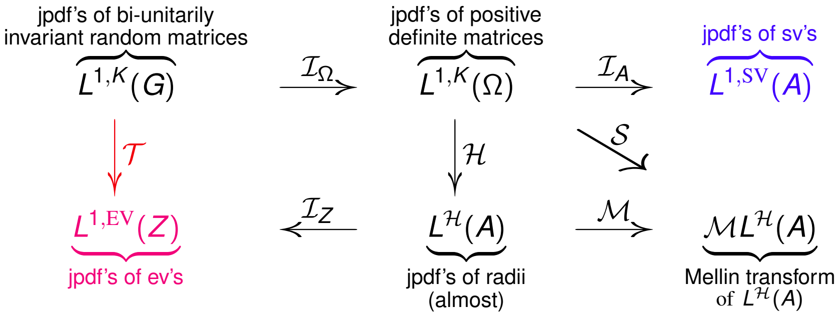
$$\mathcal{I}_Z f_A(z) \propto |\Delta_n(z)|^2 \det{}^{n-1}|z| f_A(|z|), \qquad \mathcal{I}_Z^{-1} f_Z(a) \propto \frac{\oint f_Z(\sqrt{a}\Phi) d^*\Phi}{\mathrm{Perm}\left[a_i^{(2j+n-3)/2}\right]}$$

  (Φ: complex phases of the eigenvalues)

- multivariate Mellin transform:

$$\mathcal{M} f_A(s) \propto \int \mathrm{Perm}[a_i^{s_j-1}] f_A(a) da, \quad \mathcal{M}^{-1}([\mathcal{M} f_A]; a) \propto \int \mathrm{Perm}[a_i^{-s}] \mathcal{M} f_A(s) ds$$

The Idea

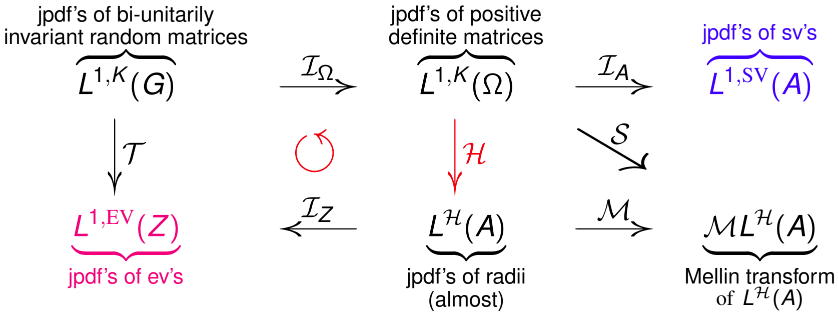


$$\mathcal{T} f_G(z) \propto |\Delta_n(z)|^2 \left(\prod_{j=1}^{n} |z_j|^{2n-2j} \right) \int_T f_G(zt)\, dt$$

This is the crucial operator we are looking for!

Bijectivity? Explicit Representation?

The Idea

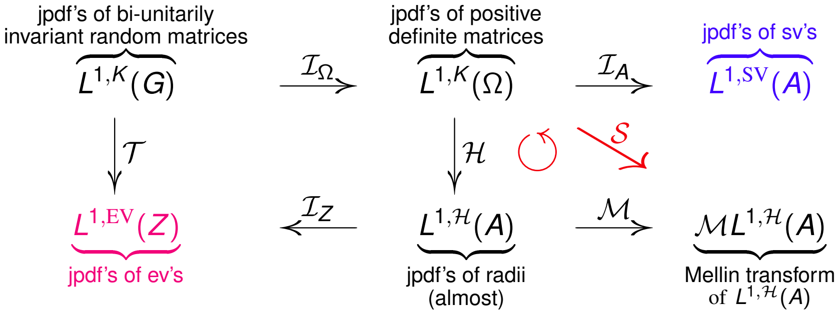


- Harish-transform (Harish-Chandra ('58))

$$\mathcal{H}f_\Omega(a) \propto \left(\prod_{j=1}^{n} a_j^{(n-2j+1)/2}\right) \int_T f_\Omega(t^* a t) dt$$

- factorization (Kieburg, Kösters ('16)): $\mathcal{H} = \mathcal{I}_Z^{-1} \mathcal{T} \mathcal{I}_\Omega^{-1}$

The Idea



- spherical-transform (Harish-Chandra ('58))

$$\mathcal{S}f_\Omega(s) \propto \int f_\Omega(y)\varphi(y,s)dy/\det^n y$$

- spherical function (Gelfand, Naĭmark ('50))

$$\varphi(y,s) \propto \frac{\det[(\lambda_i(y))^{s_j+(n-1)/2}]}{\Delta_n(s)\Delta_n(\lambda(y))}$$

- $\mathcal{S}$ is invertible (Harish-Chandra ('58))
- factorization (Harish-Chandra ('58)): $\mathcal{S} = \mathcal{M}\mathcal{H}$

Theorem: SEV-Transform $\mathcal{R}$

The SEV-(singular value-eigenvalue) transform

$$\mathcal{R} = \mathcal{T}\mathcal{I}_{\Omega}^{-1}\mathcal{I}_{A}^{-1} : \overbrace{L^{1,\mathrm{SV}}(A)}^{\text{jpdf's of sv's}} \longrightarrow \overbrace{L^{1,\mathrm{EV}}(Z)}^{\text{jpdf's of ev's}}$$

is bijective and has the explicit representation:

$$\begin{aligned} f_{\mathrm{EV}}(z) &= \mathcal{R} f_{\mathrm{SV}}(z) \\ &= \frac{\prod_{j=0}^{n-1} j!}{(n!)^2\pi^n}|\Delta_n(z)|^2 \lim_{\epsilon\to0}\int_{\mathbb{R}^n}\zeta_1(\epsilon(s-\imath\varrho'))\mathrm{Perm}\left[|z_b|^{-2(c+\imath s_c)}\right]_{b,c=1,\ldots,n} \\ &\quad\times\left(\int_A f_{\mathrm{SV}}(a)\frac{\det[a_b^{c+\imath s_c}]_{b,c=1,\ldots,n}}{\Delta_n(\varrho'+\imath s)\Delta_n(a)}\prod_{j=1}^{n}\frac{da_j}{a_j}\right)\prod_{j=1}^{n}\frac{ds_j}{2\pi} \end{aligned} \tag{3.3}$$

$$\begin{aligned} f_{\mathrm{SV}}(a) &= \mathcal{R}^{-1} f_{\mathrm{EV}}(a) \\ &= \frac{\pi^n}{(n!)^2\prod_{j=0}^{n-1} j!}\Delta_n(a)\lim_{\epsilon\to0}\int_{\mathbb{R}^n}\zeta_n\left(\epsilon\left(s-\imath\varrho'+\imath\frac{n-1}{2}\mathbb{1}_n\right)\right)\Delta_n(\varrho'+\imath s) \\ &\quad\times\det[a_b^{-c-\imath s_c}]_{b,c=1,\ldots,n}\left(\int_A\frac{\mathrm{Perm}[a'^{c+\imath s_c}_b]_{b,c=1,\ldots,n}}{\mathrm{Perm}[a'^{c-1}_b]_{b,c=1,\ldots,n}}\right. \\ &\quad\left.\times\left(\int_{[\mathrm{U}(1)]^n} f_{\mathrm{EV}}(\sqrt{a'}\Phi)\prod_{j=1}^{n}\frac{d\varphi_j}{2\pi}\right)\prod_{j=1}^{n}\frac{da'_j}{a'_j}\right) \end{aligned} \tag{3.4}$$

Please, don't try to read this!

Corollary:
Polynomial Ensembles of Derivative Type

- $f_{\rm sv}$ is polynomial ensemble of derivative type:

$$f_{\rm sv}(a)=\mathcal{R}^{-1}f_{\rm ev}(a)\propto\Delta_n(a)\det[(-a_i\partial_{a_i})^{j-1}\omega(a_i)]$$

$\Updownarrow$ **+** bi-unitary invariance of $g=k_1\sqrt{a}k_2\in G$

- $f_{\rm ev}$ has the form:

$$f_{\rm ev}(z)=\mathcal{R}f_{\rm sv}(z)\propto|\Delta_n(z)|^2\prod_{j=1}^{n}\omega(|z_j|^2)$$

Note, the arrow works in both directions!

Corollary:
Implications for the Spectral Statistics

Determinantal point process:

- joint density of squared singular values

$$f_{\rm sv}(a) = \det\left[K_{\rm sv}(a_i, a_j) = \sum_{l=0}^{n-1} p_l(a_i) q_l(a_j)\right]$$

- joint density of eigenvalues

$$f_{\rm ev}(z) = \det\left[K_{\rm ev}(z_i, \bar{z}_j) = \sqrt{\omega(|z_i|^2)\omega(|z_j|^2)} \sum_{l=0}^{n-1} \frac{(z_i \bar{z}_j)^l}{c_l}\right]$$

$\Rightarrow$**Relations:**

- polynomials: $p_l(a) = \dfrac{1}{2}\displaystyle\int_0^\infty dr \int_{-\pi}^{\pi} d\varphi (a e^{\imath\varphi} - r)^l K_{\rm ev}(\sqrt{r}, \sqrt{r} e^{-\imath\varphi})$
- weights: $q_l(a) = \dfrac{1}{2\, l!}(-\partial_a)^l \displaystyle\int_{-\pi}^{\pi} d\varphi e^{\imath l\varphi} K_{\rm ev}(\sqrt{a}, \sqrt{a} e^{-\imath\varphi})$
- kernel: $K_{\rm sv}(a_1, a_2) = \dfrac{1}{2}\partial_{a_2}^n \displaystyle\int_0^{a_2} dr \int_{-\pi}^{\pi} d\varphi (a_2 - a_1 e^{\imath\varphi})^{n-1} K_{\rm ev}(\sqrt{r}, \sqrt{r} e^{-\imath\varphi})$

Corollary: Singular Values times Unitary Matrix

- positive definite diagonal matrix $a \in A$ distributed by $f_{\mathrm{sv}} \in L^{1,\mathrm{SV}}(A)$

- considering either of the random matrices:
  (a) $g = k_1 a k_2$, with unitary matrices $k_1, k_2 \in K$ Haar distributed
  (b) $g = ak$ or $g = ka$, with unitary matrix $k \in K$ Haar distributed
  (c) $g = k_0 ak$ or $g = kak_0$, with unitary matrices $k_0 \in K$ fixed and $k \in K$ Haar distributed

$\Rightarrow$ joint density of the eigenvalues of g is

$$f_{\mathrm{ev}} = \mathcal{R} f_{\mathrm{sv}}$$

We do not need full bi-unitary invariance!

Further Results

(a) extends to signed densities and distributions

(b) generalization to deformations breaking the bi-unitary invariance

$$f_G(g) = f_G^{(K)}(g) D_G(g)$$

with $f_G^{(K)}(g) = f_G^{(K)}(k_1 g k_2)$ and $D_G(g) = D_G(g_0^{-1} g g_0)$ for all $k_1, k_2 \in K = \mathrm{U}(n)$ and $g_0, g \in G = \mathrm{Gl}(n)$

$\Rightarrow$ joint densities:

$$\begin{aligned} f_{\mathrm{ev}}(z) &= D_G(z) \mathcal{T} f_G^{(K)}(z) \\ f_{\mathrm{sv}}(a) &= |\Delta_n(a)|^2 f_G^{(K)}(\sqrt{a}) \int_K D_G(\sqrt{a}k) d^*k \end{aligned}$$

(c) products of polynomial ensembles of derivative type $\rightarrow$ semi-group

(d) semi-group action on polynomial ensembles

$\Rightarrow$ transformation law of kernels ála Claeys, Kuilaars, Wang ('15)

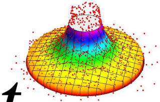
Recent developments in RMT
$\Rightarrow$ RMT enters a new Era!

image from de.best-wallpaper.net

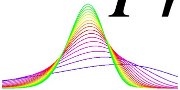
Announcement!

- **Organizers:**
  - Peter Forrester
  - Mario Kieburg
  - Roland Speicher
- **When:** August 22nd - 26th 2016 after summer school
- **Where:** ZiF next to Bielefeld University
- **Homepage:** http://www2.physik.uni-bielefeld.de/rpm_2016.html

Random



Product



Matrices

New Developments

&

Applications

Thank you for your attention!