Institute for Applied Mathematics
WS 2023/24
Prof. Dr. Anton Bovier,
Manuel Esser

Markov Processes
Sheet 1

Due on October 18, 2023

Exercise 1 *[5 Pt]*

Let X_n, $n \in \mathbb{N}$ be i.i.d random variables with $\mathbb{P}(X_1 = 1) = 1/2 = \mathbb{P}(X_1 = -1)$. Let $S_0 = 0$ and $S_n = \sum_{k=1}^n X_k$ for $n \in \mathbb{N}$.

1. Show that $\{S_n\}_{n \in \mathbb{N}_0}$ is a Markov process.
2. Compute the generator of S_n.
3. Use the martingale problem to show that $\{\frac{1}{3}S_n^3 - \sum_{l=1}^n S_l\}_{n \in \mathbb{N}}$ is a martingale.

Exercise 2 *[6 Pt]*

Suppose that $P(x, dy)$ is a transition kernel on a measurable state space $(S, \mathcal{B})$, and μ is an invariant measure w.r.t. P. We say that μ satisfies the *detailed balance condition w.r.t. P* iff

$$\int\int \mu(dx)\, P(x, dy)\, f(x, y) \;=\; \int\int \mu(dy)\, P(y, dx)\, f(x, y) \quad \text{for all measurable } f : S \times S \to \mathbb{R}_+.$$

a) Show that a measure that satisfies the detailed balance condition is invariant.

b) Suppose that (X_n) is a stationary Markov chain with one step transition kernel P and with initial distribution μ. Show that $X_n \sim \mu$ for all $n \geq 0$.

c) Now let $p \in (0, 1)$, and consider a Markov chain with state space $\mathbb{Z}_+$ and transition probabilities $P(x, x + 1) = p$ for $x \geq 0$, $P(x, x - 1) = q := 1 - p$ for $x \geq 1$, and $P(0, 0) = q$.

   (i) Find a nontrivial invariant measure.
   (ii) Show that if $p < q$ then there is a unique invariant probability measure.
   (iii) Show that if $p \geq q$ then an invariant probability measure does not exist.

Exercise 3 *[5 Pt]*

A process $(X_n)_{n\in\mathbb{N}}$ is called predictable w.r.t. a filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$ if X_n is measurable w.r.t. $\mathcal{F}_{n-1}$ for any $n \in \mathbb{N}$. Show that:

a) A predictable, discrete-time martingale is almost surely constant.

b) For a nonnegative martingale $(X_n)_{n\in\mathbb{N}}$ we have almost surely:

$$X_n(\omega) = 0 \quad \Rightarrow \quad X_{n+k}(\omega) = 0 \text{ for all } k \geq 0 \,.$$

Exercise 4 *[4 Pt]*

Let $p \in (\frac{1}{2}, 1)$, and consider a Markov chain $(X_n)_{n\in\mathbb{N}}$ with state space $\mathbb{Z}$ and transition probabilities $P(x, x+1) = p$ and $P(x, x-1) = q := 1 - p$ for $x \in \mathbb{Z}$. Let

$$u(x) \;:=\; E_x\left[\sum_{n=0}^{\infty} a^{X_n}\right] , \qquad a > 0 \,.$$

Show that $u(x+1) \;=\; a \cdot u(x)$.