Institute for Applied Mathematics
WS 2023/24
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Markov Processes
Sheet 0

These exercises will be discussed in the first tutorial. You are encouraged to study them in advance but you do not have to submit any solutions.

Exercise 1 *[0 Pt]*

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $\mathcal{G} \subseteq \mathcal{F}$ a sub-σ-algebra, and $X : \Omega \to \mathbb{R}_+$ a non-negative random variable.

1. Define the conditional expectation $\mathbb{E}[X|\mathcal{G}]$.
2. Suppose that there exists a decomposition of Ω into disjoint sets $\mathcal{A}_1, \dots, A_n$ such that $\mathcal{G} = \sigma(\{\mathcal{A}_1, \dots, A_n\})$. Show that
$$\mathbb{E}[X|\mathcal{G}] = \sum_{i:\mathbb{P}[A_i]>0} \mathbb{E}[X|A_i] \mathbb{1}_{A_i}$$
is a version of the conditional expectation of X given $\mathcal{G}$.

Exercise 2 *[0 Pt]*

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

a) Give the definition (or an equivalent characterization) of a family of uniformly integrable random variables.

b) Let $\mathcal{C}$ be a family of random variables on the given probability space. Moreover, let $g : [0, \infty) \to (0, \infty)$ be such that $g(r)/r \to \infty$ as $r \to \infty$ and assume that $\sup_{X \in \mathcal{C}} \mathbb{E}[g(|X|)] < \infty$. Show that $\mathcal{C}$ is uniformly integrable.

c) Let $(X_n)_{n\in\mathbb{N}}$ be a martingale adapted to the filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$. Let T be a stopping time such that $\mathbb{E}[|X_T|] < \infty$ and $\lim_{n\to\infty} E[|X_n|\mathbb{1}_{T>n}] = 0$. Show that $(X_{T\wedge n})_{n\in\mathbb{N}}$ is uniformly integrable.

Exercise 3 *[0 Pt]*

The Wright-Fisher model describes the evolution of a population of individuals with phenotype A or B. The total population size at each generation is kept constant and is equal to N. The Wright-Fisher model is a stochastic process $(X_n)_{n\in\mathbb{N}_0}$ with state space $S = \{0, \dots, N\}$, where

$$X_n = \text{number of individuals of type A in the nth generation.}$$

Thus the number of individuals of type B is just $N - X_n$. In this model, the evolution is given as follows. Given a population at time n, each of the descendants (population at time $n + 1$) takes the type from a randomly (with a uniform distribution) chosen parent of the nth generation. Show that

1. $(X_n)_{n\in\mathbb{N}}$ is a Markov Chain and compute the transition probability,
2. $(X_n)_{n\in\mathbb{N}}$ is a martingale,
3. the states 0 and N are absorbing,
4. $\mathbb{E}_x[\tau_{0,N}] < \infty$, for all $x \in 0, \dots, N$.
5. Compute (using e.g. Doob's Optional Stopping Theorem) the probability that the process started at $x \in 0, \dots, N$ is absorbed in N (resp. in 0).

Exercise 4 *[0 Pt]*

Please ensure you have registered for both the course and for the exercise classes on ecampus. If you want to attend the exams, do not forget to register on BASIS in time. For any questions regarding the organization of the lecture, write an email to manuel.esser@uni-bonn.de