Institute for Applied Mathematics
WS 2023/24
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Markov Processes
Sheet 10

Due on December 20, 2023

Exercise 1 *[10 Pt]*

Consider a linear operator G that is given by

$$Gf(x) = \frac{1}{2}x(1-x)f''(x) \qquad \text{for all polynomials } f : [0,1] \to \mathbb{R}.$$

(i) Show that the closure $\bar{G}$ of G on $C([0,1])$ exists, and that $\bar{G}$ is the generator of a Feller-Dynkin semi-group $(P_t)_{t\in[0,\infty)}$.

(ii) Let $(X_t)_{t\in[0,\infty)}$ be the corresponding Markov process for $(P_t)_{t\in[0,\infty)}$ (cf. Theorem 3.34 of the lecture notes). Suppose that $(X_t)_{t\in[0,\infty)}$ has continuous paths, and let $\tau = \inf\{t \geq 0 \,|\, X_t = 0 \text{ or } X_t = 1\}$. Show that for all $x \in [0,1]$,

(a) $\mathbb{E}^x\left[\int_0^\infty X_t(1-X_t)\,dt\right] = x(1-x)$,

(b) $\mathbb{P}^x[X_\tau = 1] = x$,

(c) $\mathbb{E}^x[\tau] = -2x\log x - 2(1-x)\log(1-x)$.

Exercise 2 *[5 Pt]*

Let $u, \alpha : \mathbb{R} \to \mathbb{R}$ be bounded and twice continuously differentiable and let $f \in C^2(\mathbb{R}_+ \times \mathbb{R})$ be a bounded solution to

$$\partial_t f = \alpha \partial_x f + \frac{1}{2}\partial_x^2 f, \qquad f(0,\cdot) = u.$$

Show that

$$f(t,x) = \mathbb{E}_x\left[u(B_t)\exp\left(\int_0^t \alpha(s)\mathrm{d}B_s - \frac{1}{2}\int_0^t \alpha^2(s)\mathrm{d}s\right)\right],$$

where $(B_t)_{t\geq 0}$ is, under $\mathbb{P}_x$, Brownian motion starting in x.

Hint: You may use (without proof) the following result, which is a version of Girsanov's theorem.
Let B be a Brownian motion under $\mathbb{P}$ and define the processes

$$W_t := B_t - \int_0^t \alpha(s)\mathrm{d}s$$
$$Z_t := \exp\left(\int_0^t \alpha(s)\mathrm{d}B_s - \frac{1}{2}\int_0^t \alpha^2(s)\mathrm{d}s\right),$$

which is a martingale in the present case. Moreover define, for fixed $T \geq 0$, the measure

$$\widetilde{\mathbb{P}}(A) := \mathbb{E}[\mathbb{1}_A Z_T], \qquad A \in \mathcal{F}_T.$$

Then the process $(W_t,\ t \in [0,T])$ is a Brownian motion under $\widetilde{\mathbb{P}}$.

Exercise 3 *[5 Pt]*

Let $(B_t)_{t\geq 0}$ be a d-dimensional Brownian motion and let $(X_t)_{t\geq 0} = (X_t^{(1)}, \ldots, X_t^{(d)})_{t\geq 0}$ be a solution to the SDE

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}B_t \quad t \geq 0, \qquad X_0 = (x^{(1)}, \ldots, x^{(d)}) \in \mathbb{R}^d,$$

where we assume (for simplicity) that $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ are bounded and Lipschitz continuous. Determine, for $i, j \in \{1, \ldots, n\}$, the limits

$$\lim_{t\to 0} \frac{1}{t}\mathbb{E}\left[X_t^{(i)} - x^{(i)}\right] \qquad \text{and} \qquad \lim_{t\to 0} \frac{1}{t}\mathbb{E}\left[(X_t^{(i)} - x^{(i)})(X_t^{(j)} - x^{(j)})\right].$$