**Institute for Applied Mathematics**
**WS 2023/24**
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# Markov Processes
# Sheet 11

**Due on January 10, 2023**

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With this sheet we like to get familiar with some properties of weak convergence that are collected in the Appendix (i.e. Chapter 7) of the lecture notes. Since main tools (e.g. Prohorov's Theorem) are used in many applications, it is worthwhile to study the material thoroughly.

**Exercise 1** *[10 Pt]*

Consider the setting and the notation of Section 7.3 from the lecture notes.

(i) Show that the functions $x_n = \mathbb{1}_{[1-1/n,\infty)}$ converge to the function $x_\infty = 1_{[1,\infty)}$ with respect to the metric $d$ (cf. equation (7.3.3) from the lecture notes).

(ii) For all $n \in \mathbb{N}$, let $x_n = \mathbb{1}_{[0,1/2^n)}$.

  (a) Show that $(x_n)_n$ is a Cauchy sequence with respect to the metric $d_{J_1}$ (cf. equation (7.3.17) from the lecture notes).

  (b) Show that $(x_n)_n$ does not converge with respect to the metric $d_{J_1}$.

  (c) Show that $(x_n)_n$ is not Cauchy with respect to the metric $d$.

(iii) Let $(x_n)_n$ be some sequence in $D_E[0,\infty)$ that converges to $x_\infty$ with respect to the metric $d$. Moreover suppose that $x_\infty$ is continuous. Show that then $x_n \to x_\infty$ also in the sense to uniform convergence on compact intervals.

**Exercise 2** *[10 Pt]*

Let $(\mu_n)_{n\in\mathbb{N}} \subset \mathcal{M}_1(S)$ and $\mu \in \mathcal{M}_1(S)$, where $S$ is a Polish space. Show that the following conditions are equivalent:

(i) $\mu_n \to \mu$ weakly;

(ii) $\limsup_{n\to\infty} \mu_n(F) \le \mu(F)$, for every closed $F \subseteq S$;

(iii) $\liminf_{n\to\infty} \mu_n(G) \ge \mu(G)$, for every open $G \subseteq S$;

(iv) $\lim_{n\to\infty} \mu_n(A) = \mu(A)$, for every $\mu$-continuity set $A \subseteq S$.

*Remark:* A set $A \subseteq S$ is called a $\mu$-continuity set, iff $A \in \mathcal{B}(S)$ and $\mu(\partial A) = 0$.

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**The Markov Processes team wishes you a merry Christmas and a good start into the new year!**