Portfolio Optimization and Martingale Measures.

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Abstract.
The paper studies connections between risk aversion and martingale measures in a discrete–time incomplete financial market. An investor is considered whose attitude towards risk is specified in terms of the index b of constant proportional risk aversion. Then dynamic portfolios are admissible if the terminal wealth is positive. It is assumed that the return (risk) processes are bounded. Sufficient (and nearly necessary) conditions are given for the existence of an optimal dynamic portfolio which chooses portfolios from the interior of the set of admissible portfolios. This property leads to an equivalent martingale measure defined through the optimal dynamic portfolio and the index $0 < b \le 1$. Moreover, the option pricing formula of Davis is given by this martingale measure. In case of b = 1, i.e. in case of the log–utility, the optimal dynamic portfolio defines the numeraire portfolio.

Abbreviated title: Portfolios and martingales.

Key words: incomplete financial market, martingale measure, optimal dynamic portfolio, risk aversion, numeraire portfolio, dynamic programming.

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§1 Introduction and Summary.

In discrete time t = 0,1,...,T, an incomplete financial market is studied which is free of arbitrage opportunities but incomplete. Then one has several choices of equivalent martingale measures. It is known (see Harrison & Kreps 1979, Harrison & Pliska 1981) that each martingale measure corresponds to a consistent price system. Thus in incomplete markets, no preference independent pricing of contingent claims is possible.

In the market 1+d assets can be traded. One of them with price process $\{B_t, 0\le t\le T\}$ is called the bond and serves as a numeraire. The other d assets are called stocks and are described by the d–dimensional price process $\{S_t, 0\le t\le T\}$. An investor is considered whose attitude towards risk is specified in terms of a utility function $U(x)) = x^{1-b}/(1-b)$ for $0<b<1$ and $U(x) = \ln x$ for b=1, then $U'(x) = x^{-b}$, $0<b\le 1$. There $b = -x\cdot U''(x)/U'(x)$ is the measure of constant proportional risk aversion. The utility being defined only for a nonnegative wealth, a dynamic portfolio is called admissible if its (self–financed) terminal wealth $V_T(x)$ depending on the initial wealth x is nonnegative. It is convenient to specify a dynamic portfolio by a portfolio process π where the invested quantities are given in terms of percentages. Then one has the useful relation

$V_T^{\pi}(x) = x \cdot V_T^{\pi}(1)$. The investor's objective is to maximize the expected utility of terminal wealth and the maximum utility is given by

(1.1) $$J(x) = \sup_{\pi} E\left[U(V_T^{\pi}(x))\right] = E\left[U(V_T^{\pi^*}(x))\right]$$

where the solution π^* to the optimization problem is independent of x and J is of the same form as U. Utility optimization is now a classical subject. However, here we are interested in optimal portfolios which can be chosen from the interior of the set of admissible portfolios. Hakansson (1971) has such a result under the assumption that the underlying space is finite and $U(0+) = -\infty$ as in the case b=1. Here we present a result which makes use of $U'(0+) = \infty$, thus including all cases $0 < b \leq 1$.

Optimal 'interior' portfolios can be used to define equivalent martingale measures. Since the martingale property is a local one both in time and in space, martingale measures can be constructed by local optimization problems. This was done by Rogers (1994) in order to prove the existence of a martingale measure under the no–arbitrage condition. The latter condition naturally plays an important role here, too. It was already used in a similar form by Hakansson (1971) under the name "no–easy–money–condition". However, here we apply global optimization to obtain an equivalent martingale measure Q^b by

(1.2) $$dQ^b = \text{const} \cdot B_T \cdot (V_T^{\pi^*}(1))^{-b} = \text{const} \cdot B_T \cdot U'(V_T^{\pi^*}(1)) \quad \text{where } \pi^* \text{ is optimal.}$$

Grandits (1999) considers martingale measures of the form (1.2) where b<0. For a proof of the existence of an optimal dynamic portfolio, the classical techniques of discrete–time dynamic programming are used here. When applying the martingale method for portfolio optimization, the assumption $U'(0+) = \infty$ is also used in order to define the inverse of U' (see Karatzas & Kou 1996, Davis 1997). In the case of the log–utility, π^* the the numeraire portfolio (Long 1990).

Davis (1997) explains how to base option pricing on an optimal dynamic portfolio by use of a "marginal rate of substitution" argument. One of the main results of the present paper is that in our situation the option pricing formula of Davis is given by a martingale measure, namely Q^b as specified by (1.2). A similar result can be found in Karatzas & Kou (1996, Theorem 7.4).

§2 The financial market model.

The evolution of the price of 1+d assets will be modelled by a 1–dimensional and a d–dimensional stochastic process $\{B_t, t=0,1,...,T\}$ and $\{S_t, t=0,1,...,T\}$ on some probability space $(\Omega,\mathfrak{F},P)$ where T is the time horizon. Then S_0 is deterministic and B_t and the components S_t^k , $1 \leq k \leq d$, of S_t are assumed to be positive. The asset with price process $\{B_t\}$ is called the bond. We assume (w.l.o.g.) $B_0 = 1$. The **interest rates** of the bond are then defined through

(2.1) $$B_t =: (1+r_1) \cdot ... \cdot (1+r_t) .$$

The other d assets are called stocks. In many cases B_t will be determistic or predictable. But we don't need such an assumption for the theory of this paper.

The information structure will be represented by random variables H_t with values in some space $(\Omega_t,\mathfrak{F}_t)$,$0\le t\le T$, which are assumed to be Borel subsets of some Polish space which is convenient for applying selection theorems. There H_t describes the **history** of the market at time t where H_0 is a given constant, $\Omega_0 := \{0\}$ say. Previous histories are never forgotten; therefore we assume that H_{t-1} is a measurable function of H_t. Then we have a filtration $\{\sigma(H_t),\ 0\le t\le T\}$ on $(\Omega,\mathfrak{F})$ and a stochastic process $\{Z_t\}$ is **adapted** if $Z_t = \zeta_t(H_t)$ for some measurable function ζ_t. This implies that Z_0 is deterministic. It is natural to assume that $\{r_t\}$ and $\{S_t\}$ are adapted.

As was shown by Harrison and Kreps (1979), one can use a reduction to the case where the interest rates of the bond are zero upon defining the **discounted stock price process** $\{\check{S}_t\}$ by

(2.2) $$\check{S}_t = (\check{S}^1_t,...,\check{S}^d_t),\ \check{S}^k_t := S^k_t / B_t,\ \text{, k= 1,...,d, t = 0,...,T.}$$

For any vector–valued process $\{Z_t\}$ define the backward increment by $\Delta Z_t := Z_t - Z_{t-1}$. Further, we write $x^\top \cdot y$ for the inner product of $x,y \in \mathbb{R}^d$.

The **relative risk process** $\{R_t = (R^1_t,...,R^d_t),\ 1\le t\le T\}$ (see Karatzas & Kou 1996) is defined by

(2.3) $$1 + R^k_t := 1 + \Delta\check{S}^k_t / \check{S}^k_{t-1} = (1 + \Delta S^k_t/S^k_{t-1}) / (1 + r_t)$$

where $\{\ \Delta S^k_t/S^k_{t-1}.\ 1\le t\le T\ \}$ is the **return process** corresponding to $\{S^k_t,\ 0\le t\le T\}$ (see Pliska 1997 § 3.2). Then we get

(2.4) $$\check{S}^k_t = \check{S}^k_{t-1}\cdot(1 + R^k_t) = S^k_0\cdot(1 + R^k_1)\cdot ...\cdot(1 + R^k_t)\ .$$

A possible choice for H_t would be $H_t := (r_1,R_1,...,r_t,R_t)$ where R^k_m can also be replaced by $\Delta S^k_m/S^k_{m-1}$ in view of (2.3). A dynamic portfolio is here described by a **portfolio process** given by an $\mathbb{R}^d$–valued adapted stochastic process $\pi = \{\pi_t,\ 0\le t<T\}$ (see Karatzas & Kou 1996). During $(t-1,t]$ the investor holds a portfolio $\pi_{t-1} = (\pi^1_{t-1},...,\pi^d_{t-1})$ where π^k_{t-1} represents the proportion of the wealth (value) which is invested in the k–th stock. We will consider self–financing portfolio processes. Then the proportion π^o_t of the wealth (value) which is invested in the bond in $[t,t+1)$ is given by $\pi^o_t = 1 - \sum_{k=1}^d \pi^k_t$. Given the initial wealth x, the **value process** $\{V^\pi_t(x)\}$ and the **discounted value process** $\{\check{V}^\pi_t(x)\}$ are defined through

(2.5) $$V^\pi_t(x) := B_t\cdot\check{V}^\pi_t(x) := B_t\cdot x\cdot(1 + \pi_0^\top\cdot R_1)\cdot ...\cdot(1 + \pi_{t-1}^\top\cdot R_t) = x\cdot V^\pi_t(1)\ .$$

We assume the **no–arbitrage condition** (cf. Schachermayer 1992, Jacod & Shiryaev 1998):

(NA) For $1\le t\le T$ and any π: $\pi_{t-1}^\top\cdot R_t \ge 0$ a.s. implies $\pi_{t-1}^\top\cdot R_t = 0$ a.s. .

A portfolio process π is called **admissible** if $V_T^{\pi}(1) \geq 0$. Hence π is admissible if

(2.6) $\quad 1 + \pi_{t-1}^{\top} \cdot R_t \geq 0$ a.s. for all t.

[In order to describe a trading strategy ξ with value process $\{V_t^{\xi}\}$, as used in many papers, in terms of a portfolio process it is necessary that $V_{t-1}^{\xi}(x) = 0$ implies $V_t^{\xi}(x) = 0$. This property however follows from (NA) for an admissible trading strategy.] Let us write Π for the set of admissible portfolio processes π. We will assume boundedness of the risk process which is also used in continuous time (cf. Karatzas & Kou 1996).

2.7 Boundedness Assumption. r_t and the components of R_t are bounded and $1 + r_t \geq \beta_t$ for some constant $\beta_t > 0$, $1 \leq t \leq T$.

When dealing with conditional expectations of random variables Z we will use both $\zeta(h) = E[Z \mid H_t=h]$ defined on Ω_t and $E[Z \mid \sigma(H_t)](\omega)$ defined on Ω where $E[Z \mid \sigma(H_t)] = \zeta(H_t)$. We will consider martingales w.r.t. the filtration $\{\sigma(H_t)\}$. Then a probability measure Q on $(\Omega,\mathfrak{F})$ is a **martingale measure** if $\{\check{S}_t\}$ forms a martingale under Q, which is equivalent to

(2.8) $\quad E_Q[R_t^k \mid \sigma(H_{t-1})] = 0 \quad ,1 \leq k \leq d, \quad \forall\, 1 \leq t \leq T.$

From the'Fundamental Theorem of Asset Pricing' (see Dalang et al 1990, Schachermayer 1992, Rogers 1994, Jacod & Shiryaev 1998) we know that there exists a martingale measure Q which is equivalent to P if and only if (NA) holds.

§3 The one–period model.

3.1 Definition. Σ_1 is the support of R_1, i.e. Σ_1 is the smallest closed subset A in $\mathbb{R}^d$ such that $R_1 \in A$ a.s. . $\mathscr{L}_1$ is the smallest linear space in $\mathbb{R}^d$ containing Σ_1 [i.e. $R_1 \in \mathscr{L}_1$ a.s.].

Then it is easy to see that the (NA)–condition may be written as:

$(NA)_1$ $\quad$ For all $\vartheta \in \mathbb{R}^d$: $\vartheta^{\top} \cdot \sigma \geq 0 \quad \forall\, \sigma \in \Sigma_1$ implies $\vartheta^{\top} \cdot \sigma = 0 \quad \forall\, \sigma \in \Sigma_1$.

3.2 Definition.

$$\Theta_1 := \{\, \vartheta \in \mathbb{R}^d;\ 1 + \vartheta^{\top}\sigma \geq 0,\ \forall\, \sigma \in \Sigma_1 \},$$

$$\Theta_1^o := \{\, \vartheta \in \Theta_1;\ 1 + \vartheta^{\top}\sigma > 0,\ \forall\, \sigma \in \Sigma_1 \},$$

$$\partial\Theta_1 := \{\, \vartheta \in \Theta_1;\ 1 + \vartheta^{\top}\sigma = 0 \ \text{ for some } \sigma \in \Sigma_1 \}.$$

Let $B(0,\rho)$ be the ball around 0 with radius ρ.

3.3 Lemma. (a) $\Theta_1 := \{ \vartheta \in \mathbb{R}^d;\ \ 1 + \vartheta^\top \cdot R_1 \geq 0 \ \text{ a.s.}\}$ and Θ_1 is convex.
(b) Σ_1 is compact; if $\Sigma_1 \subset B(0,\rho)$, say, then $B(0,1/\rho) \subset \Theta_1$.

The proof is easy. Because R_1 is bounded, so is Σ_1. An easy consequence of 3.3(b) is:

3.4 Lemma. Θ_1^o is an open subset and Θ_1 and $\partial\Theta_1$ are closed subsets of $\mathbb{R}^d$.

According to the following property, we can restrict attention to portfolios $\vartheta \in \ \Theta_1 \cap \mathscr{L}_1$.

(3.5) If $\Gamma_1 : \mathbb{R}^d \mapsto \mathscr{L}_1$ denotes the orthogonal projection, then
$\vartheta^\top \cdot \sigma = (\Gamma_1 \vartheta)^\top \cdot \sigma$ for $\sigma \in \ \mathscr{L}_1 \supset \Sigma_1$.

3.6 Remark. Let $\mathscr{H}_1$ be the smallest affine hyperplane containing Σ_1 and thus the convex hull $\text{conv}(\Sigma_1)$ of Σ_1. $\mathscr{H}_1$ reduces to one point if and only if Σ_1 is a singleton set.
Then the following conditions are equivalent:

(1) condition $(NA)_1$ holds;
(2) $0 \in B(0,\varepsilon_1) \cap \mathscr{H}_1 \subset \text{conv}(\Sigma_1)$ for some $\varepsilon_1 > 0$;
(3) $B(0,\varepsilon_1) \cap \mathscr{L}_1 \subset \text{conv}(\Sigma_1)$ for some $\varepsilon_1 > 0$;

The equivalence of (1) and (2) was proved by Jacod & Shiryaev (1998, Theorem 3).
"(2) ⇒ (3)" Assume that (2) holds. Then $0 \in \mathscr{H}_1$ and thus $\mathscr{H}_1 = \mathscr{L}_1$.
"(3) ⇒ (2)" If (3) holds, then $0 \in B(0,\varepsilon_1) \cap \mathscr{L}_1 \subset \text{conv}(\Sigma_1) \subset \mathscr{H}_1$ and we again have $\mathscr{H}_1 = \mathscr{L}_1$. □

3.7 Lemma. For $\varepsilon_1 > 0$ as in 3.6(3), one has: $\Theta_1 \cap \mathscr{L}_1 \subset B(0,1/\varepsilon_1)$.

Proof. From $1 + \vartheta^\top \cdot \sigma \geq 0 \quad \forall\, \sigma \in \Sigma_1$ we obviously obtain $1 + \vartheta^\top \cdot \sigma \geq 0 \quad \forall\, \sigma \in \text{conv}(\Sigma_1)$ and thus $1 + \vartheta^\top \cdot \sigma \geq 0 \quad \forall\, \sigma \in B(0,\varepsilon_1) \cap \mathscr{L}_1$. If the latter relation holds for $\vartheta \in \mathscr{L}_1$, we can choose $\sigma = -\frac{1}{\|\vartheta\|}\, \varepsilon_1\, \vartheta$ and obtain $\varepsilon_1 \cdot \|\vartheta\| \leq 1$. □
Since Θ_1 and $\mathscr{L}_1$ are closed, the following result follows from Lemma 3.7.

3.8 Proposition. $\Theta_1 \cap \mathscr{L}_1$ is compact.

A similar statement as in 3.8 was given by Hakansson (1971, Lemma p. 435).

We shall use the following properties of the utility function U:

(3.9) $U : [0,\infty) \mapsto [-\infty,\infty)$ is real valued, increasing, strictly concave and differentiable on $(0,\infty)$ with derivative $U'(x)$. Further $U(0) = U(0+) \geq -\infty$, $U'(0) := U'(0+) = \infty$.

In particular we know that U is continuous on $[0,\infty)$ and :

(3.10) $\mu \mapsto U'(x(1+\mu z))\cdot z$ in decreasing on $\{\mu \in \mathbb{R};\ 1+\mu z > 0\}$ for $x \geq 0$.

For a one–period model, a portfolio process π just specifies one portfolio $\vartheta \in \Theta$. We note that then $V_1^{\pi}(1) = B_1 \cdot (1 + \vartheta^{\top} \cdot R_1)$ is a bounded random variable. Hence the following expressions are well–defined:

(3.11) $I(\vartheta) := E\left[U(B_1 \cdot [1 + \vartheta^{\top} \cdot R_1])\right]$ for $\vartheta \in \Theta_1$.

In view of (3.5), the proof of the following properties is easy .

3.12 Lemma. (a) I is upper semi–continuous;

(b) $I(\vartheta)$ attains the maximum on Θ_1 where the maximum point can be chosen in the compact set $\Theta_1 \cap \mathscr{L}_1$.

The following lemma follows from strict concavity of U.

3.13 Lemma. The maximum point of I in $\Theta_1 \cap \mathscr{L}_1$ is unique.

Now, the following result is one of the main steps.

3.14 Proposition. If $\vartheta \in \Theta_1$, $\vartheta \neq 0$, such that $E\left[U'(B_1 \cdot [1 + \vartheta^{\top} \cdot R_1])\, B_1\, \vartheta^{\top} \cdot R_1\right] < 0$, then ϑ is no maximum point of I.

Proof. We may assume $I(\vartheta) > -\infty$. Choose $0 \leq \mu < \lambda < 1$. Then by concavity:

$$\frac{1}{\lambda - \mu}\{I(\lambda\vartheta) - I(\mu\vartheta)\} = E\left[\frac{1}{\lambda-\mu}\left\{U(B_1 \cdot [1 + \lambda\, \vartheta^{\top} \cdot R_1]) - U(B_1 \cdot [1 + \mu\, \vartheta^{\top} \cdot R_1])\right\}\right]$$
$$\leq E\left[U'(B_1 \cdot [1 + \mu\, \vartheta^{\top} \cdot R_1])\, B_1\, \vartheta^{\top} \cdot R_1\right].$$

Now, $I(\lambda\vartheta) = E\left[U(B_1 \cdot [1 + \lambda\, \vartheta^{\top} \cdot R_1]) \cdot \mathbf{1}_{\{\vartheta^{\top} R_1 \geq 0\}}\right] + E\left[U(B_1 \cdot [1 + \lambda\, \vartheta^{\top} \cdot R_1]) \cdot \mathbf{1}_{\{\vartheta^{\top} R_1 < 0\}}\right]$.

Since U is increasing and continuous, we obtain from the monotone convergence theorem: $I(\lambda\vartheta) \to I(\vartheta)$ as $\lambda \uparrow 1$ and hence by (3.10)

$$\frac{1}{1-\mu}\{I(\vartheta) - I(\mu\vartheta)\} \leq E\left[U'(B_1 \cdot [1 + \mu\, \vartheta^{\top} R_1])\, B_1\, \vartheta^{\top} \cdot R_1\right]$$
$$\leq E\left[U'(B_1)\, B_1\, \vartheta^{\top} \cdot R_1\right] \leq \|\vartheta\| \cdot M \cdot U'(\beta_1) < \infty$$

where $\beta_1 > 0$ is a lower bound for B_1, and M is some upper bound for $B_1 \cdot \|R_1\|$.

Again, by (3.10) we can use the monotone convergence theorem to obtain for $\mu \uparrow 1$

$$E\left[U'(B_1 \cdot [1 + \mu\, \vartheta^{\top} \cdot R_1])\, B_1\, \vartheta^{\top} \cdot R_1\right] \downarrow E\left[U'(B_1 \cdot [1 + \vartheta^{\top} \cdot R_1])\, B_1\, \vartheta^{\top} \cdot R_1\right] < 0$$

by assumption. Thus ϑ cannot be a maximum point of I. $[]$

From the proof of Proposition 3.14 we learn that

(3.15) $\quad E\left[U'(B_1 \cdot [1 + \vartheta^\top \cdot R_1])\, B_1\, \vartheta^\top \cdot R_1\right] \le E\left[U'(B_1)\, B_1\, \vartheta^\top \cdot R_1\right]$

which is useful for the following condition:

(3.16) $\quad E\left[U'(B_1 \cdot [1 + \vartheta^\top \cdot R_1])\, B_1\, \vartheta^\top \cdot R_1\right] < 0 \quad$ for each $\vartheta \in \partial\Theta_1 \cap \mathscr{L}_1$.

One can look upon condition (3.16) as a kind of no–arbitage condition. For convenience, let us consider the case: B_1=1. Then we know from the (NA) that $\vartheta^\top \cdot R_1$ takes on both positive and negative values for $\vartheta \in \mathscr{L}_1 \setminus \{0\}$. However in spite of (NA), one of these events may have arbitrarily small probability which would cause an arbitrage opportunity in practise. If $E\left[\vartheta^\top \cdot R_1\right] < 0$, then we conclude from (3.15) that $E\left[U'(1 + \vartheta^\top R_1)\, \vartheta^\top \cdot R_1\right] < 0$ and (3.16) holds. If $E\left[\vartheta^\top \cdot R_1\right] = 0$ then $I(\vartheta) = E\left[U(1 + \vartheta^\top \cdot R_1)\right] < U(1) = I(0)$ by Jensen's inequality and ϑ cannot be a maximum point. If $E\left[\vartheta^\top \cdot R_1\right] > 0$, then positive values of $\vartheta^\top \cdot R_1$ have more probability than negative values. Now condition (3.16) requires that the negative values should not have too little probability in the following sense. If the values $\vartheta^\top \cdot R_1$ are multiplied by a weight $U'(1 + \vartheta^\top \cdot R_1)$ which is decreasing in $\vartheta^\top \cdot R_1$ and even $U'(1 + \vartheta^\top \cdot R_1) \approx \infty$ for $\vartheta^\top \cdot R_1 \approx -1$ (what is possible for $\vartheta \in \partial\Theta_1$), then the expectation $E\left[U'(1 + \vartheta^\top \cdot R_1)\, \vartheta^\top \cdot R_1\right]$ should be negative (see also 3.18, 3.19 below).

Now the next theorem immediately follows from the lemmata 3.12b, 3.13, and 3.14.

3.17 Theorem. Suppose condition (3.16) is fulfilled, then the maximum of I is attained at a unique point ϑ^* in $\Theta_1^o \cap \mathscr{L}_1$.

Unfortunately, the criterion (3.16) which will lead to a first order condition in Theorem 3.21 below is not as simple as those of Leland (1972), Bertsekas (1974) who considered utility functions defined on the real line. The proof of Proposition 3.14 shows that the condition (3.16) comes close to being necessary as was also stated by Bertsekas (1974). We will give another sufficient condition for (3.16). It is satisfied if Σ_1 is finite or if Σ_1 is a polytope, i.e. a bounded polyhedral set as in the important case where Σ_1 is a rectangle. We will then assume that each ball $B(\sigma,\varepsilon)$ around a vertex σ with radius $\varepsilon>0$ has enough probability.

3.18 Proposition. Condition (3.16) is fulfilled if there exists some finite subset Σ_o of Σ_1 such that $\Sigma_1 \subset \mathrm{conv}(\Sigma_o)$ and $U'(x \cdot \varepsilon) \cdot P\left[R_1 \in B(\sigma,\varepsilon)\right] \to \infty$ as $\varepsilon \downarrow 0$ for $x > 0$, $\sigma \in \Sigma_o$.

Proof. Let be $\vartheta \in \partial\Theta_1$. Then one has the relation:

$$\inf_{\sigma\in\Sigma_1} 1 + \vartheta^\top\cdot\sigma = \min_{\sigma\in\Sigma_o} 1 + \vartheta^\top\cdot\sigma = 1 + \vartheta^\top\cdot\sigma_o = 0 \quad \text{for some } \sigma_o \in \Sigma_o.$$

Further, we then have for $\sigma \in B(\sigma_o,\varepsilon)$: $1 + \vartheta^\top\cdot\sigma = \vartheta^\top\cdot(\sigma - \sigma_o) \le \|\vartheta\|\cdot\varepsilon \le \frac{1}{2}$ for sufficiently small ε and hence $\{R_1 \in B(\sigma_o,\varepsilon)\} \subset \{1 + \vartheta^\top\cdot R_1 \le \|\vartheta\|\cdot\varepsilon\} \subset \{\vartheta^\top\cdot R_1 \le -\frac{1}{2}\}$. Now

$$E\left[U'(B_1\cdot[1+\vartheta^\top\cdot R_1])\,B_1\,\vartheta^\top\cdot R_1\right] \le E\left[\mathbf{1}_{\{R_1\in B(\sigma_o,\varepsilon)\}}\cdot U'(B_1\cdot[1+\vartheta^\top\cdot R_1])\,B_1\,\vartheta^\top\cdot R_1\right] +$$

$$E\left[\mathbf{1}_{\{\vartheta^\top R_1 > 0\}}\cdot U'(B_1\cdot[1+\vartheta^\top\cdot R_1])\,B_1\,\vartheta^\top\cdot R_1\right]$$

$$\le -\tfrac{1}{2}\,E\left[\mathbf{1}_{\{R_1\in B(\sigma_o,\varepsilon)\}}\cdot U'(B_1\cdot\|\vartheta\|\cdot\varepsilon)\cdot B_1\right] + E\left[\mathbf{1}_{\{\vartheta^\top R_1 > 0\}}\cdot U'(B_1)\cdot B_1\,\vartheta^\top\cdot R_1\right]$$

$$\le -\tfrac{1}{2}\,M\cdot U'(\beta_1\cdot\|\vartheta\|\cdot\varepsilon)\cdot P\left[R_1 \in B(\sigma_o,\varepsilon)\right] + U'(\beta_1)\cdot\|\vartheta\|\cdot M \to -\infty \quad \text{as } \varepsilon \downarrow 0,$$

where $\beta_1>0$ is a lower bound for B_1, and M is an upper bound for B_1 and $B_1\cdot\|R_1\|$. We obtain:

(3.18a) $E\left[U'(B_1\cdot[1+\vartheta^\top\cdot R_1])\,B_1\,\vartheta^\top\cdot R_1\right] = -\infty$ for $\vartheta \in \partial\Theta_1$. []

If Σ_1 is finite, we may choose $\Sigma_o = \Sigma_1$ and have $P\left[R_1 \in B(\sigma,\varepsilon)\right] \ge P\left[R_1 = \sigma\right] > 0$.

3.19 Corollary. If R_1 takes on only finitely many values, then condition (3.16) holds.

3.20 Lemma. I is partially differentiable on Θ_1^o with partial derivatives

$$\partial_k I(\vartheta) = E\left[U'(B_1\cdot[1+\vartheta^\top\cdot R_1])\cdot B_1\cdot R_1^k\right] .$$

Proof. Let be $\vartheta \in \Theta_1^o$ such that $\min_{\sigma\in\Sigma_1} 1 + \vartheta^\top\sigma =: 2\varepsilon > 0$, say, ζ the k–th unit vector in $\mathbb{R}^d$, $0<\delta<1$ such that $\delta\cdot M \le \varepsilon$ where M is an upper bound for $B_1\cdot\|R_1\|$ and $\|R_1\|$. Then $\zeta^\top\cdot R_1 = R_1^k$ and $1 + \vartheta^\top R_1 - \delta\cdot|\zeta^\top\cdot R_1| \ge 2\varepsilon - \delta\cdot M \ge \varepsilon$. For $0 < |\mu| < \delta$, we have

$$\frac{1}{\mu}\{I(\vartheta+\mu\,\zeta) - I(\vartheta)\} = E\left[\frac{1}{\mu}\{U(B_1\cdot[1+(\vartheta+\mu\,\zeta)^\top\cdot R_1]) - U(B_1\cdot[1+\vartheta^\top\cdot R_1])\}\right]$$

where the integrand on the right–hand side is bounded by

$$U'(B_1\cdot[1+\vartheta^\top R_1 - \delta\cdot R_1^k])\cdot B_1\cdot R_1^k \le U'(B_1\cdot\varepsilon)\cdot B_1\cdot\|R_1\| \le U'(\beta_1\cdot\varepsilon)\cdot x\cdot M$$

where $\beta_1>0$ is a lower bound for B_1. Now we can apply the dominated convergence theorem. []

3.21 Theorem. Assume that condition (3.16) is satisfied. Then the following **first order condition** holds: $E\left[U'(B_1\cdot[1+\vartheta^{*\top}\cdot R_1])\cdot B_1\cdot R_1^k\right] = 0$ for $1\le k\le d$ where ϑ^* is the unique maximum point of I in $\Theta_1^o \cap \mathcal{L}_1$.

Proof. A maximum point ϑ^* can be chosen in the open set Θ_1^o by Theorem 3.17. Further I is partially differentiable in Θ_1^o by Lemma 3.19. Therefore we know that $\partial_k I(\vartheta^*) = 0$ for $1\le k\le d$. []

§4 The multi–period model.

For $t \geq 1$ we choose regular versions of the conditional distributions $P[(r_t,R_t) \in \cdot \,|\, H_{t-1}=h]$ of (r_t,R_t) given the past $H_{t-1} = h \in \Omega_{t-1}$. Then for fixed h, we can apply the results of §3. Now we need the support $\Sigma_t(h)$ of $P[R_t \in \cdot \,|\, H_{t-1}=h]$. Furthermore, let $\mathcal{L}_t(h)$ be the smallest linear space in $\mathbb{R}^d$ containing $\Sigma_t(h)$. Finally we set

(4.1)
$$\Theta_t(h) := \{ \vartheta \in \mathbb{R}^d;\ \ 1 + \vartheta^\top \cdot \sigma \geq 0 \ \ \forall\, \sigma \in \Sigma_t(h)\}\,,$$
$$\Theta_t^o(h) := \{ \vartheta \in \Theta_t(h);\ \ 1 + \vartheta^\top \cdot \sigma > 0 \ \ \forall\, \sigma \in \Sigma_t(h)\}\,,$$
$$\partial\Theta_t(h) := \{ \vartheta \in \Theta_t(h);\ \ \exists\, \sigma \in \Sigma_t(h) \text{ with } 1 + \vartheta^\top \cdot \sigma = 0\}.$$

4.2 Lemma. Let $\mathbb{P}$ denote the compact metric space of all d×d orthogonal projection matrices and define the mapping $\Gamma_t : \Omega_{t-1} \mapsto \mathbb{P}$ such that $\Gamma_t(h)$ is the orthogonal projection on $\mathcal{L}_t(h)$. Then $\Gamma_t(h)\vartheta$ is measurable in (ϑ,h) and:

(4.3)
$$P[\ \vartheta^\top \cdot R_t = (\Gamma_t(h)\vartheta)^\top \cdot R_t \,|\, H_{t-1}=h] = 1, \quad h \in \Omega_{t-1}.$$

Proof. Rogers (1994, Proposition 2.4) proved that Γ_t is measurable on a set B_{t-1} where $P[H_{t-1} \in B_{t-1}] = 1$. This fact is sufficient for the application below. But an analysis of the proof shows that one can here choose $B_{t-1} = \Omega_{t-1}$. Thus $\Gamma_t(h)\vartheta$ is measurable in h und continuous in ϑ, hence measurable in (h,ϑ). □

4.4 Lemma. $\{(h,\vartheta) \in \Omega_{t-1}\times\mathbb{R}^d;\ \vartheta \in \Theta_t(h)\cap\mathcal{L}_t(h)\}$ is a measurable subset of $\Omega_{t-1}\times\mathbb{R}^d$.

Proof. The result follows Lemma 4.2 since $\{(h,\vartheta) \in \Omega_{t-1}\times\mathbb{R}^d;\ \vartheta \in \Theta_t(h)\cap\mathcal{L}_t(h)\}$
$= \{(h,\vartheta) \in \Omega_{t-1}\times\mathbb{R}^d;\ P[\ 1 + \vartheta^\top \cdot R_t \geq 0 \ \,|\, H_{t-1}=h] = 1\ ,\ \Gamma_t(h)\vartheta = \vartheta\ \}$. □

It is known that the no–arbitrage condition also holds locally a.s. (see Dalang et al. 1990, Lemma 2.3 , Pliska 1997 (3.22), Jacod & Shiryaev 1998, Schäl 1999 §2). By an appropriate choice of the conditional distributions we may assume that the no–arbitrage condition (NA) locally holds everywhere. Here we will assume a slightly stronger condition which can be looked upon as a kind of uniform no–arbitrage condition in view of 3.6(3):

4.5 Uniform (NA)–Assumption. For all $1 \leq t \leq T$ there exists some $\varepsilon_t > 0$ such that
$$B(0,\varepsilon_t)\cap\mathcal{L}_t(h) \subset \text{conv}(\Sigma_t(h)) \quad \forall\, h \in \Omega_{t-1}.$$

4.6 Remark. Assumption 4.5 is equivalent to the usual (NA)–condition in each of the following cases: (1) Ω is finite; (2) $\Sigma_t(h)$ is independent of $h \in \Omega_{t-1}$ for $1 < t \leq T$. □

4.7 Lemma. (a) $\Theta_t(h)\cap\mathscr{L}_t(h) \subset B(0,1/\varepsilon_t) \quad \forall\, h \in \Omega_{t-1}$.

(b) For any $\pi \in \Pi$, define $\pi' \in \Pi$ by $\pi'_t = \Gamma_t(H_t)\pi_t$, then $\pi'_t \in \Theta_t(H_t)\cap\mathscr{L}_t(H_t)$ and $V_t^{\pi}(x) = V_t^{\pi'}(x)$ a.s. $1\le t\le T$, where $V_t^{\pi'}(x)$ is a bounded random variable.

Proof. Part (a) is an immediate consequence of Lemma 3.7 and Assumption 4.5. Now we can conclude that $\|\pi'_t\| \le \max\{1/\varepsilon_t,\ 1\le t\le T\}$. Then part (b) follows from Lemma 4.2. $[]$

Throughout this section, we will use the following assumption on the utility function:

4.8 Assumption: $U(x) = U_b(x) = \begin{cases} \ln x & \text{for } b=1 \\ \frac{1}{1-b}\, x^{1-b} & \text{for } 0<b<1 \end{cases}$, $x\ge 0$, $\quad U'_b(x) = 1/x^b$, $x>0$,

where $b = -x\cdot U''(x)/U'(x)$ is the measure of constant proportional risk aversion.

The following notation will be used where $E[\,...\,|\,H_0=h] = E[\,...\,]$.

(4.9)
$$I_t^{\pi}(x,h) = E\big[U\big(x\cdot \Pi_{m=t+1}^{T}[1+r_m]\cdot[1+\pi_{m-1}^{\top}\cdot R_m]\big)\,\big|\,H_t=h\big],$$
$$J_t(x,h) = \sup_{\pi\in\Pi} I_t^{\pi}(x,h)\,, \quad h\in\Omega_t,$$
$$I_T^{\pi}(x,h) = J_T(x,h) = U(x) \quad \text{for all } \pi\in\Pi.$$

There $I_t^{\pi}(x,h)$ $[J_t(x,h)]$ is the [maximal] expected utility of the terminal wealth given the history h and the wealth x at time t. We remark that $I_0^{\pi}(x)$ and $J_0(x)$ do not depend on h. Let us now introduce the well–known operators from dynamic programming where we write here:

(4.10) $\mathscr{B}_t := \{f : [0,\infty)\times\Omega_t \mapsto [-\infty,\infty)\ ;\ f \text{ is measurable, } \sup_{0\le x\le M, h\in\Omega_t} f(x,h) < \infty\ \ \forall\, M>0\}$.

(4.11) $L_{t-1}f(x,\vartheta,h) := E\big[f\big(x\cdot[1+r_t]\cdot[1+\vartheta^{\top}\cdot R_t], H_t\big)\,\big|\,H_{t-1}=h\big]$, $\vartheta\in\Theta_t(h), h\in\Omega_{t-1}, f\in\mathscr{B}_t$.

4.12 Proposition. (a) For $x\ge 0$, $h\in\Omega_t$, $0\le t\le T$ one has

(i) in the case $0<b<1$: $J_t(x,h) = x^{1-b}\cdot J_t(1,h)$,

(ii) in the case $b=1$: $J_t(x,h) = \ln x + J_t(1,h)$.

(b) $\sup_{h\in\Omega_t} J_t(1,h) < \infty$, hence $J_t \in \mathscr{B}_t$.

(c) **Optimality equation:** $J_t(x,h) = \sup_{\vartheta\in\Theta_{t+1}(h)\cap\mathscr{L}_{t+1}(h)} L_t J_{t+1}(x,\vartheta,h)$, $t\ge 0$, $h\in\Omega_t$.

(d) There exists a unique measurable function $\delta_t : \Omega_t \mapsto \mathbb{R}^d$ such that $\delta_t(h) \in \Theta_{t+1}(h)\cap\mathscr{L}_{t+1}(h)$ and $L_t J_{t+1}(1,\delta_t(h),h) = J_t(1,h)$.

(e) We have $J_t(x,H_t) = I_t^{\pi^*}(x,H_t)$ if we define

(4.13) $\pi_t^* := \delta_t(H_t)$.

(f) If (r_{t+1},R_{t+1}) and H_t are independent, $1\le t<T$, then $\delta_t(h)$ and $J_t(x,h)$ do not depend on $h\in\Omega_t$, $0\le t<T$.

The situation in 4.12(f) is assumed in most papers on portfolio optimization.

4.14 Definition. We define $V_t^* := V_t^{\pi^*}(1)$ and call $\{V_t^*,\ 0\le t\le T\}$ the **optimal value process.**

Proof of 4.12**.** First assume $0<b<1$. For the proof of (a) – (e) we will use backward induction by t. For t=T part (a) and (b) are obvious and part (c) and (e) are not defined.
Now assume that (a) – (e) hold for some $1\le t+1\le T$ and define

$$J_t^*(x,h) := \sup_{\vartheta\in\Theta_{t+1}(h)\cap\mathscr{L}_{t+1}(h)} L_tJ_{t+1}(x,\vartheta,h),\ t\ge 0,\ h\in\Omega_t.$$

Then by (a): $L_tJ_{t+1}(x,\vartheta,h) = x^{1-b}\cdot L_tJ_{t+1}(1,\vartheta,h)$ where

$$L_tJ_{t+1}(1,\vartheta,h) = E\left[(1+r_{t+1})^{1-b}\cdot(1+\vartheta^\top\cdot R_{t+1})^{1-b}\cdot J_{t+1}(1,H_{t+1})\,|\,H_t=h\right].$$

From the boundedness of r_{t+1}, R_{t+1}, and $\Theta_{t+1}(h)\cap\mathscr{L}_{t+1}(h)$ (see Lemma 4.7a) we obtain

$$J_t^*(x,h) = x^{1-b}\cdot J_t^*(1,h),\ \sup_{h\in\Omega_t} J_t^*(1,h) < \infty.$$

Obviously $L_tJ_{t+1}(1,\vartheta,h)$ is measurable in (ϑ,h). In order to apply the results of §3 we define a probability measure P_t for fixed h by

$$E_t[X] := E\left[X\cdot J_{t+1}(1,H_{t+1})\,|\,H_t=h\right]/E\left[J_{t+1}(1,H_{t+1})\,|\,H_t=h\right]$$

where $J_{t+1}(1,h)\ge U(\beta)>0$ with $\beta := \prod_{m=t+2}^{T}\beta_m > 0$ as lower bound for $\prod_{m=t+2}^{T}[1+r_m]$. Then

$$I(\vartheta) := E_t\left[U(\left[1+r_{t+1}\right]\cdot\left[1+\vartheta^\top\cdot R_{t+1}\right])\right] = (1-b)^{-1}\cdot L_tJ_{t+1}(1,\vartheta,h)/E\left[J_{t+1}(1,H_{t+1})\,|\,H_t=h\right].$$

Since P_t has the same null sets as $P[\,\cdot\,|\,H_t=h]$, R_{t+1} has the same support $\Sigma_{t+1}(h)$ under P_t. Now we conclude from Lemma 3.12a that $L_tJ_{t+1}(1,\vartheta,h)$ is upper semi–continuous in ϑ. Furthermore by Proposition 3.8, $\Theta_{t+1}(h)\cap\mathscr{L}_{t+1}(h)$ is compact. Thus we know by Lemma 3.13 that there is a unique $\vartheta^* = \delta_t(h)$ in $\Theta_{t+1}(h)\cap\mathscr{L}_{t+1}(h)$ which is a maximum point of the function $\vartheta\mapsto L_tJ_{t+1}(1,\vartheta,h)$ defined on $\mathbb{R}^d$. In view of Lemma 4.4 we can now apply a selection theorem in order to conclude that δ_t is measurable (see Bertsekas & Shreve 1978, Proposition 7.33). Then we have for any π

$$\begin{aligned} I_t^{\pi}(x,h) &= E\left[\,E\left[U(x\cdot\textstyle\prod_{m=t+1}^{T}[1+r_m]\cdot[1+\pi_{m-1}^\top\cdot R_m])\,|\,H_{t+1}\right]\,|\,H_t=h\right] \\ &= E\left[\,x^{1-b}\cdot[1+r_{t+1}]^{1-b}\cdot[1+\pi_t^\top\cdot R_{t+1}]^{1-b}\cdot I_{t+1}^{\pi}(1,H_{t+1})\,|\,H_t=h\right] \\ &\le E\left[\,x^{1-b}\cdot[1+r_{t+1}]^{1-b}\cdot[1+\pi_t^\top\cdot R_{t+1}]^{1-b}\cdot J_{t+1}(1,H_{t+1})\,|\,H_t=h\right] \end{aligned}$$

$= E\left[\, J_{t+1}(x\cdot\left[1+r_{t+1}\right]\cdot\left[1+\pi_t^{\top}\cdot R_{t+1}\right],H_{t+1})\;\middle|\, H_t=h\right]$

$\le J_t^*(x,h) = L_t J_{t+1}(x,\delta_t(h),h)$.

Thus $I_t^{\pi}(x,H_t) \le J_t^*(x,H_t) = L_t I_{t+1}^{\pi^*}(x,\pi_t^*,H_t) = I_t^{\pi^*}(x,H_t)$ and hence

$J_t^*(x,H_t) = J_t(x,H_t) = I_t^{\pi^*}(x,H_t)$. Obviously $I_t^{\pi^*}(x,H_t)$ depends on π^* only through $\pi_t^*,...,\pi_{T-1}^*$.

For b=1, it is known that one can construct a stepwise optimal ("myopic") portfolio process. This follows from the fact that by (2.11):

$$E\left[\, \ln(V_T^{\pi}(x))\right] = \ln(x) + \sum_{t=1}^{T} E\left[\ln(1+r_t)\right] + \sum_{t=1}^{T} E\left[\ln(1+\pi_{t-1}^{\top}\cdot R_t)\right]$$

where $E\left[\ln(1+\pi_{t-1}^{\top}\cdot R_t)\right] = E\left[\, E\left[\ln(1+\pi_{t-1}^{\top}\cdot R_t)\,\middle|\, H_{t-1}\right]\,\right]$.

Now one can apply the results of §3 directly to $I(\vartheta) := E\left[\ln(1+\vartheta^{\top}\cdot R_t)\,\middle|\, H_{t-1}=h\right]$ and separately for each $1\le t\le T$.

In the situation of (f), we know that $E\left[f(r_{t+1},R_{t+1})\,\middle|\, H_t=h\right] = E\left[f(r_{t+1},R_{t+1})\right]$ is independent of h for any function f and the result follows from the proof above. []

In place of condition (3.16), we now have to make the following

4.15 Assumption. For $1 \le t \le T$ and for each $\vartheta \in \partial\Theta_t(h)\cap\mathscr{L}_t(h)$:

(i) $E\left[(1+r_t)^{1-b}\cdot(1+\vartheta^{\top}R_t)^{-b}\,\vartheta^{\top}\cdot R_t\cdot J_t(1,H_t)\,\middle|\, H_{t-1}=h\right] < 0$ in the case $0<b<1$ or

(ii) $E\left[(1+\vartheta^{\top}R_t)^{-1}\,\vartheta^{\top}\cdot R_t\,\middle|\, H_{t-1}=h\right] < 0$ in the case b=1.

Now we can conclude from the proof of Proposition 4.12 and the results of §3 (3.17 and 3.21):

4.16 Corollary. For $1\le t\le T$, $h\in\Omega_{t-1}$, we have for the function δ_{t-1} of 4.12(d):

(a) $\delta_{t-1}(h) \in \Theta_t^o(h)\cap\mathscr{L}_t(h)$ and the first order condition:

(i) $E\left[(1+r_t)^{1-b}\cdot(1+\delta_{t-1}(h)^{\top}\cdot R_t)^{-b}\,R_t^k\cdot J_t(1,H_t)\,\middle|\, H_{t-1}=h\right] = 0$ in the case $0<b<1$,

(ii) $E\left[(1+\delta_{t-1}(h)^{\top}\cdot R_t)^{-1}\cdot R_t^k\,\middle|\, H_{t-1}=h\right] = 0$ in the case b=1.

(b)

(i) $J_{t-1}(1,h) = E\left[(1+r_t)^{1-b}\cdot(1+\delta_{t-1}(h)^{\top}\cdot R_t)^{1-b}\cdot J_t(1,H_t)\,\middle|\, H_{t-1}=h\right] =$

$E\left[(1+r_t)^{1-b}\cdot(1+\delta_{t-1}(h)^{\top}\cdot R_t)^{-b}\cdot J_t(1,H_t)\,\middle|\, H_{t-1}=h\right]$ in the case $0<b<1$;

(ii) $E\left[(1+\delta_{t-1}(h)^{\top}\cdot R_t)^{-1}\,\middle|\, H_{t-1}=h\right] = 1$ in the case b=1.

Proof. It is sufficient to prove (b). This part however follows from (a) writing

$(1+\delta_{t-1}(h)^{\top}\cdot R_t)^{1-b} = (1+\delta_{t-1}(h)^{\top}\cdot R_t)^{-b}\cdot(1+\sum_{k=1}^{d}\delta_{t-1}^k(h)\cdot R_t^k)$. For case (i) we also have to apply Proposition 4.12(d). []

4.17 Lemma. $V^*_t > 0$ a.s., $0 \le t \le T$, and $0 < c := E\left[B_T \cdot V^{*-b}_T\right] = E\left[V^{*1-b}_T\right] < \infty$.

Proof. From 4.16(a) we know that $1 + \pi^{*\top}_{t-1} \cdot R_t > 0$ a.s. $\forall$ t and thus $V^*_t > 0$ a.s. $\forall$ t . Now we only consider the case $0<b<1$. By induction we are going to prove

(4.17a) $E\left[B_t \cdot V^{*-b}_t \cdot J_t(1,H_t)\right] = E\left[V^{*1-b}_t \cdot J_t(1,H_t)\right]$ in the case $0<b<1$.

Then the result follows by setting t=T. For t=1 (4.17a) directly follows from Corollary 4.16b. Now assume that (4.17a) holds for t–1. Then we obtain again by Corollary 4.16b:

$E\left[B_{t-1} \cdot V^{*-b}_{t-1} \cdot J_{t-1}(1,H_{t-1})\right]$
$= E\left[B_{t-1} \cdot V^{*-b}_{t-1} \cdot E\left[(1+r_t)^{1-b} \cdot (1 + \pi^{*\top}_{t-1} \cdot R_t)^{-b} \cdot J_t(1,H_t) \,|\, H_{t-1}\right]\right]$
$= E\left[B_{t-1} \cdot V^{*-b}_{t-1} \cdot (1+r_t)^{1-b} \cdot (1 + \pi^{*\top}_{t-1} \cdot R_t)^{-b} \cdot J_t(1,H_t)\right] = E\left[B_t \cdot V^{*-b}_t \cdot J_t(1,H_t)\right]$ and
$E\left[V^{*1-b}_{t-1} \cdot J_{t-1}(1,H_{t-1})\right] =$
$E\left[V^{*1-b}_{t-1} \cdot E\left[(1+r_t)^{1-b} \cdot (1 + \pi^{*\top}_{t-1} \cdot R_t)^{1-b} \cdot J_t(1,H_t) \,|\, H_{t-1}\right]\right]$
$= E\left[V^{*1-b}_{t-1} \cdot (1+r_t)^{1-b} \cdot (1 + \pi^{*\top}_{t-1} \cdot R_t)^{1-b} \cdot J_t(1,H_t)\right] = E\left[V^{*1-b}_t \cdot J_t(1,H_t)\right]$. []

4.18 Theorem. Let $\{V^*_t\}$ be the optimal value process and c as in 4.17. The process $\{Z_t,\ 0 \le t \le T\}$ is defined by $Z_t := (1-b)\ B_t \cdot V^{*-b}_t \cdot J_t(1,H_t)$ for $b<1$ and $Z_t := B_t \cdot V^{*-1}_t$ for b=1, in particular $Z_T = B_T \cdot V^{*-b}_T$, then:

(a) one obtains an equivalent martingale measure Q^b on $(\Omega,\mathfrak{F})$ by $dQ^b = \frac{1}{c} \cdot Z_T\, dP$;

(b) $\{Z_t,\ 0 \le t \le T\}$ a positive martingale under P; hence $(Z_t/Z_0,\ 0 \le t \le T)$ is the density process of dQ^b/dP .

Proof. From Corollary 4.16 we obtain:

(4.19a) $E\left[(1+r_t)^{1-b} \cdot (1 + \pi^{*\top}_{t-1} \cdot R_t)^{-b}\ R^k_t \cdot J_t(1,H_t) \,|\, \sigma(H_{t-1})\right] = 0$ in the case $0<b<1$
$E\left[(1 + \pi^{*\top}_{t-1} \cdot R_t)^{-1} \cdot R^k_t \,|\, H_{t-1}=h\right] = 0$ in the case b=1.

(4.19b) $J_{t-1}(1,H_{t-1}) = E\left[(1+r_t)^{1-b} \cdot (1 + \pi^{*\top}_{t-1} \cdot R_t)^{1-b} \cdot J_t(1,H_t) \,|\, \sigma(H_{t-1})\right]$
$= E\left[(1+r_t)^{1-b} \cdot (1 + \pi^{*\top}_{t-1} \cdot R_t)^{-b} \cdot J_t(1,H_t) \,|\, \sigma(H_{t-1})\right]$ in the case $0<b<1$;
$E\left[(1 + \pi^{*\top}_{t-1} \cdot R_t)^{-1} \,|\, \sigma(H_{t-1})\right] = 1$ in the case b=1.

Now part (b) immediately follows from (4.19b). From (4.19a) we obviously obtain $E\left[R^k_t \cdot Z_t \,|\, \sigma(H_{t-1})\right] = 0$, $1 \le m \le t$, $1 \le k \le d$. Since R^k_t is bounded, we now know from (4.17a) that $R^k_t \cdot Z_t$ is integrable. By use of Bayes' rule we finally get part (a). []

In Schäl (1999a) the case of a more general utility U is studied where the uniform no–arbitrage assumption is replaced by a compactness condition. Schäl (1999b) explains the easier situation of of a utility U defined on the whole real line $\mathbb{R}$.

Let $\{Y_t, 0\le t\le T\}$ be defined by $Y_t := V_t^* \cdot B_t^{-1} \cdot Z_t$. As in the proof of Theorem 4.18, it immediately follows from (4.19b) that also $\{Y_t, 0\le t\le T\}$ is a positive martingale under P and we obtain:

4.20 Corollary. Let $(V_t^*, 0\le t\le T)$ be the optimal value process, then $(V_t^*/B_t, 0\le t\le T)$ is a martingale under Q^b (as defined in 4.18).

Let c be given by 4.17. If we define $\tilde{Q}^b$ by $d\tilde{Q}^b = \frac{1}{c} Y_T \, dP$ and replace B_t by V_t^* (change of numeraire), then it easily follows from Corollary 4.20 that $\{V_t^{*-1} \cdot (B_t, S_t), 0\le t\le T\}$ is a (1+d)–dimensional martingale under $\tilde{Q}^b$ (see Geman, El Karoui & Rochet 1995). This result is interesting for the case b=1 where $Y_t = 1$ and hence $\tilde{Q} = P$. Then $\{V_t^{*-1} \cdot (B_t, S_t), 0\le t\le T\}$ is a martingale under the given physical probability measure P and π^* is called the **numeraire portfolio** (see Long 1990, Artzner 1997) and the change of measure can be replaced by a change of numeraire.

4.21 Corollary (numeraire portfolio). For b=1, $\{V_t^{*-1} \cdot (B_t, S_t), 0\le t\le T\}$ is a (1+d)–dimensional martingale under P.

Corollary 4.21 provides the new condition (4.15) for the existence of the numeraire portfolio. Moreover, one gets another sufficient condition from 3.18. In Korn & Schäl 1999, an example is given where no numeraire portfolio exists and condition (4.15) is violated. In the same paper, it is shown for b=1 that the martingale measure Q^b coincides with the minimal martingale measure (see Föllmer & Schweizer 1991) if and only if the market is complete. The proof carries over to the case of a general index $0<b\le 1$.

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