Institute for Applied Mathematics
WS 2023/24
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Markov Processes
Sheet 7

Due on November 29, 2023

Exercise 1 *[7 Pt]*

Let $(X_t)_{t\geq 0}$ be a Brownian motion started in $X_0 = 0$. Our aim is to apply the duality method established in Chapter 3.5.2 of the lecture notes to compute, for $p \in \mathbb{N}$, the moments $m_p = \mathbb{E}_x[(X_t)^p]$. To this end:

(i) Find proper functions $f : \mathbb{R}^2 \to \mathbb{R}$ and $\beta : \mathbb{Z} \to \mathbb{R}$ and an integer-valued process $(Y_t)_{t\geq 0}$ such that X and Y are dual with respect to $(f, 0, \beta)$.

(ii) Construct the process Y explicitly using jump-times.

(iii) Consider the event when the dual process can give a contribution to the duality equation and compute the moments m_p.

Exercise 2 *[7 Pt]*

Let $(X_r(t),\, t \geq 0)$ be the reflected Brownian motion with generator

$$G_r f = \frac{1}{2} f'' \qquad \text{and} \qquad \mathcal{D}(G_r) = \{f \in C_0([0,\infty)) : f', f'' \in C_0([0,\infty)), f'(0) = 0\}$$

and let $(X_a(t),\, t \geq 0)$ be the absorbed Brownian motion with generator

$$G_a f = \frac{1}{2} f'' \qquad \text{and} \qquad \mathcal{D}(G_a) = \{f \in C_0([0,\infty)) : f', f'' \in C_0([0,\infty)), f''(0) = 0\}\,.$$

Moreover, take and odd function $g \in C^2(\mathbb{R}) \cap C_0(\mathbb{R})$, i.e. $g(-z) = -g(z)$, and define $f(x, y) = g(x + y) + g(x - y)$.

(i) Show that the processes X_r and X_a are dual with respect to $(f, 0, 0)$.

(ii) Use this duality to show that

$$\mathbb{P}\left(X_a(t) > y | X_a(0) = x\right) = \mathbb{P}\left(X_r(t) < x | X_r(0) = y\right).$$

Hint: Represent the function $f(x, y) = \mathbb{1}_{x>y}$ with proper g. Then use an approximation to ensure $g_n \in C^2(\mathbb{R}) \cap C_0(\mathbb{R})$.

Exercise 3 *[6 Pt]*

Consider a Feller process X on $\mathbb{R}$, whose generator is given by

$$Gf = \frac{1}{2}f'' - f'$$

for C^2-functions f with compact support. For all $b \in \mathbb{R}$ let $\tau_b = \inf\{t \geq 0 \,|\, X_t = b\}$. You may suppose that $\tau_0 < \infty$ a.s. (Dont need that really. Rather need that paths are continuous.)

(i) Use the martingale problem to show that $\lim_{b\to\infty} b\,\mathbb{P}^x(\tau_b < \tau_0) = 0$ for $x > 0$.

(iii) Use the martingale problem to compute $\mathbb{E}_x\tau_0$ for $x > 0$.