Institute for Applied Mathematics
WS 2023/24
Prof. Dr. Anton Bovier,
Manuel Esser

Markov Processes
Sheet 9

Due on December 13, 2023

Exercise 1 *[10 Pt]*

Suppose that $X(t) = (X_1(t), \dots, X_n(t))$ consists of independent Brownian motions, and let

$$Y(t) = \|X(t)\| = \sqrt{\sum_{j=1}^{n} X_j^2(t)}.$$

(i) Show that if $f \in C^2([0,\infty))$ has compact support and satisfies $f'(0) = 0$, then it belongs to the domain of the generator G of Y, and that

$$Gf(y) = \frac{1}{2} f''(y) + \beta \frac{f'(y)}{y}$$

for some choice of β, and determine β.

(ii) Find a natural martingale related to Y to show that, for $x \in D$,

$$p_{\text{out}} = \mathbb{P}_x\left[X \text{ exits } D \text{ through the outer boundary}\right] = \begin{cases} \frac{\ln|x| - \ln r}{\ln R - \ln r} & : n = 2, \\ \frac{|x|^{2-n} - r^{2-n}}{R^{2-n} - r^{2-n}} & : n \geq 3, \end{cases}$$

where $D := \{x \in \mathbb{R}^n : r < |x| < R\}$.

Remark: Processes with generators of this type are called Bessel processes - so named because one form of Bessel's differential equation can be written as $\mathcal{L}f + f = 0$.

Exercise 2 *[10 Pt]*

Let B be a one dimensional Brownian motion, $x > 0$ and let $\mu, \sigma \in \mathbb{R}$ with $\sigma \neq 0$. Define the process $(X_t)_{t \in \mathbb{R}_+}$ by

$$X_t = x\, e^{(\mu - \frac{\sigma^2}{2})t + \sigma B_t}.$$

(i) Show that X is the unique strong solution of a stochastic differential equation.

(ii) Give the precise form of a generator G such that X is a solution of the martingale problem for G.