Institute for Applied Mathematics
WS 2023/24
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Markov Processes
Sheet 4

Due on November 8, 2023

Exercise 1 *[5 Pt]*

Define the family of operators $P_t, t \in \mathbb{R}_+$, on $C_0(\mathbb{R})$ by $P_t f(x) = f(x+t)$. Show that $P_t, t \in \mathbb{R}_+$, is a strongly continuous contraction semigroup and that its generator $(G, \mathcal{D}(G))$ is given by

$$Gf = f' \qquad \text{and} \qquad \mathcal{D}(G) = \{f \in C_0(\mathbb{R}) : f' \in C_0(\mathbb{R})\}.$$

Exercise 2 *[5 Pt]*

Let B be a one-dimensional Brownian motion and define the so-called *reflected Brownian motion* X by

$$X_t := |B_t|, \quad \forall t \geq 0.$$

Define the family, $P_t, t \in \mathbb{R}_+$, of operators by $P_t f(x) := \mathbb{E}^x[f(X_t)]$ for $f \in C_0(\mathbb{R})$. Show that its generator $(G, \mathcal{D}(G))$ is given by

$$Gf = \frac{1}{2} f'' \qquad \text{and} \qquad \mathcal{D}(G) = \{f \in C_0([0,\infty)) : f', f'' \in C_0([0,\infty)), f'(0) = 0\}.$$

Exercise 3 *[5 Pt]*

Let G be the linear operator on $C_0(\mathbb{R})$ defined by

$$Gf = f''' \qquad \text{and} \qquad \mathcal{D}(G) = \{f \in C_0(\mathbb{R}) : f', f'', f''' \in C_0(\mathbb{R})\}.$$

Show that G is *not* the generator of a strongly continuous contraction semi-group.
Hint: You may show that Theorem 3.20 (ii) is contradicted if you suppose that G would be the generator of a strongly continuous contraction semi-group.

Exercise 4 *[5 Pt]*

Let G be a closed operator on a Banach space B_0, and let $\rho(G)$ be its resolvent set. Let $\lambda, \mu \in \rho(G)$. Show that the corresponding resolvents $R_\lambda = (\lambda - G)^{-1}$ and $R_\mu = (\mu - G)^{-1}$ satisfy the resolvent identity.