

Markov Processes Sheet 5

Due on November 15, 2023

Exercise 1

[4 Pt]

A linear operator $\mathcal{L} : \mathcal{D}(\mathcal{L}) \subset B \rightarrow B$ on a Banach space B is called closable if the closure of its graph $\{(f, \mathcal{L}f) : f \in \mathcal{D}(\mathcal{L})\}$ is again the graph of a linear operator.

(i) Show that the operator \mathcal{L} is closable if and only if

$$\forall (f_n)_{n \in \mathbb{N}} \subset \mathcal{D}(\mathcal{L}) : f_n \rightarrow 0, \mathcal{L}f_n \rightarrow g \implies g = 0$$

(ii) Give an example of a linear operator that is not closable.

Exercise 2

[6 Pt]

Let the family $\{\mathbb{P}_x\}_{x \in \mathbb{R}}$ of probability measures be such that

- for $x \neq 0$, \mathbb{P}_x denotes the law of the one-dimensional Brownian motion starting from x , and
- \mathbb{P}_0 is the law of the process, which is constant 0, i.e. \mathbb{P}_0 is a Dirac measure.

Let X be a process with law $\{\mathbb{P}_x\}_{x \in \mathbb{R}}$.

a) Show that X satisfies the Markov property, i.e. for all bounded random variables η ,

$$\mathbb{E}_x[\theta_t \eta | \mathfrak{F}_t] = \mathbb{E}_{X_t}[\eta] \quad \mathbb{P}_x - \text{a.s.}$$

b) Show that X does not satisfy the strong Markov property.

Additionally: Can X be a Feller-process? Can we see this also from the properties of the induced semigroup $(P_t)_{t \geq 0}$?

Exercise 3

[10 Pt]

Let $\{\mathbb{P}_x\}_{x \in \mathbb{R}}$ be the law of the one-dimensional Brownian motion B . Let $0 < b < a$ and let $t \geq 0$.

a) Use the strong Markov property to show that

$$\mathbb{P}_0 \left[\sup_{0 \leq s \leq t} B_s > a, B_t < b \right] = \mathbb{P}_0 [B_t > 2a - b].$$

b) Show that $\sup_{0 \leq s \leq t} B_s$ and $|B_t|$ have the same distribution.

c) Show that $\sup_{0 \leq s \leq t} B_s - B_t$ and $|B_t|$ have the same distribution.

d) Let $\tau = \inf\{t \geq 0 \mid B_t = 0\}$ and let $x \neq 0$. Show that

$$\mathbb{P}_x [\tau < t] = \int_0^t \frac{|x|}{\sqrt{2\pi z^3}} e^{-\frac{x^2}{2z}} dz.$$