Institute for Applied Mathematics
WS 2023/24
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Markov Processes
Solution 8

Due on December 6, 2023

Exercise 1 *[7 Pt]*

Let S be a locally compact Polish space, and let $(P_t)_{t\geq 0}$ be a Feller-Dynkin semigroup acting on the space $C_0(S)$. You may suppose that $P_t 1 = 1$ for all $t \geq 0$. We say that a probability measure μ on S is *stationary* for $(P_t)_{t\geq 0}$ if $\mu P_t = \mu$ for all $t > 0$, i.e., if

$$\int f \mathrm{d}\mu = \int P_t f \mathrm{d}\mu \qquad \text{for all } f \in C_0(S) \text{ and } t \geq 0.$$

Let $\mathcal{M}_1(S)$ denote the set of all probability measures on S, and let $\mathcal{J} \subset \mathcal{M}_1(S)$ denote the set of all stationary probability measures.

(i) Show that if $\mu = \lim_{t\to\infty} \nu P_t$ exists (in the weak sense) for some $\nu \in \mathcal{M}_1(S)$, then $\mu \in \mathcal{J}$.

(ii) Show that if $\mu = \lim_{n\to\infty} t_n^{-1} \int_0^{t_n} \nu P_t \mathrm{d}t$ exists for some $\nu \in \mathcal{M}_1(S)$ and some $t_n \uparrow \infty$, then $\mu \in \mathcal{J}$.

(iii) Suppose that S is compact. Show that $\mathcal{J}$ is a compact subset (with respect to weak convergence) of $\mathcal{M}_1(S)$.

Exercise 2 *[8 Pt]*

The semigroup of the Ornstein-Uhlenbeck process on $\mathbb{R}$ is given by

$$(P_t f)(x) \;=\; (2\pi)^{-1/2} \int f\left(\mathrm{e}^{-t} x + \sqrt{1-\mathrm{e}^{-2t}}\, y\right) \mathrm{e}^{-y^2/2}\, \mathrm{d}y \qquad \text{for } f \in C_0(\mathbb{R}).$$

(i) Let G denote the generator corresponding to $(P_t)_{t\in[0,\infty)}$. Show that

$$(Gf)(x) \;=\; f''(x) - x f'(x) \qquad \text{for any } f \in C^2(\mathbb{R}) \text{ with compact support.}$$

(ii) Show that the standard normal distribution γ is the unique stationary distribution for $(P_t)_{t\in[0,\infty)}$

Hint: You may use Exercise 1, and you may verify pointwise convergence of $P_t f(x)$ as $t \to \infty$ for fixed $f \in C_0(\mathbb{R})$ and $x \in \mathbb{R}$. The latter is also useful to show the uniqueness claim!

Exercise 3 *[5 Pt]*

Let S be a locally compact Polish space, and let G be the generator on $C_0(S)$ of a Feller-Dynkin semi-group $(P_t)_{t\in[0,\infty)}$.

(i) Let $f \in \mathcal{D}(G)$. Show that $P_t f \in \mathcal{D}(G)$ and

$$\frac{\mathrm{d}}{\mathrm{d}t} P_t f = P_t G f = G P_t f.$$

(ii) Let $f \in C_0(S)$. Show that

$$\lim_{n\to\infty} \left(1 - \frac{t}{n} G\right)^{-n} f = P_t f.$$

Hint: You may show that for $f \in \mathcal{D}(G)$,

$$\left(1 - \frac{t}{n} G\right)^{-n} f = \mathbb{E}\left[P_{\frac{t}{n}\sum_{i=1}^n \xi_i} f\right],$$

where $(\xi_i)_i$ are i.i.d. exponentially distributed random variables with mean 1.