Institute for Applied Mathematics
WS 2023/24
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Markov Processes
Sheet 3

Due on November 1, 2023

Exercise 1 *[5+5+5 Pt]*

Let X be a one-dimensional Brownian motion. Define the family, $P_t, t \in \mathbb{R}_+$, of operators by $P_t f(x) := \mathbb{E}^x[f(X_t)]$ for $f \in B(\mathbb{R}, \mathbb{R})$.

(i) Show that $P_t, t \in \mathbb{R}_+$, is an honest, normal Markov semi group.

(ii) Recall the definition of $C_0(\mathbb{R})$ (see page 29 from the lecture notes). We equip $C_0(\mathbb{R})$ with the supremum norm, which turns it into a Banach space. Show that $P_t, t \in \mathbb{R}_+$, is a strongly continuous contraction semi group on $C_0(\mathbb{R})$.

Hint: Note that functions in $C_0(\mathbb{R})$ are uniformly continuous.

(iii) (Difficult!) Explain why the statement of (ii) is not true if $C_0(\mathbb{R})$ would be replaced by $C_b(\mathbb{R})$ (without the requirement of vanishing at infinity).

Hint: You may consider the function

$$f(x) = \begin{cases} |x - 2Kn|n & : x \in [2Kn - 1/n, 2Kn + 1/n] \text{ for some } n \in \mathbb{N}, \\ 1 & : \text{else}, \end{cases}$$

where $K \in \mathbb{R}_+$ is a suitably chosen constant!

Exercise 2 *[5 Pt]*

Let $(X_t)_{t\in\mathbb{R}_+}$ be a Markov jump process constructed as in Section 3.1 of the lecture notes. Suppose that the state space S is finite and equipped with the discrete topology. Define the family, $P_t, t \in \mathbb{R}_+$, of operators on $B(S, \mathbb{R})$ by $P_t f(x) := \mathbb{E}^x[f(X_t)]$ for $f \in B(S, \mathbb{R})$. Show that $P_t, t \in \mathbb{R}_+$, is an honest, normal and strongly continuous contraction Markov semi group.