Institute for Applied Mathematics
WS 2023/24
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Markov Processes
Sheet 6

Due on November 22, 2023

Exercise 1 *[4 Pt]*

Let (S, d) be a Polish space. Let X and Y be S-valued random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω is assumed to be a Polish space. Furthermore, let $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra of $\mathcal{F}$. Suppose that $M \subset C_b(S)$ is separating and countable and that

$$\mathbb{E}[f(X)|\mathcal{G}] = f(Y) \qquad \text{a.s. for all } f \in M.$$

Show that $X = Y$ a.s.

Definition. A set $M \subset C_b(S)$ is called *separating* if whenever μ and ν are probability measures on S and

$$\int_S f\, d\mu = \int_S f\, d\nu \qquad \text{for all } f \in M,$$

then $\mu = \nu$.

Exercise 2 *[8 Pt]*

Suppose $c : \mathbb{R} \to (0, \infty)$ is strictly positive and continuous.

(i) Consider a Feller-Dynkin Markov process whose generator, when restricted to C^2-functions with compact support, is given by

$$Gf(x) = \frac{1}{2}c(x)f''(x).$$

By applying the martingale problem to an appropriate function, show that if $a < x < b$ and τ is the hitting time of $\{a, b\}$, then

$$\mathbb{E}_x\tau = \int_a^b \frac{2}{c(z)} \frac{(x \wedge z - a)(b - x \vee z)}{b - a} dz.$$

Remark. You may suppose that $\tau < \infty$ a.s. (Dont need that really. Rather need that paths are continuous.)

(ii) Use part (i) to show that if τ is the hitting time of 0, then for $x > 0$, $\mathbb{E}_x\tau < \infty$ if and only if $\int_0^\infty \frac{1}{c(z)}dz < \infty$.

Exercise 3 *[8 Pt]*

Let B be the one-dimensional Brownian motion, and let τ be the hitting time of 0.

(i) Let the process X_a be defined by

$$X_a(t) = \begin{cases} B(t) & : t < \tau, \\ 0 & : t \geq \tau. \end{cases}$$

Show that the generator $(G_a, D(G_a))$ of X_a is given by

$$G_a f = \frac{1}{2}f'' \qquad \text{and} \qquad \mathcal{D}(Ga) = \{f \in C_0([0,\infty)) : f', f'' \in C_0([0,\infty)), f''(0) = 0\}.$$

(ii) Define the operator $(G, D(G))$ by

$$Gf = \frac{1}{2}f'' \qquad \text{and} \qquad \mathcal{D}(G) = \{f \in C_0([0,\infty)) : f', f'' \in C_0([0,\infty)), f'(0) = f''(0) = 0\}.$$

Show that $(G, D(G))$ is not the generator of a Feller-Dynkin semi-group.

Hint: Recall what was the generator of $X_r(t) = |B(t)|,\ t \geq 0$.