

GRADIENT FLOWS FOR SEMICONVEX FUNCTIONS ON METRIC MEASURE SPACES – EXISTENCE, UNIQUENESS AND LIPSCHITZ CONTINUITY

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ABSTRACT. Given any continuous, lower bounded and κ -convex function V on a metric measure space (X, d, m) which is infinitesimally Hilbertian and satisfies some synthetic lower bound for the Ricci curvature in the sense of Lott-Sturm-Villani, we prove existence and uniqueness for the (downward) gradient flow for V . Moreover, we prove Lipschitz continuity of the flow w.r.t. the starting point

$$d(x_t, x'_t) \leq e^{-\kappa t} d(x_0, x'_0).$$

Throughout this paper, let (X, d) be a locally compact geodesic space and let m be a locally finite measure with full topological support. We always assume that the metric measure space (X, d, m) satisfies the $\text{RCD}(K, \infty)$ -condition for some finite number $K \in \mathbb{R}$. Recall that this means that the Boltzmann entropy w.r.t. m is weakly K -convex on the L^2 -Wasserstein space of probability measures on (X, d) and that the Cheeger energy on (X, d, m) is a quadratic functional on $L^2(X, m)$ [11], [6], [3], [5]. Among many others, it implies that $\int_X e^{-C' d^2(x, x')} m(dx) < \infty$ for some $C' \in \mathbb{R}, x' \in X$. Note that the latter in turn implies $\text{Ent}(\mu|m) \geq -C'' - C' W_2^2(\mu, \delta_{x'})$ for each $\mu \in \mathcal{P}_2(X)$.

Moreover, let a function $V : X \rightarrow (-\infty, +\infty]$ be given which is continuous (w.r.t. the topology of the extended real line) and bounded from below by $-C_0 - C_1 d^2(\cdot, x_0)$ for some $C_0, C_1 \in \mathbb{R}, x_0 \in X$. Put $X_0 = \{V < \infty\}$ and $X' = \overline{X_0}$.

Definition. (i) Given a number $\kappa \in \mathbb{R}$, we say that V is weakly κ -convex if for each $x_0, x_1 \in X$ there exists a geodesic $\gamma : [0, 1] \rightarrow X$ connecting them (i.e. $\gamma_0 = x_0, \gamma_1 = x_1$) such that

$$V(\gamma_t) \leq (1-t)V(\gamma_0) + tV(\gamma_1) - \kappa t(1-t)|\dot{\gamma}|^2/2 \quad (1)$$

for each $t \in [0, 1]$. If (1) holds for all geodesics γ in X and all $t \in [0, 1]$ then V is called strongly K -convex.

(ii) Given $x_0 \in X$, an EVI_κ -gradient flow for V starting in x_0 is a curve $(x_t)_{t \geq 0}$ in X emanating in x_0 such that for all $z \in X_0$ and a.e. $t > 0$

$$\frac{d}{dt} \frac{1}{2} d^2(x_t, z) + \kappa d^2(x_t, z) \leq V(z) - V(x_t). \quad (2)$$

The main result of this paper is

Theorem. If V is weakly κ -convex then for each $x_0 \in X'$ there exists a unique EVI_κ -gradient flow for V starting in x_0 .

It is well-known – and easy to see – that the existence of an EVI_κ -gradient flow for V starting in x_0 implies its uniqueness and the 'contraction' property (or 'expansion' bound)

$$d(x_t, x'_t) \leq e^{-\kappa t} d(x_0, x'_0) \quad (3)$$

for each pair of points $x_0, x'_0 \in X'$ and associated flows $(x_t)_{t \geq 0}$ and $(x'_t)_{t \geq 0}$. Moreover, every EVI_κ -gradient flow is also a gradient flow in the sense of energy dissipation

$$V(x_t) = V(x_s) - \frac{1}{2} \int_s^t (|\dot{x}_r|^2 + |\nabla^- V|^2) dr \quad (4)$$

for all $0 \leq s \leq t$. In particular, $t \mapsto V(x_t)$ is non-increasing.

The Theorem will be proven according to the following line of argumentation in the subsequent lemmata:

- There exists an EVI_{κ_n} -gradient flow for the functional $S_n(\mu) = \frac{1}{n}\text{Ent}(\mu|m) + \int V d\mu$ on $\mathcal{P}_2(X')$ with $\kappa_n = \frac{1}{n}K + \kappa$.
- In the limit $n \rightarrow \infty$ this yields an EVI_κ -gradient flow for the functional $S(\mu) = \int V d\mu$ on $\mathcal{P}_2(X')$.
- The EVI_κ -flow for S is non-diffusive. In particular, the flow starting in a Dirac mass will not spread out: it is a Dirac mass in a moving point.

Lemma 1. *There exists an EVI_{κ_n} -gradient flow for the functional $S_n(\mu) = \frac{1}{n}\text{Ent}(\mu|m) + \int V d\mu$ on $\mathcal{P}_2(X')$ with $\kappa_n = \frac{1}{n}K + \kappa$.*

Proof. (i) Let us begin with some general remarks. Since X is geodesic and locally compact, it is 'proper' in the sense that all closed balls are compact. This in turn implies that closed balls in $\mathcal{P}_2(X)$ are compact. In particular, $\mathcal{P}_2(X)$ is locally compact.

(ii) As a first step, we replace the space X by X' and the measure m by $m' := 1_{X_0}m$. Weak κ -convexity of the function V implies weak convexity of the set X_0 . Together with the fact that $\text{RCD}(K, \infty)$ implies strong K -convexity of the Boltzmann entropy $\text{Ent}(\cdot|m)$ on $\mathcal{P}_2(X)$, it yields weak K -convexity of the Boltzmann entropy $\text{Ent}(\cdot|m')$ on $\mathcal{P}_2(X')$. Moreover, the Cheeger energy for (X', d, m') is still quadratic. (The relaxed gradients w.r.t. m and m' , resp., are local operators, they coincide on X_0 . And $X' \setminus X_0$ is a m' -zero set.) Thus (X', d, m') satisfies the condition $\text{RCD}(K, \infty)$, cf. [3], Thm. 4.19 (iii).

(iii) Next we consider the weighted mms (X', d, m'_1) with $m'_1 = e^{-V}m'$. Since V is lower semicontinuous and bounded from below by $-C_0 - C_1d^2(\cdot, x_0)$, it follows that the functional $S : \mu \mapsto \int V d\mu$ on $\mathcal{P}_2(X')$ is lower semicontinuous and bounded from below by $-C_0 - C_1W_2^2(\cdot, \delta_{x_0})$. (For the proof of the lower semicontinuity, approximate V from below by continuous $V_i \nearrow V$.)

The weak κ -convexity of V on X implies that S is weakly κ -convex on $\mathcal{P}_2(X')$. Indeed, if $\mu_0 = \sum_{i=1}^{\infty} \lambda_i \cdot \delta_{x_i}$ and $\mu_1 = \sum_{i=1}^{\infty} \lambda_i \cdot \delta_{y_i}$ for some $\lambda_i \geq 0$ and $x_i, y_i \in X'$ are given in such a way that $q = \sum_{i=1}^{\infty} \lambda_i \cdot \delta_{(x_i, y_i)}$ is an optimal coupling of them and if z_i is a t -intermediate point of x_i, y_i (i.e. $\frac{1}{t}d(x_i, z_i) = \frac{1}{1-t}d(y_i, z_i) = d(x_i, y_i)$) with $V(z_i) \leq (1-t)V(x_i) + tV(y_i) - \frac{\kappa}{2}(t(1-t)d^2(x_i, y_i))$ for each $i \in \mathbb{N}$ then $\mu_t := \sum_{i=1}^{\infty} \lambda_i \cdot \delta_{z_i}$ is a t -intermediate point of μ_0, μ_1 and

$$S(\mu_t) \leq (1-t)S(\mu_0) + tS(\mu_1) - \frac{\kappa}{2}(t(1-t)W_2^2(\mu_0, \mu_1)).$$

For arbitrary discrete measures $\mu_0 = \sum_{i=1}^{\infty} \alpha_i \cdot \delta_{x_i}$ and $\mu_1 = \sum_{j=1}^{\infty} \beta_j \cdot \delta_{y_j}$, each optimal coupling of them can be written as $q = \sum_{i,j=1}^{\infty} \lambda_{ij} \cdot \delta_{(x_i, y_j)}$ for suitable $\lambda_{ij} \geq 0$ with $\sum_{k=1}^{\infty} \lambda_{ik} = \alpha_i$ and $\sum_{k=1}^{\infty} \lambda_{kj} = \beta_j$. Replacing $\alpha_i \delta_{x_i}$ and $\beta_j \delta_{y_j}$ by $\sum_{k=1}^{\infty} \lambda_{ik} \delta_{x_i}$ and $\sum_{k=1}^{\infty} \lambda_{kj} \delta_{y_j}$, resp., and re-labeling the indices, we may assume without restriction that μ_0, μ_1 and q are always given as above.

Given general $\mu_0, \mu_1 \in \mathcal{P}_2(X')$, choose $\mu_s^n \in \mathcal{P}_2(X')$ for $s \in \{0, 1\}$ of the above type with $\mu_s^n \rightarrow \mu_s$ and $\int V d\mu_s^n \rightarrow \int V d\mu_s$ as $n \rightarrow \infty$ and for given $t \in (0, 1)$ choose t -intermediate points μ_t^n of μ_0^n, μ_1^n satisfying the above inequality for κ -convexity. Local compactness of $\mathcal{P}_2(X')$ and W_2^2 -boundedness of $(\mu_t^n)_{n \in \mathbb{N}}$ implies that for each $t \in (0, 1)$ there exists a converging subsequence, again denoted by $\mu_t^n, n \in \mathbb{N}$, of t -intermediate points of μ_0^n, μ_1^n with limit point μ_t which then must be a t -intermediate point of μ_0, μ_1 . Lower semicontinuity of S implies

$$\begin{aligned} S(\mu_t) &\leq \liminf_{n \rightarrow \infty} S(\mu_t^n) \\ &\leq \liminf_{n \rightarrow \infty} \left[(1-t)S(\mu_0^n) + tS(\mu_1^n) - \frac{\kappa}{2}(t(1-t)W_2^2(\mu_0^n, \mu_1^n)) \right] \\ &= \left[(1-t)S(\mu_0) + tS(\mu_1) - \frac{\kappa}{2}(t(1-t)W_2^2(\mu_0, \mu_1)) \right]. \end{aligned}$$

This is the weak κ -convexity of S on $\mathcal{P}_2(X')$.

Since $\text{Ent}(\mu|m') + \int V d\mu = \text{Ent}(\mu|e^{-V}m')$ and since $\text{Ent}(\mu|m')$ is strongly K -convex on $\mathcal{P}_2(X')$ this also yields the weak $(K + \kappa)$ -convexity of $\text{Ent}(\mu|e^{-V}m')$ on $\mathcal{P}_2(X')$ or, in other words, the $\text{CD}(K + \kappa)$ -condition for (X', d, m'_1) .

(iv) The next step will be to prove that the Cheeger energy for (X', d, m'_1) is quadratic or, equivalently, that the relaxed gradients in (X', d, m'_1) are quadratic. The assumption on the

quadratic lower bound for V guarantees the requested integrability condition for m'_1 . Moreover, the local boundedness of V guarantees that on each ball, the measures m' and m'_1 are equivalent with Radon-Nikodym density bounded from above and below (away from 0). Thus according to [2], Lemma 4.11, the relaxed gradients w.r.t. m' and w.r.t. m'_1 coincide. Since the former already was proven to be quadratic this finally proves the $\text{RCD}(K + \kappa)$ -condition for (X', d, m'_1) .

(v) These observations will now be applied with $n \cdot V$ in the place of V which then yields that the space $(X', d, e^{-nV} m')$ satisfies the $\text{RCD}(K + n\kappa, \infty)$ -condition. The latter in turn implies the existence of an $\text{EVI}_{\tilde{\kappa}_n}$ -gradient flow for the functional $\tilde{S}_n(\mu) = \text{Ent}(\mu | e^{-nV} m') = \text{Ent}(\mu | m') + n \int V d\mu$ on $\mathcal{P}_2(X')$ with $\tilde{\kappa}_n = K + n\kappa$, [3]. Replacing \tilde{S}_n by $S_n = \frac{1}{n} \tilde{S}_n$ and rescaling time by the factor n yields the claim. \square

Lemma 2. *There exists an EVI_{κ} -gradient flow for the functional $S(\mu) = \int V d\mu$ on $\mathcal{P}_2(X')$.*

Proof. (i) Let $(\mu_t^n)_{t>0}$ denote the EVI_{κ_n} -gradient flow for the functional $S_n(\mu) = \frac{1}{n} \text{Ent}(\mu | m) + \int V d\mu$ on $\mathcal{P}_2(X')$ with $\kappa_n = \frac{1}{n} K + \kappa$, unique and well-defined for each starting point $\mu_0 \in \mathcal{P}_2(X')$. The property (2) ($\forall t > 0$) – with S_n and W_2 in the place of V and d – is obviously equivalent to the property

$$\frac{1}{2} W_2^2(\mu_t^n, \nu) e^{2\kappa_n t} - \frac{1}{2} W_2^2(\mu_s^n, \nu) e^{2\kappa_n s} \leq \frac{1}{2\kappa_n} (e^{2\kappa_n t} - e^{2\kappa_n s}) [S_n(\nu) - S_n(\mu_t^n)] \quad (5)$$

($\forall 0 \leq s < t$). In the case $\kappa = 0$, the prefactor $\frac{1}{2\kappa} (e^{2\kappa t} - e^{2\kappa s})$ has to be replaced by $(t - s)$.

(ii) Note that our assumptions on m and V imply that $S_n(\nu) \leq S_1(\nu) + C'' + C' W_2^2(\nu, \delta_{x'}) \leq C^*$ whenever $S_1(\nu) < \infty$ and $-S_n(\mu_t^n) \leq C'' + C' W_2^2(\mu_t^n, \delta_{x'}) + C_0 + C_1 W_2^2(\mu_t^n, \delta_{x_0})$. The differential version of (5) (or, in other words, (2) with S_n and W) allows to estimate uniformly in n

$$\begin{aligned} \frac{d}{dt} \frac{1}{2} W_2^2(\mu_t^n, \nu) &\leq -\kappa W_2^2(\mu_t^n, \nu) + S(\nu) - S(\mu_t^n) \\ &\leq C + c \cdot \frac{1}{2} W_2^2(\mu_t^n, \nu) \end{aligned}$$

which leads to the uniform growth bound

$$\frac{1}{2} W_2^2(\mu_t^n, \nu) \leq e^{ct} \cdot \frac{1}{2} W_2^2(\mu_0, \nu) + (e^{ct} - 1) \cdot C/c$$

with constants C and c which may depend on ν but not on n . Thus for each $t \in (0, 1)$, the sequence $(\mu_t^n)_{n \in \mathbb{N}}$ is W_2 -bounded. Local compactness of $\mathcal{P}_2(X')$ implies that there exists some limit point μ_t such that $\mu_t^n \rightarrow \mu_t$ weakly – at least after passing to a subsequence. A diagonal argument allows to choose this subsequence in such a way that convergence holds for all rational $t \geq 0$.

(iii) The quadratic lower bound (in terms of W_2) for $\mu \mapsto \text{Ent}(\mu | m)$ and the lower semicontinuity $\mu \mapsto \int V d\mu$ imply that $\liminf_{n \rightarrow \infty} S_n(\mu_t^n) \geq S(\mu_t)$. Moreover, $\lim_{n \rightarrow \infty} S_n(\nu) = S(\nu)$ provided $\text{Ent}(\nu) < \infty$. Thus for all such ν and for all rational $0 \leq s < t$

$$\frac{1}{2} W_2^2(\mu_t, \nu) e^{2\kappa t} - \frac{1}{2} W_2^2(\mu_s, \nu) e^{2\kappa s} \leq \frac{1}{2\kappa} (e^{2\kappa t} - e^{2\kappa s}) [S(\nu) - S(\mu_t)] \quad (6)$$

or, equivalently,

$$\frac{d}{dt} \frac{1}{2} W_2^2(\mu_t, \nu) + \kappa W_2^2(\mu_t, \nu) \leq S(\nu) - S(\mu_t) \quad (7)$$

where the time derivative is taken along the discrete time parameter set.

(iv) The next step is to extend (6) (and in turn therefore also (7)) to all $\nu \in \mathcal{P}_2(X')$, i.e. to get rid of the constraint $\text{Ent}(\nu | m) < \infty$. That is, given $\nu \in \mathcal{P}_2(X')$ we have to find a sequence of $\nu_n \in \mathcal{P}_2(X')$ with $\text{Ent}(\nu_n | m') < \infty$ such that $W_2(\nu_n, \nu) \rightarrow 0$ and $\int V d\nu_n \rightarrow \int V d\nu$ as $n \rightarrow \infty$.

One easily verifies that it suffices to prove this approximation result for functions V which are uniformly bounded from below. Indeed, assume that for each $i \in \mathbb{N}$ there exists a sequence of measures $\nu_{i,n} \in \mathcal{P}_2(X')$ with $\text{Ent}(\nu_{i,n} | m') < \infty$ such that $W_2(\nu_{i,n}, \nu) \leq \frac{1}{n}$ and $\int V d\nu_{i,n} \leq$

$\int V_i d\nu + \frac{1}{n}$ for all n where $V_i := \max\{V, -i\}$. Then the sequence $\nu_{n,n}$ will do the job for V :

$$\liminf_{n \rightarrow \infty} \int V d\nu_{n,n} \leq \liminf_{n \rightarrow \infty} \left[\int V_n d\nu_{n,n} - \frac{1}{n} \right] \leq \liminf_{n \rightarrow \infty} \left[\int V_n d\nu \right] = \int V d\nu.$$

Similarly, a simple monotone convergence argument shows that it suffices to prove the approximation results for measures ν which are compactly supported in X_0 since the assertions (6) and (7) are void if ν is not supported in X_0 .

Thus, let us now assume that ν has compact support K in X_0 and that $V \geq 0$. Fix $\epsilon > 0$ such that $K' := \overline{B}_\epsilon(K)$ is compact in X_0 . Let $\bar{\nu}_t$, $t > 0$, be the gradient flow in $\mathcal{P}_2(X')$ for the entropy $\text{Ent}(\cdot|m')$ starting in ν or, in other words, the heat distribution on X' after time t for the initial distribution ν . Put $\nu_t(\cdot) = \frac{1}{\alpha_t} \bar{\nu}_t(K' \cap \cdot)$ where $\alpha_t = \bar{\nu}_t(K')$. Then $\alpha_t \rightarrow 1$ as $t \rightarrow 0$. (For instance, this will follow from the integrated Gaussian estimate [10] and the fact that d is the intrinsic metric for the Cheeger energy [4].) Thus

$$\limsup_{t \rightarrow 0} \int V d\nu_t = \limsup_{t \rightarrow 0} \frac{1}{\alpha_t} \int_{K'} V d\bar{\nu}_t \leq \int_{K'} V d\nu = \int V d\nu$$

since $\bar{\nu}_t \rightarrow \nu$ weakly and since V is bounded and continuous on K' . Moreover, obviously $\text{Ent}(\nu_t|m') < \infty$ for all $t > 0$ and $W_2(\nu_t, \nu) \rightarrow 0$ as $t \rightarrow 0$.

(v) Assume that the accumulation points of $(\mu_t^n)_{n \in \mathbb{N}}$ are not unique, say $\tilde{\mu}_t$ is another accumulation point (again for simplicity along one fixed subsequence for all rational t). Applying the previous estimate twice (first with $\tilde{\mu}_t$ in the place of ν , then with μ_t in the place of ν and $\tilde{\mu}_t$ in the place of μ_t) yields $\frac{d}{dt} \frac{1}{2} W_2^2(\mu_t, \tilde{\mu}_t) + \kappa W_2^2(\mu_t, \tilde{\mu}_t) \leq 0$ which implies $W_2(\mu_t, \mu'_t) = 0$. This proves the uniqueness of the accumulation points for all rational t . Finally, applying the estimate (6) (or (7)) to $\nu = \mu_s$ we obtain local Lipschitz continuity in $t \in (0, \infty) \cap \mathbb{Q}$ of the flow $(\mu_t)_t$ which allows to extend it to all t , still satisfying all the previous estimates.

(vi) Now let us consider two flows starting in different points, say $\mu_0, \mu'_0 \in \mathcal{P}_2(X')$. Again from (7) we may deduce $\frac{d}{dt} \frac{1}{2} W_2^2(\mu_t, \mu'_t) + \kappa W_2^2(\mu_t, \mu'_t) \leq 0$ which immediately yields the W_2 -expansion bound

$$W_2(\mu_t, \mu'_t) \leq e^{-\kappa t} W_2(\mu_0, \mu'_0). \quad (8)$$

□

Lemma 3. For each $\mu_0 \in \mathcal{P}_2(X')$ the EVI_κ -flow for S satisfies

$$\text{Var}_2(\mu_t) \leq e^{-2\kappa t} \text{Var}_2(\mu_0)$$

for all $t > 0$ where $\text{Var}_2(\nu) := \int \int d^2(x, y) \nu(dx) \nu(dy)$.

Proof. The EVI_κ -property (6) for $(\mu_t)_{t \geq 0}$ applied to $\nu = \delta_y$ states

$$\frac{1}{2} \int d^2(x, y) d\mu_t(x) e^{2\kappa t} - \frac{1}{2} \int d^2(x, y) d\mu_s(x) e^{2\kappa s} \leq \frac{1}{2\kappa} (e^{2\kappa t} - e^{2\kappa s}) [V(y) - S(\mu_t)]. \quad (9)$$

Integrating this w.r.t. $\mu_t(dy)$ or $\mu_s(dy)$, resp., yields

$$\frac{1}{2} \text{Var}_2(\mu_t) e^{2\kappa t} - \frac{1}{2} \int \int d^2(x, y) d\mu_s(x) d\mu_t(x) e^{2\kappa s} \leq 0 \quad (10)$$

and

$$\frac{1}{2} \int \int d^2(x, y) d\mu_s(x) d\mu_t(x) e^{2\kappa t} - \frac{1}{2} \text{Var}_2(\mu_s) e^{2\kappa s} \leq \frac{1}{2\kappa} (e^{2\kappa t} - e^{2\kappa s}) [S(\mu_s) - S(\mu_t)]. \quad (11)$$

Adding up these two inequalities and letting $s \rightarrow t$ we obtain

$$\frac{d}{dt} \frac{1}{2} \text{Var}_2(\mu_t) e^{2\kappa t} \leq 0$$

since $S(\mu_s) \rightarrow S(\mu_t)$. This proves the claim. □

Corollary 1. For each $x_0 \in X'$ there exists a unique curve $(x_t)_{t \geq 0}$ such that $\mu_t = \delta_{x_t}$ is the EVI_κ -gradient flow for S starting in $\mu_0 = \delta_{x_0}$.

The curve $(x_t)_{t \geq 0}$ is the unique EVI_κ -gradient flow for V starting in x_0 .

Proof. Put $\mu_0 = \delta_{x_0}$. Then $\text{Var}_2(\mu_0) = 0$ and thus, according to the previous lemma, also $\text{Var}_2(\mu_t) = 0$. This implies that each of the measures μ_t is supported by a single point, say x_t .

Moreover, the fact that the curve $(\delta_{x_t})_{t \geq 0}$ in $\mathcal{P}_2(X')$ is the EVI_κ -gradient flow for S starting in $\mu_0 = \delta_{x_0}$ immediately implies that the curve $(x_t)_{t \geq 0}$ in X' is the (unique) EVI_κ -gradient flow for V starting in x_0 . (To verify, (2) just apply (7) to Diracs $\nu = \delta_{\{z\}}$.) \square

Corollary 2. *The following properties are equivalent:*

- (i) V is weakly κ -convex;
- (ii) V is strongly κ -convex;
- (iii) V is κ -convex in EVI -sense, i.e. for each $x_0 \in X'$ there exists a curve $(x_t)_{t \geq 0}$ in X' such that for all $z \in X_0$ and all $t > 0$

$$\frac{d}{dt} \frac{1}{2} d^2(x_t, z) + \kappa d^2(x_t, z) \leq V(z) - V(x_t).$$

Note that the implications (iii) \Rightarrow (ii) \Rightarrow (i) hold true in any geodesic space, see [5].

Remark. Our main result extends previous results in various respects.

In the setting of Alexandrov spaces (i.e. metric spaces with synthetic lower bounds for the sectional curvature), the existence, uniqueness and Lipschitz continuity of gradient flows for semiconvex functions has been derived in an unpublished manuscript by Perelman and Petrunin [8]. The proof has been worked out and published by Plaut [9] and Lytchak [7]. This result requires no local compactness of the state space. The function V , however, is always required to be bounded and Lipschitz continuous.

In the setting of Hilbert spaces (and to some extent also for metric spaces), a far reaching theory of gradient flows for lower semicontinuous functions (with suitable lower growth bounds and compact sublevel sets) has been developed by Brezis as well as by De Giorgi and his school, see in particular the monograph [1] by Ambrosio-Gigli-Savare. In this context, however, the state space X has to be flat (or of nonpositive curvature).

In the setting of metric measure spaces with synthetic Ricci bounds, currently no direct proof – without passing to the Wasserstein space but instead using more detailed properties of the Hessian of V , cf. [13] – is known to the author.

APPENDIX 1

Let us finally add some observations which are not directly related to our main result but which might be of independent interest.

Denote the unique EVI_κ -flow for V starting in $x_0 \in X'$ by $x_t = \Phi_t(x_0), t > 0$. Then for each $t > 0$ the map $\Phi_t : X' \rightarrow X'$ is Lipschitz continuous (with Lipschitz constant $e^{-\kappa t}$). The family of maps $\Phi_t, t > 0$ is a semigroup w.r.t. composition: $\Phi_t \circ \Phi_s = \Phi_{t+s}$.

Corollary 3. *For each $\mu_0 \in \mathcal{P}_2(X')$ the (unique) EVI_κ -flow for S starting in μ_0 is obtained via push forward as*

$$\mu_t = (\Phi_t)_* \mu_0. \tag{12}$$

That is, $\int f(x) d\mu_t(x) = \int f(\Phi_t(x)) d\mu_0(x)$ for each bounded Borel function f on X .

This is a straightforward consequence of the following lemma which states that the EVI_κ -flow for S is σ -additive in μ .

Lemma 4. *Assume that $\mu_0 = \sum_{i=1}^n \lambda_i \mu_0^i$ for some probability measures μ_0^i and positive numbers λ_i with $\sum_i \lambda_i = 1$ where $n \in \mathbb{N}$ or $n = \infty$. For each i , let $(\mu_t^i)_{t > 0}$ be the EVI_κ -flow for S starting in μ_0^i . Then*

$$\mu_t = \sum_{i=1}^n \lambda_i \mu_t^i \tag{13}$$

is the EVI_κ -flow starting in μ_0 .

Proof. Let us define μ_t as in (13) and for fixed t let q_t be an optimal coupling of μ_t and ν . Observe that $\mu_t^i = f_t^i \mu_t$ for suitable bounded nonnegative functions f_t^i on X . Put $dq_t^i(x, y) = f_t^i(x) dq_t(x, y)$ and $\nu_t^i(\cdot) = q_t^i(X, \cdot)$. Then ν and q_t can be represented as

$$\nu = \sum_{i=1}^n \lambda_i \nu^i, \quad q_t = \sum_{i=1}^n \lambda_i q_t^i$$

where for each i the probability measure q^i is an optimal coupling of μ_t^i and ν^i (since sub-couplings of optimal couplings are optimal). Thus

$$W_2^2(\mu_t, \nu) = \sum_i \lambda_i W_2^2(\mu_t^i, \nu_t^i). \quad (14)$$

On the other hand, for each $s > t$

$$W_2^2(\mu_s, \nu) \leq \sum_i \lambda_i W_2^2(\mu_s^i, \nu_t^i). \quad (15)$$

(One should not expect equality since in general the optimal decomposition of ν induced by the decomposition $\mu_t = \sum_{i=1}^n \lambda_i \mu_t^i$ will not be optimal for the coupling with $\mu_s = \sum_{i=1}^n \lambda_i \mu_s^i$.) The EVI_κ -property for $(\mu_s^i)_{s>t}$ (with observation point ν_t^i) implies

$$\lim_{s \searrow t} \frac{1}{2(s-t)} \left[W_2^2(\mu_s^i, \nu_t^i) - W_2^2(\mu_t^i, \nu_t^i) \right] + \kappa W_2^2(\mu_t^i, \nu_t^i) \leq S(\nu_t^i) - S(\mu_t^i). \quad (16)$$

Adding up the last three inequalities yields

$$\begin{aligned} & \limsup_{s \searrow t} \frac{1}{2(s-t)} \left[W_2^2(\mu_s, \nu_t) - W_2^2(\mu_t, \nu_t) \right] + \kappa W_2^2(\mu_t, \nu_t) \\ & \leq \sum_i \lambda_i \left[\lim_{s \searrow t} \frac{1}{2(s-t)} \left(W_2^2(\mu_s^i, \nu_t^i) - W_2^2(\mu_t^i, \nu_t^i) \right) + \kappa W_2^2(\mu_t^i, \nu_t^i) \right] \\ & \leq \sum_i \lambda_i \left[S(\nu_t^i) - S(\mu_t^i) \right] \\ & = \left[S(\nu_t) - S(\mu_t) \right]. \end{aligned}$$

For the last equality, we made use of the additivity of S . □

APPENDIX 2

Finally, let us consider EVI -gradient flows in a more general setting. Let (X, d) be an arbitrary complete geodesic space and let $V : X \rightarrow (-\infty, +\infty]$ be a continuous function, finite on a dense subset X_0 of X , and let a family of maps $\Phi_t : X \rightarrow X$ be given such that for each $x \in X$ the curve $(\Phi_t(x))_{t \geq 0}$ is the unique EVI_κ -gradient flow for V (for some fixed $\kappa \in \mathbb{R}$). This gradient flow comes with the 'natural' parametrization

$$\frac{d}{dt} V(\Phi_t) = -|\nabla^- V(\Phi_t)|^2 = -|\dot{\Phi}_t|^2$$

We will now have a closer look on a reparametrization $(\Psi_a(x))_{a \in \mathbb{R}}$ of this flow with $\frac{\partial}{\partial a} V(\Psi_a) = 1$. For $a \in \mathbb{R}$ and $x \in X$ we put

$$T_a(x) = \inf \{ t \geq 0 : V(\Phi_t(x)) \leq a \}$$

(with $\inf \emptyset := \infty$) and if $T_a(x) < \infty$

$$\Psi_a(x) = \Phi_{T_a(x)}(x).$$

Continuity of $x \mapsto V(x)$ and $t \mapsto \Phi_t(x)$ for $t > 0$ imply that $V(\Psi_a(x)) = a$ provided $0 < T_a(x) < \infty$.

Proposition. *For all $x, x' \in X$ and all $a \in \mathbb{R}$ with $T_a(x), T_a(x') < \infty$*

$$d(\Psi_a(x), \Psi_a(x')) \leq \exp\left(-\kappa \frac{T_a(x) + T_a(x')}{2}\right) \cdot d(x, x'). \quad (17)$$

Proof. Given $x, x' \in X$ and $a \in \mathbb{R}$ put $T = T_a(x) \wedge T_a(x')$ and $S = |T_a(x) - T_a(x')|$. Assume without restriction that $T_a(x) \leq T_a(x')$. Then $T_a(x) = T$, $T_a(x') = T + S$ and $\Psi_a(x) = \Phi_T(x)$, $\Psi_a(x') = \Phi_{T+S}(x')$. Applying the basic contraction property (3) with $t = T$ yields

$$d(\Phi_T(x), \Phi_T(x')) \leq e^{-\kappa T} d(x, x'). \quad (18)$$

Next we apply (2) to the flow $(\Phi_{T+t}(x'))_{t>0}$ with observation point $z = \Phi_T(x)$. It states that

$$\frac{d}{dt} \frac{1}{2} d^2(\Phi_T(x), \Phi_{T+t}(x')) + \kappa d^2(\Phi_T(x), \Phi_{T+t}(x')) \leq V(\Phi_T(x)) - V(\Phi_{T+t}(x')) \leq 0$$

for $t \in [0, S]$ which immediately yields

$$d(\Phi_T(x), \Phi_{T+S}(x')) \leq e^{-\kappa S/2} d(\Phi_T(x), \Phi_T(x')).$$

Together with (18) we thus obtain

$$d(\Phi_T(x), \Phi_{T+S}(x')) \leq e^{-\kappa(T+S/2)} d(x, x') = \exp\left(-\frac{\kappa}{2}(T_a(x) + T_a(x'))\right) \cdot d(x, x').$$

This is the claim. \square

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