Institute for Applied Mathematics
WS 2023/24
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Markov Processes
Sheet 2

Due on October 25, 2023

In the following exercises we consider the same setting as in Section 3.1 of the lecture notes. More precisely, we consider a Markov jump process $(X_t)_{t\in\mathbb{R}_+}$ on a finite state space S with generator G and transition kernel P_t. The holding times $(m(x))_{x\in S}$ satisfy the same boundedness assumption as in Section 3.1. Additionally, we define for $x, y \in S$, the transition probabilities by

$$p_t(x,y) = P_t(x,\{y\}).$$

Exercise 1 *[4 Pt]*

Let ϕ be a bounded function on S. State an explicit solution $u(t,x)$ of the discrete heat equation given by

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t,x) = (Gu)(t,x), \qquad u(0,x) = \phi(x).$$

Exercise 2 *[6 Pt]*

Suppose that $S = \{0,1\}$. Let $\beta, \delta \geq 0$ and $\beta + \delta > 0$. Let for $f : S \to \mathbb{R}$, Gf be given by

$$\begin{pmatrix} Gf(0) \\ Gf(1) \end{pmatrix} = \begin{pmatrix} -\beta & +\beta \\ +\delta & -\delta \end{pmatrix} \begin{pmatrix} f(0) \\ f(1) \end{pmatrix}.$$

1. Show that $-G(0,\{0\}) = G(0,\{1\}) = \beta$ and $-G(1,\{1\}) = G(1,\{0\}) = \delta$.

2. Show that the corresponding transition probabilities are

$$\begin{array}{ll} p_t(0,0) = \frac{\delta}{\beta+\delta} + \frac{\beta}{\beta+\delta}e^{-t(\beta+\delta)}\ , & p_t(0,1) = \frac{\beta}{\beta+\delta}\left[1 - e^{-t(\beta+\delta)}\right], \\ p_t(1,1) = \frac{\beta}{\beta+\delta} + \frac{\delta}{\beta+\delta}e^{-t(\beta+\delta)}\ , & p_t(1,0) = \frac{\delta}{\beta+\delta}\left[1 - e^{-t(\beta+\delta)}\right]. \end{array} \qquad (1)$$

3. Show directly that the transition probabilities given in (1) satisfy the Chapman-Kolmogorov equations and that

$$\frac{\mathrm{d}}{\mathrm{d}t}p_t(x,y)|_{t=0} = G(x,\{y\}).$$

Exercise 3 *[10 Pt]*

Consider the same setting as in Exercise 2.

(i) Use Lemma 3.3 from the lecture notes to show that

– $p_t(0,1) = \int_0^t \beta e^{-\beta(t-s)} p_s(1,1)\mathrm{d}s$,

– $p_t(1,1) = e^{-\delta t} + \int_0^t \delta e^{-\delta(t-s)} p_s(0,1)\mathrm{d}s$.

(ii) For $\lambda > 0$, let $\mathcal{M}p_t(0,1)(\lambda)$ and $\mathcal{M}p_t(1,1)(\lambda)$ denote the Laplace transforms of $p_t(0,1)$ and $p_t(1,1)$ respectively. Show that the following identity holds:

$$\mathcal{M}p_t(0,1)(\lambda) = \frac{\beta}{\beta+\lambda}\mathcal{M}p_t(1,1)(\lambda).$$

(iii) Use (i) and (ii) to show that $p_t(0,1) = \frac{\beta}{\beta+\delta}\left[1 - e^{-t(\beta+\delta)}\right]$.