Institute for Applied Mathematics
WS 2023/24
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Markov Processes
Sheet 12

Due on January 17, 2024
This is an additional recap-sheet and the points are not considered for the admission to the exam. Nevertheless, you are encouraged to hand in your solutions to get a feedback on your submissions.

Exercise 1 *[1000* Pt]*

Ensure to register for the exams officially via BASIS in time. The preclusive deadline is January 19. This also applies in case you only like to take the exam in March. Without registration in BASIS you cannot participate in any exam. In case of any difficulties, please directly contact the BaMa-office.
To schedule the single examination appointments, we will place a separate booking tool on eCampus. Further information on this follow after we decided about the admissions. Please regularly check the lecture webpage.

Exercise 2 *[10* Pt]*

Suppose that E is a Hilbert space with norm $\|f\| = (f,f)^{1/2}$, and L is a densely defined linear operator on E.

a) Define the *adjoint operator* $(L^*, \mathrm{Dom}(L^*))$. What does it mean that L is *self-adjoint*?

b) Show that if L is *self-adjoint* then it generates a strongly continuous contraction semigroup on E if and only if L is *negative definite*, i.e.

$$(f, Lf) \leq 0 \qquad \text{for all } f \in \mathrm{Dom}(L).$$

Remark: In this case, the semigroup generated by L is given by $P_t = e^{tL}$, where the exponential is defined by spectral theory, see e.g. Reed & Simon: Methods of modern mathematical physics, Vol. I and II.

Exercise 3 *[10* Pt]*

For $0 < c < \infty$, consider the operator G_c defined by

$$G_c f = \frac{1}{2} f'' \qquad \text{and} \qquad \mathcal{D}(G_c) = \{f \in C_0([0,\infty)) : f', f'' \in C_0([0,\infty)), f'(0) = cf''(0)\}.$$

Show that G_c is the operator of a strongly continuous contraction semigroup.
Remark: The corresponding process is called Brownian motion with sticky boundary at zero and we already know the limiting cases $c \downarrow 0$ and $c \uparrow \infty$.