ON THE EQUIVALENCE OF THE ENTROPIC CURVATURE-DIMENSION CONDITION AND BOCHNER'S INEQUALITY ON METRIC MEASURE SPACES

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ABSTRACT. We prove the equivalence of the curvature-dimension bounds of Lott-Sturm-Villani (via entropy and optimal transport) and of Bakry–Émery (via energy and Γ_2-calculus) in complete generality for infinitesimally Hilbertian metric measure spaces. In particular, we establish the full Bochner inequality on such metric measure spaces. Moreover, we deduce new contraction bounds for the heat flow on Riemannian manifolds and on mms in terms of the L^2-Wasserstein distance.

CONTENTS

1. INTRODUCTION

Bochner's inequality is one of the most fundamental estimates in geometric analysis. It states that

$$\frac{1}{2}\Delta|\nabla u|^2 - \langle \nabla u, \nabla \Delta u\rangle \geq K \cdot |\nabla u|^2 + \frac{1}{N} \cdot |\Delta u|^2 \tag{1.1}$$

for each smooth function u on a Riemannian manifold (M,g) provided $K \in \mathbb{R}$ is a lower bound for the Ricci curvature on M and $N \in (0,\infty]$ is an upper bound for the dimension of M. The main results of this paper is an analogous Bochner inequality on metric measure spaces (X,d,m) with linear heat flow and satisfying the (reduced) curvature-dimension condition. Indeed, we will also prove the converse: if the heat flow on a mms (X,d,m) is linear then an appropriate version of (1.1) (for the canonical gradient and Laplacian on X) will imply the reduced curvature-dimension condition. Besides that, we also derive new, sharp W_2-contraction results for the heat flow as well as pointwise gradient estimates and prove that each of them is equivalent to the curvature-dimension condition. That way, we obtain a complete one-to-one correspondence between the

Date: July 30, 2013.

Eulerian picture captured in the Bochner inequality and the Lagrangian interpretation captured in the curvature-dimension inequality.

The **curvature-dimension condition** $\mathrm{CD}(K,N)$ was introduced by Sturm in [39]. It was later adopted and slightly modified by Lott & Villani, see also the elaborate presentation in the monograph [40]. The $\mathrm{CD}(K,N)$-condition for finite N is a sophisticated tightening up of the much simpler $\mathrm{CD}(K,\infty)$-condition introduced as a synthetic Ricci bound for metric measure spaces independently by Sturm [39] and Lott & Villani [29]. From the very beginning, a disadvantage of the $\mathrm{CD}(K,N)$-condition for finite N was the lack of a local-to-global result. To overcome this drawback, Bacher & Sturm [9] introduced the *reduced curvature-dimension condition* $\mathrm{CD}^*(K,N)$ which has a local-to-global property and which is equivalent to the local version of $\mathrm{CD}(K,N)$. The curvature-dimension condition $\mathrm{CD}(K,N)$ has been verified for Riemannian manifolds [39], Finsler spaces [31], Alexandrov spaces [35], [42], cones [8] and warped products of Riemannian manifolds [24]. Actually, in all these cases the conditions $\mathrm{CD}(K,N)$ and $\mathrm{CD}^*(K,N)$ turned out to be equivalent.

A completely different approach to generalized curvature-dimension bounds was set forth in the pioneering work of Bakry and Émery [10]. It applies to the general setting of Dirichlet forms and the associated Markov semigroups and is formulated using the (iterated) *carré du champ* operators built from the generator of the semigroup. This **energetic curvature-dimension condition** $\mathrm{BE}(K,N)$ has proven a powerful tool in particular in infinite dimensional situations. It yields hypercontractivity of the semigroup and has successfully been used to derive functional inequalities like the logarithmic Sobolev inequalities in a variety of examples. Among the remarkable analytic consequences of the Bakry–Émery condition $\mathrm{BE}(K,\infty)$ we single out the point-wise gradient estimates for the semigroup H_t. It implies that for any f in a large class of functions

$$\Gamma(H_t f) \;\leq\; \mathrm{e}^{-2Kt}\, H_t\Gamma(f)\,,$$

where Γ is the carré du champ operator.

The relation between the two notions of curvature bounds based on optimal transport and Dirichlet forms has been studied in large generality by Ambrosio, Gigli and Savaré in a series of recent works [4, 5], see also [2]. The key tool of their analysis is a powerful calculus on metric measure spaces which allows them to match the two settings. Starting from a metric measure structure they introduce the so called Cheeger energy which takes over the role of the 'standard' Dirichlet energy and is obtained by relaxing the L^2-norm of the slope of Lipschitz functions. A key result is the identification of the L^2-gradient flow of the Cheeger energy with the Wasserstein gradient flow of the entropy. This is the mms equivalent of the famous result by Jordan–Kinderlehrer–Otto [23] and allows one to define unambiguously a heat flow in metric measure spaces.

We say that a metric measure space is *infinitesimally Hilbertian* if the heat flow is linear. This is equivalent to the Cheeger energy being the associated Dirichlet form. We denote its domain by $W^{1,2}$. Under the assumption of linearity of the heat flow, Ambrosio–Gigli–Savaré prove that $\mathrm{CD}(K,\infty)$ implies $\mathrm{BE}(K,\infty)$ and the converse also holds under an additional regularity assumption. Combining linearity of the heat flow with the $\mathrm{CD}(K,\infty)$ condition leads to the **Riemannian curvature condition** $\mathrm{RCD}(K,\infty)$ introduced in [4]. This concept again turns out to be stable under Gromov–Hausdorff convergence and tensorization.

Recently, also **Bochner's inequality** has been extended to singular spaces. Ohta & Sturm [32] proved it for Finsler spaces and Gigli, Kuwada & Ohta [21] and Zhang & Zhu [43] for Alexandrov spaces. Finally, Ambrosio, Gigli & Savaré established the Bochner inequality without the dimension term (i.e. with $N=\infty$) in $\mathrm{RCD}(K,\infty)$ spaces. However, in the classical setting, the full strength of Bochner's inequality only comes to play if also the dimension effect is taken into account, i.e. with finite N. This can be seen for example from the famous results of Li–Yau [28] who derive from it a differential Harnack inequality, eigenvalue estimates for the Laplacian and Gaussian heat kernel bounds.

We prove the equivalence of curvature-dimension bounds via optimal transport and via the Bakry–Émery approach in full generality for infinitesimally Hilbertian metric measure spaces. In particular, we establish the full Bochner inequality on such metric measure spaces.

Our approach strongly relies on properties and consequences of a new curvature-dimension condition, the so-called **entropic curvature dimension condition** $\mathrm{CD}^e(K,N)$. It simply states that the Boltzmann entropy Ent is (K,N)-convex on the Wasserstein space $\mathscr{P}_2(X,d)$. Here a function s on an interval $I\subset\mathbb{R}$ is called (K,N)-convex if

$$s'' \;\geq\; K+\frac{1}{N}\cdot(s')^2\,. \tag{1.2}$$

holds in distribution sense. A function S on a geodesic space is called (K,N)-convex if it is (K,N)-convex along each unit speed geodesic – or at least along each curve within a class of unit speed geodesics which connect each pair of points in X. This way, (K,N)-convexity is a weak formulation of

$$\mathrm{Hess}\,S \;\geq\; K+\frac{1}{N}\big(\nabla S\otimes\nabla S\big)\,. \tag{1.3}$$

Our first result is the following

Theorem 1 (Theorem 3.12)**.** *For a essentially non-branching mms (see Definition 3.10) the entropic curvature-dimension condition* $\mathrm{CD}^e(K,N)$ *is equivalent to the reduced curvature-dimension condition* $\mathrm{CD}^*(K,N)$*.*

We say that a metric measure space satisfies the **Riemannian curvature-dimension condition** $\mathrm{RCD}^*(K,N)$ if it is infinitesimally Hilbertian and satisfies $\mathrm{CD}^e(K,N)$ or $\mathrm{CD}^*(K,N)$. This notion turns out to have the natural stability properties. Namely, we prove (see Theorems 3.22, 3.23, 3.25) that the $\mathrm{RCD}^*(K,N)$ condition is preserved under measured Gromov–Hausdorff convergence as well as under tensorization of metric measure spaces and that it has a local–to–global property.

The geometric intuition coming from the analysis of (K,N)-convex functions and their gradient flows leads to a new form of the **Evolution Variation Inequality** $\mathrm{EVI}_{K,N}$ on the Wasserstein space taking into account also the effect of the dimension bound. Until now, the notion of $\mathrm{EVI}_{K,N}$ gradient flow was known only without dimension term (i.e. with $N=\infty$). These Evolution Variational Inequalities first appeared in the setting of Hilbert spaces where they characterize uniquely the gradient flows of K-convex functionals. In a general metric setting and in connection with optimal transport these inequalities have been extensively studied in [34, 16, 4]. In particular, it turned out that $\mathrm{RCD}(K,\infty)$ spaces can be characterized by the fact that the heat flow is an $\mathrm{EVI}_{K,\infty}$ gradient flow of the entropy. Here we obtain a reinforcement of this result. Namely, the new Riemannian curvature-dimension condition $\mathrm{RCD}^*(K,N)$ is equivalent to the existence of an $\mathrm{EVI}_{K,N}$ gradient flow of the entropy in the following sense.

Theorem 2 (Definition 2.14, Theorem 3.17)**.** *A mms* (X,d,m) *satisfies* $\mathrm{RCD}^*(K,N)$ *if and only if* (X,d) *is a length space,* m *satisfies an integrability condition* (3.6) *and every* $\mu_0\in\mathscr{P}_2(X,d)$ *is the starting point of a curve* $(\mu_t)_{t\geq0}$ *in* $\mathscr{P}_2(X,d)$ *such that for any other* $\nu\in\mathscr{P}_2(X,d)$ *and a.e.* $t>0$*:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{s}_{K/N}\left(\frac{1}{2}W_2(\mu_t,\nu)\right)^2+K\cdot\mathfrak{s}_{K/N}\left(\frac{1}{2}W_2(\mu_t,\nu)\right)^2\;\leq\;\frac{N}{2}\left(1-\frac{U_N(\nu)}{U_N(\mu_t)}\right). \tag{1.4}$$

Here $U_N(\mu)=\exp\Big(-\frac{1}{N}\mathrm{Ent}(\mu)\Big)$ *and* $\mathfrak{s}_\kappa(r)=\sqrt{1/\kappa}\sin\big(\sqrt{\kappa}r\big)$ *provided* $\kappa>0$ *and* $\mathfrak{s}_\kappa(r)=\sqrt{1/(-\kappa)}\sinh\big(\sqrt{-\kappa}r\big)$, $\mathfrak{s}_0(r)=r$ *for* $\kappa<0$ *resp.* $\kappa=0$*.*

This curve is unique, in fact, it is the heat flow which we denote in the following by $\mu_t=H_t\mu_0$.

The Evolution Variation Inequality $\mathrm{EVI}_{K,N}$ as stated above immediately implies new, sharp contraction estimates (or, more precisely, expansion bounds) in Wasserstein metric for the heat flow.

Theorem 3 (Theorem 2.19, Theorem 4.1 and Proposition 2.12)**.** *Let (X,d,m) be a $\mathrm{RCD}^*(K,N)$ space. Then for any $\mu,\nu\in\mathscr{P}_2(X,d)$ and $s,t>0$:*

$$\mathfrak{s}_{K/N}\left(\frac{1}{2}W_2(H_t\mu,H_s\nu)\right)^2 \le e^{-K(s+t)}\cdot\mathfrak{s}_{K/N}\left(\frac{1}{2}W_2(\mu,\nu)\right)^2+\frac{N}{K}\Big(1-e^{-K(s+t)}\Big)\frac{\big(\sqrt{t}-\sqrt{s}\big)^2}{2(s+t)}\,. \tag{1.5}$$

The latter implies the slightly weaker bound

$$W_2(H_t\mu,H_s\nu)^2 \le \mathrm{e}^{-K\tau(s,t)}\cdot W_2(\mu,\nu)^2+2N\frac{1-\mathrm{e}^{-K\tau(s,t)}}{K\tau(s,t)}\big(\sqrt{t}-\sqrt{s}\big)^2\,,$$

where $\tau(s,t)=2(t+\sqrt{ts}+s)/3$. In the particular case $t=s$ this reduces to the well-known estimate $W_2(H_t\mu,H_t\nu)\le\mathrm{e}^{-Kt}\cdot W_2(\mu,\nu)$.

Due to the work of Kuwada [27], it is well known that W_2-expansion bounds are intimately related to pointwise gradient estimates. The next result is a particular case of a more general equivalence that will be the subject of a forthcoming publication [26].

Theorem 4 (Theorem 4.3)**.** *Assume that the mms (X,d,m) is infinitesimally Hilbertian and satisfies a regularity assumption (Assumption 4.2). If the W_2-expansion bound* (1.5) *holds then for any f of finite Cheeger energy:*

$$|\nabla\,\mathrm{H}_t f|_w^2+\frac{4Kt^2}{N\big(e^{2Kt}-1\big)}|\Delta\,\mathrm{H}_t f|^2 \le e^{-2Kt}\,\mathrm{H}_t\left(|\nabla f|_w^2\right)\quad m\text{-a.e.} \tag{1.6}$$

Note that Assumption 4.2 is the same as what is assumed in [5] and it is always satisfied if (X,d,m) is $\mathrm{RCD}(K',\infty)$ for any $K'\in\mathbb{R}$. Hence, Theorem 3 and Theorem 4 imply in particular that (1.6) holds on a $\mathrm{RCD}^*(K,N)$ space. Here $|\nabla f|_w$ denotes the weak upper gradient of f introduced in [6]. This kind of gradient estimate has first been established by Bakry and Ledoux [11] in the setting of Γ-calculus. It is new in the framework of metric measure spaces and allows us to establish the Bochner formula for the canonical gradients and Laplacians on mms.

Theorem 5 (Theorem 4.8)**.** *Assume that the mms (X,d,m) is infinitesimally Hilbertian and satisfies the gradient estimate* (1.6)*. Then for all $f\in D(\Delta)$ with $\Delta f\in W^{1,2}(X,d,m)$ and all $g\in D(\Delta)$ bounded and non-negative with $\Delta g\in L^\infty(X,m)$ we have*

$$\frac{1}{2}\int\Delta g|\nabla f|_w^2\mathrm{d}m-\int g\langle\nabla(\Delta f),\nabla f\rangle\mathrm{d}m\ge K\int g|\nabla f|_w^2\mathrm{d}m+\frac{1}{N}\int g\big(\Delta f\big)^2\mathrm{d}m\,. \tag{1.7}$$

Theorem 6 (Proposition 4.9, Theorem 4.19)**.** *Assume that the mms (X,d,m) is infinitesimally Hilbertian and satisfies Assumption 4.2. Then the Bochner inequality* $\mathrm{BE}(K,N)$ (1.7) *implies the entropic curvature-dimension condition $\mathrm{CD}^e(K,N)$.*

Thus we have closed the circle. All the previous key properties are equivalent to each other, at least if we require the heat flow to be linear.

Theorem 7 (Summary)**.** *Let (X,d,m) be an infinitesimally Hilbertian metric measure space. Then the following properties are equivalent:*

(i) $\mathrm{CD}^*(K,N)$,
(ii) $\mathrm{CD}^e(K,N)$,
(iii) *(X,d) is a length space,* (3.6) *and the existence of the $\mathrm{EVI}_{K,N}$ gradient flow of the entropy starting from every $\mu\in\mathscr{P}_2(X,d)$.*

If one of them is satisfied, we obtain the following:

(iv) *The W_2-expansion bound* (1.5),
(v) *The Bakry–Ledoux pointwise gradient estimate* $\mathrm{BL}(K,N)$ (1.6),
(vi) *The Bochner inequality* $\mathrm{BE}(K,N)$ (1.7).

Moreover, under Assumption 4.2, all of properties (i)–(vi) *are equivalent.*

Remark. Finally, let us point out – on a more heuristic level – two remarkable links between (K,N)-convexity and the Bakry-Émery condition $\mathrm{BE}(K,N)$:

(I) The (K,N)-convexity of a function V on a Riemannian manifold (M,g) can be interpreted as the $BE(K,N)$-condition for the re-scaled drift diffusion

$$dX_t = \sqrt{2\alpha}\, dB_t - \nabla V(X_t)\, dt \tag{1.8}$$

in the limit of vanishing diffusion.

(II) The $\mathrm{BE}(K,N)$-condition for the Brownian motion or heat flow on M is equivalent to the (K,N)-convexity of the function $S = \mathrm{Ent}(.)$ on the Wasserstein space $\mathscr{P}_2(M)$.

Both links are related to each other since the heat flow is the solution to the ODE ("without diffusion")

$$d\mu_t = -\nabla S(\mu_t)\, dt$$

on $\mathscr{P}_2(M)$ (regarded as infinite dimensional Riemannian manifold). The link (II) is the main result of this paper.

To see (I), note that in the case $\alpha > 0$, equilibration and regularization effects of the stochastic dynamic (1.8) can be formulated in terms of the Bakry-Émery estimate for the generator $L = \alpha\Delta - \nabla V \cdot \nabla$ of the associated transition semigroup $\mathrm{H}_t\, u(x) = \mathbb{E}_x[u(X_t)]$. The law of X_t evolves according to the dual semigroup $(\mathrm{H}_t^*)_{t>0}$ with generator $L^*u = \alpha\Delta u + \mathrm{div}(u\cdot\nabla V)$. Assume that the manifold M has dimension $\leq n$ and Ricci curvature $\geq k$. Then the time-changed operator $\widetilde{L} := \frac{1}{\alpha}L$ satisfies the Bakry-Émery condition $\mathrm{BE}(\frac{1}{\alpha}K, \frac{1}{\alpha}N)$ provided

$$\mathrm{Hess}\, V - \frac{1}{N-\alpha n}(\nabla V \otimes \nabla V) \geq K - \alpha k \tag{1.9}$$

[Prop. 4.21]. In the Wasserstein picture, the $\mathrm{BE}(\frac{1}{\alpha}K, \frac{1}{\alpha}N)$-condition for $\widetilde{L}$ translates into the $(\frac{1}{\alpha}K, \frac{1}{\alpha}N)$-convexity of the functional $\widetilde{S}(\mu) = \mathrm{Ent}(\mu) + \frac{1}{\alpha}\int V\, d\mu$ [Thm. 7]. The latter in turn is equivalent to the (K,N)-convexity of $S(\mu) = \alpha\, \mathrm{Ent}(\mu) + \int V\, d\mu$ on $\mathscr{P}_2(M)$ [Lemma 2.9].

Note that this also makes perfectly sense for $\alpha = 0$ in which case the associated gradient flow equation on the Wasserstein space $\mathscr{P}_2(M)$ reads

$$d\mu_t = -\nabla V\, dt.$$

Obviously, this precisely describes the evolution on M determined by the semigroup $(\mathrm{H}_t^*)_{t>0}$ with generator $L^*u = \mathrm{div}(u\cdot\nabla V)$. Equilibration and regularization for this evolution are characterized by the parameters K and N in the bound (1.9) for $\alpha = 0$, i.e.

$$\mathrm{Hess}\, V - \frac{1}{N}(\nabla V \otimes \nabla V) \geq K.$$

This is the (K,N)-convexity of V on M.

Organization of the article. First we illustrate the new concept of (K,N)-convexity in a smooth and finite dimensional setting. Since many of the arguments which relate geodesic convexity, the Evolution Variational Inequality and space-time expansion bounds for the gradient flow are of a purely metric nature we study (K,N)-convexity, $\mathrm{EVI}_{K,N}$ and its consequences in the general setting of metric spaces in Section 2. In Section 3 we turn to the study of (K,N)-convexity of the entropy on the Wasserstein space. The entropic curvature-dimension condition is introduced in Section 3.1 and its basic properties are established. In particular we prove equivalence with the reduced curvature-dimension condition for essentially non-branching spaces. In Section 3.3 we prove that the entropic curvature-dimension condition plus linearity of the heat flow is equivalent to the existence of an $\mathrm{EVI}_{K,N}$ gradient flow of the entropy which leads to the Riemannian curvature-dimension condition. Here we also prove the stability results for $\mathrm{RCD}^*(K,N)$. Finally, in Section 4 we prove the equivalence of the entropic curvature-dimension condition, space-time Wasserstein expansion bounds, pointwise gradient estimates and the Bochner inequality for infinitesimally Hilbertian metric measure spaces. As applications, new functional inequalities deduced from $\mathrm{CD}^e(K,N)$ are studied in Section 3.4 and the sharp Lichnerowicz bound for $\mathrm{RCD}^*(K,N)$ spaces is established in Section 4.3.

2. (K,N)-convex functions and their EVI gradient flows

2.1. Gradient flows and (K,N)-convexity in a smooth setting. In order to illustrate the concept of (K,N)-convexity of the entropy and the consequences for its gradient flow, we consider in this section a smooth and finite-dimensional setting.

Let M be a smooth connected and geodesically complete Riemannian manifold with metric tensor $\langle\cdot,\cdot\rangle$ and Riemannian distance d. Let $S : M \to \mathbb{R}$ be a smooth function. Given two real numbers $K \in \mathbb{R}$ and $N > 0$, we say that S is (K,N)-convex, if

$$\mathrm{Hess}\, S - \frac{1}{N}\big(\nabla S \otimes \nabla S\big) \;\geq\; K\;, \tag{2.1}$$

in the sense that for all $x \in M$ and $v \in T_xM$ we have

$$\mathrm{Hess}\, S(x)[v] - \frac{1}{N}\langle \nabla S(x), v\rangle_x^2 \;\geq\; K|v|_x^2\;.$$

Obviously, this condition becomes weaker as N increases and in the limit $N \to \infty$ we recover the notion of K-convexity, i.e. $\mathrm{Hess}\, S \geq K$. It turns out to be useful to introduce the function $U_N : M \to \mathbb{R}_+$ given by

$$U_N(x) \;=\; \exp\left(-\frac{1}{N}S(x)\right)\;.$$

A direct calculation shows that (2.1) can equivalently be written as:

$$\mathrm{Hess}\, U_N \;\leq\; -\frac{K}{N}\cdot U_N\;. \tag{2.2}$$

This condition can be thought of as a "concavity" property of U_N. As with concavity, it can be expressed in an integrated form. To this end we introduce the following functions.

Definition 2.1. *For $\kappa \in \mathbb{R}$ and $\theta \geq 0$ we define the functions*

$$\mathfrak{s}_\kappa(\theta) \;=\; \begin{cases} \frac{1}{\sqrt{\kappa}}\sin\big(\sqrt{\kappa}\theta\big)\;, & \kappa > 0\;,\\ \theta\;, & \kappa = 0\;,\\ \frac{1}{\sqrt{-\kappa}}\sinh\big(\sqrt{-\kappa}\theta\big)\;, & \kappa < 0\;,\end{cases}$$

$$\mathfrak{c}_\kappa(\theta) \;=\; \begin{cases} \cos\big(\sqrt{\kappa}\theta\big)\;, & \kappa \geq 0\;,\\ \cosh\big(\sqrt{-\kappa}\theta\big)\;, & \kappa < 0\;.\end{cases}$$

Moreover, for $t \in [0,1]$ we set

$$\sigma_\kappa^{(t)}(\theta) \;=\; \begin{cases} \frac{\mathfrak{s}_\kappa(t\theta)}{\mathfrak{s}_\kappa(\theta)}\;, & \kappa\theta^2 \neq 0 \text{ and } \kappa\theta^2 < \pi^2\;,\\ t\;, & \kappa\theta^2 = 0\;,\\ +\infty\;, & \kappa\theta^2 \geq \pi^2\;.\end{cases}$$

Lemma 2.2. *The following statements are equivalent:*

(i) *The function S is (K,N)-convex.*

(ii) *For each constant speed geodesic $(\gamma_t)_{t\in[0,1]}$ in M and all $t \in [0,1]$ we have with $d := d(\gamma_0,\gamma_1)$:*

$$U_N(\gamma_t) \;\geq\; \sigma_{K/N}^{(1-t)}\big(d\big)\cdot U_N(\gamma_0) + \sigma_{K/N}^{(t)}\big(d\big)\cdot U_N(\gamma_1)\;. \tag{2.3}$$

(iii) *For each constant speed geodesic $(\gamma_t)_{t\in[0,1]}$ in M we have that*

$$U_N(\gamma_1) \;\leq\; \mathfrak{c}_{K/N}\big(d\big)\cdot U_N(\gamma_0) + \frac{\mathfrak{s}_{K/N}\left(d\right)}{d}\cdot \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} U_N(\gamma_t)\;. \tag{2.4}$$

Proof. (i)⇒(ii): Let $(\gamma_t)_{t\in[0,1]}$ be a constant speed geodesic. Then in particular $|\dot\gamma_t|_{\gamma_t} = d$ and (2.2) immediately yields that the function $u : t \mapsto U_N(\gamma_t)$ satisfies

$$u''(t) \;\le\; -\frac{K}{N}d^2 \cdot u(t)\ . \tag{2.5}$$

The function $v : [0,1] \to \mathbb{R}$ given by the right-hand side of (2.3) has the same boundary values as u and satisfies $v'' = -(K/N)d^2 \cdot v$. A comparison principle thus yields $u \ge v$.

(ii)⇒(iii): This follows immediately by subtracting $U_N(\gamma_0)$ on both sides of (2.3), dividing by t and letting $t \searrow 0$.

(iii)⇒(i): Let $\gamma : [-1,1] \to M$ be a constant speed geodesic with $\gamma_0 = x$ and $\dot\gamma_0 = v$, i.e. $d = d(\gamma_0,\gamma_1) = |v|$. Using (2.4) for the rescaled geodesics $\gamma' : [0,1] \to M,\ t \mapsto \gamma_{\varepsilon t}$ and $\gamma'' : [0,1] \to M,\ t \mapsto \gamma_{-\varepsilon t}$ and adding up we obtain

$$U_N(\gamma_\varepsilon) + U_N(\gamma_{-\varepsilon}) - 2\mathfrak{c}_{K/N}\big(\varepsilon d\big) \cdot U_N(\gamma_0) \;\le\; 0\ .$$

Dividing by ε^2 and using the fact that $\mathfrak{c}_{K/N}\big(\varepsilon d\big) = 1 - \frac{K}{N}\varepsilon^2 d^2 + o(\varepsilon^2)$ finally yields

$$\operatorname{Hess} U_N(x)[v] \;\le\; -\frac{K}{N}|v|^2\ .$$

□

Remark 2.3. We note that the existence of a (K,N)-convex function $S : M \to \mathbb{R}$ with $K > 0$ poses strong constraints on the manifold M. In particular, it implies that the diameter of M is bounded by $\sqrt{\frac{N}{K}}\pi$. This is immediate from the characterization (2.3) and the singularity of the coefficient $\sigma_\kappa^{(t)}(\cdot)$ at $\pi/\sqrt{\kappa}$.

Lemma 2.4. *Assume that S is (K,N)-convex and differentiable. A smooth curve $x : [0,\infty) \to M$ is a solution to the gradient flow equation*

$$\dot x_t \;=\; -\nabla S(x_t) \quad \forall t > 0\ , \tag{2.6}$$

if and only if the following Evolution Variation Inequality ($EVI_{K,N}$) holds: for all $z \in M$ and all $t > 0$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{s}_{K/N}\left(\frac12 d(x_t,z)\right)^2 + K\cdot \mathfrak{s}_{K/N}\left(\frac12 d(x_t,z)\right)^2 \;\le\; \frac{N}{2}\left(1 - \frac{U_N(z)}{U_N(x_t)}\right)\ . \tag{2.7}$$

Proof. To prove the only if part, fix $t \ge 0$, $z \in M$ and a constant speed geodesic $\gamma : [0,1] \to M$ connecting x_t to z. Observe that by (2.6) and the first variation formula we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} U_N(\gamma_s) \;&=\; -\frac{1}{N}U_N(x_t)\langle \nabla S(x_t), \dot\gamma_0\rangle \;=\; \frac1N U_N(x_t)\langle \dot x_t, \dot\gamma_0\rangle\\ &=\; -\frac1N U_N(x_t)\frac{\mathrm{d}}{\mathrm{d}t}\frac12 d(x_t,z)^2\ .\end{aligned}$$

Combining this with the (K,N)-convexity condition in the form (2.4) we obtain with $d = d(x_t,z)$:

$$U_N(z) \;\le\; \mathfrak{c}_{K/N}\big(d\big)U_N(x_t) - \frac{\mathfrak{s}_{K/N}\left(d\right)}{Nd}U_N(x_t)\frac{\mathrm{d}}{\mathrm{d}t}\frac12 d(x_t,z)^2\ . \tag{2.8}$$

Using the identity

$$\frac{2}{N}\mathfrak{s}_{K/N}\left(\frac12\theta\right)^2 \;=\; \frac1K\big(1 - \mathfrak{c}_{K/N}(\theta)\big)\ , \tag{2.9}$$

it is immediate to see that the last inequality is equivalent to (2.7).

For the if part, fix $t \ge 0$ and a constant speed geodesic $\gamma : [0,1] \to M$ with $\gamma_0 = x_t$. Using the Evolution Variational inequality in the form (2.8) with $z = \gamma_\varepsilon$ for some $\varepsilon > 0$ we obtain

$$U_N(\gamma_\varepsilon) - \mathfrak{c}_{K/N}\big(\varepsilon|v|\big)U_N(\gamma_0) \;\le\; \frac{\mathfrak{s}_{K/N}\left(\varepsilon|v|\right)}{N\varepsilon|v|}U_N(\gamma_0)\langle \dot x_t, \varepsilon v\rangle\ ,$$

where $v = \dot\gamma_0$. Dividing by ε and letting $\varepsilon \searrow 0$, taking into account that $\mathfrak{c}_{K/N}(\varepsilon d) = 1 + o(\varepsilon)$ and $\mathfrak{s}_{K/N}(\varepsilon d) = \varepsilon d + o(\varepsilon^2)$, we obtain

$$\langle -\nabla S(x_t), v\rangle \ \le\ \langle \dot x_t, v\rangle\ .$$

Since the direction of $v \in T_{x_t}M$ was arbitrary we obtain (2.6). □

We conclude this section by exhibiting some 1-dimensional models of (K,N)-convex functions.

Example 2.5. Each of the following are (K,N)-convex functions. Note that the domain of definition is maximal in each case.

(i) For $N > 0$ and $K > 0$ let $S : (-\frac{\pi}{2}\sqrt{\frac{N}{K}}, \frac{\pi}{2}\sqrt{\frac{N}{K}}) \to \mathbb{R}$ defined by

$$S(x) \ =\ -N \log\cos\left(x\sqrt{K/N}\right)\ .$$

(ii) For $N > 0$ and $K = 0$ let $S : (0,\infty) \to \mathbb{R}$ defined by

$$S(x) = -N\log x\ .$$

(iii) For $N > 0$ and $K < 0$ let $S : (0,\infty) \to \mathbb{R}$ defined by

$$S(x) \ =\ -N \log\sinh\left(x\sqrt{-K/N}\right)\ .$$

(iv) For $N > 0$ and $K < 0$ let $S : (-\infty,\infty) \to \mathbb{R}$ defined by

$$S(x) \ =\ -N \log\cosh\left(x\sqrt{-K/N}\right)\ .$$

The cases (i) and (iv) of the previous example canonically extend to multidimensional spaces.

Example 2.6. Let (M, g) be a n-dimensional Riemannian manifold, $z \in M$ be any point and $N > 0$ be any real number.

(i) Then for each $K > 0$ the function

$$S(x) \ =\ -N \log\cos\left(d(x,z)\sqrt{K/N}\right)$$

defined on the open ball $\left\{x \in M :\ d(x, z) < \frac{\pi}{2}\sqrt{N/K}\right\}$ is (K,N)-convex provided the sectional curvature of the underlying space is $\le K/N$. (This in particular applies to the Euclidean space $\mathbb{R}^n$.)

(ii) For each $K < 0$ the function

$$S(x) \ =\ -N \log\cosh\left(d(x,z)\sqrt{-K/N}\right)$$

defined on all of M is (K,N)-convex provided the sectional curvature of the underlying space is $\ge K/N$. (This in particular applies to the Euclidean space $\mathbb{R}^n$.)

Indeed, analogous statements hold true on geodesic spaces with generalized bounds for the sectional curvature in the sense of Alexandrov [14].

2.2. (K,N)**-convexity in metric spaces.** We proceed our study of (K,N)-convexity in a purely metric setting. Let (X, d) be a complete and separable metric space and let $S : X \to [-\infty,\infty]$ be a functional on X. We denote by $D(S) := \{x \in X\ :\ S(x) < \infty\}$ the proper domain of S. Given a number $N \in (0,\infty)$ we define the functional $U_N : X \to [0,\infty)$ by setting

$$U_N(x) \ :=\ \exp\left(-\frac{1}{N}S(x)\right)\ . \tag{2.10}$$

Definition 2.7. *Let $K \in \mathbb{R}$, $N \in (0,\infty)$. We say that the functional S is (K,N)-convex if and only if for each pair $x_0, x_1 \in D(S)$ there exists a constant speed geodesic $\gamma : [0,1] \to X$ connecting x_0 to x_1 such that for all $t \in [0,1]$:*

$$U_N(\gamma_t) \ \ge\ \sigma^{(1-t)}_{K/N}\big(d(\gamma_0,\gamma_1)\big)\cdot U_N(\gamma_0) + \sigma^{(t)}_{K/N}\big(d(\gamma_0,\gamma_1)\big)\cdot U_N(\gamma_1)\ . \tag{2.11}$$

If (2.11) *holds for every geodesic $\gamma : [0,1] \to D(S)$ we say that S is* strongly *(K,N)-convex.*

For investigating (K,N)-convexity (especially for the strong form), the following equivalent conditions will be helpful in the sequel.

Lemma 2.8. *Let $u : X \to [0,\infty)$ be a upper semi-continuous function and $\kappa \in \mathbb{R}$. Then the following statements are equivalent:*

(i) *For each constant speed geodesic $\gamma : [0,1] \to X$ and $t \in [0,1]$, $u''(\gamma_t) \le -\kappa d(\gamma_0,\gamma_1)^2 u(\gamma_t)$ in the distributional sense, i.e.*
$$\int_0^1 \varphi''(t)u(\gamma_t)\,\mathrm{d}t \;\le\; -\kappa d(\gamma_0,\gamma_1)^2 \int_0^1 \varphi(t)u(\gamma_t)\,\mathrm{d}t$$
for any $\varphi \in C_0^\infty((0,1))$ with $\varphi \ge 0$.

(ii) *For each constant speed geodesic γ on X and $t \in [0,1]$,*
$$u(\gamma_t) \;\ge\; \sigma_\kappa^{(1-t)}(d(\gamma_0,\gamma_1)) \cdot u(\gamma_0) + \sigma_\kappa^{(t)}(d(\gamma_0,\gamma_1)) \cdot u(\gamma_1). \tag{2.12}$$

(iii) *For each constant speed geodesic γ on X, there is $\delta = \delta_\gamma > 0$ such that for all $0 \le s \le t \le 1$ with $t - s \le \delta$ and $\alpha \in [0,1]$,*
$$u(\gamma_{(1-\alpha)s+\alpha t}) \;\ge\; \sigma_\kappa^{(1-\alpha)}(d(\gamma_s,\gamma_t)) \cdot u(\gamma_s) + \sigma_\kappa^{(\alpha)}(d(\gamma_s,\gamma_t)) \cdot u(\gamma_t). \tag{2.13}$$

(iv) *For each constant speed geodesic γ on X and $t \in [0,1]$,*
$$u(\gamma_t) \;\ge\; (1-t)\cdot u(\gamma_0) + t \cdot u(\gamma_1) + \kappa d(\gamma_0,\gamma_1)^2 \int_0^1 g(t,r)u(\gamma_r)\mathrm{d}r$$
with $g(t,r) = \min\{(1-t)r, (1-r)t\}$ being the Green function on the interval $[0,1]$.

In particular, when $-\infty \notin S(X)$ and S is lower semi-continuous, S is strongly (K,N)-convex if and only if $u = U_N$ and $\kappa = K/N$ satisfies one of these conditions.

Proof. For simplicity of presentation, we denote $\theta = \theta_\gamma = d(\gamma_0,\gamma_1)$ in this proof whenever a fixed geodesic is under consideration. we also denote the restriction of γ on $[s,t]$ for $0 \le s < t \le 1$ by $\gamma^{[s,t]} : [0,1] \to X$, that is, $\gamma_r^{[s,t]} := \gamma_{(1-r)s+rt}$.

(i) $\Rightarrow$ (iv): Let us denote $u_*(s) := \int_0^1 g(s,r)u(\gamma_r)\,\mathrm{d}r$. Since we have
$$\int_0^1 \varphi''(r)u_*(r)\,\mathrm{d}r = -\int_0^1 \varphi(r)u(\gamma_r)\,\mathrm{d}r$$
for any $\varphi \in C_0^\infty((s,t))$ with $\varphi \ge 0$, (i) implies $(u(\gamma_\cdot) - \kappa\theta^2 u_*)'' \le 0$ on $[0,1]$ in the distributional sense. Thus the distributional characterization of convex functions (see [38, Theorem 1.29], for instance) yields that $u(\gamma_\cdot) - \kappa\theta^2 u_*$ coincides with a concave function a.e. and hence concave because u is upper semi-continuous. It immediately implies (iv) since $u_*(0) = u_*(1) = 1$.

(iv) $\Rightarrow$ (i): Note first that $u(\gamma_t)$ is continuous. Indeed, the condition (iv) together with the upper semi-continuity of u implies that $u(\gamma_t)$ is continuous at $t = 0, 1$. Thus the continuity follows by applying the same for $\gamma^{[0,s]}$ and $\gamma^{[s,1]}$. For $s \in (0,1)$ and $h > 0$ with $s+h, s-h \in [0,1]$, we apply (iv) to $\gamma^{[s-h,s+h]}$ and $t = 1/2$ to obtain
$$\frac{u(\gamma_{s+h}) + u(\gamma_{s-h}) - 2u(\gamma_s)}{2} \le 4\kappa h^2\theta^2 \int_0^1 g\left(\frac{1}{2}, r\right) u(\gamma_{s+(2r-1)h})\,\mathrm{d}r.$$
Then (i) follows by multiplying $\varphi \in C_0^\infty((0,1))$, integrating w.r.t. t (for sufficiently small h), dividing by h^2 and $h \to 0$ with a change of variable.

(i) $\Rightarrow$ (ii): Take $\varepsilon > 0$ and $\varphi \in C_0^\infty((0,1))$ with $\int_0^1 \varphi(x)\,\mathrm{d}x = 1$, and let
$$\widetilde{u}_\varepsilon(t) := \int_0^1 \varepsilon^{-1}\varphi(\varepsilon^{-1}(t-r))u(\gamma_r)\,\mathrm{d}r\,.$$
Then (i) implies $\widetilde{u}_\varepsilon''(t) \le -\kappa\theta^2\widetilde{u}_\varepsilon(t)$ for each $t \in [a_\varepsilon, 1]$ for some $a_\varepsilon > 0$. Note that a_ε can be chosen so that $\lim_{\varepsilon\to 0} a_\varepsilon = 0$. Thus, in the same way as in Lemma 2.2, we obtain
$$\widetilde{u}_\varepsilon((1-t)a_\varepsilon + t) \ge \sigma_\kappa^{(1-t)}(\theta)\widetilde{u}_\varepsilon(a_\varepsilon) + \sigma_\kappa^{(t)}(\theta)\widetilde{u}_\varepsilon(1).$$
By virtue of the equivalence (i) $\Leftrightarrow$ (iv), $u \circ \gamma$ is continuous and hence $\widetilde{u}_\varepsilon \to u \circ \gamma$ as $\varepsilon \to 0$ uniformly on $[0,1]$. Thus the conclusion follows by letting $\varepsilon \to 0$.

(ii) ⇒ (iii): It follows by considering (ii) for $\gamma^{[s,t]}$.
(iii) ⇒ (i): We imitate the proof of the implication (iv) ⇒ (i) by using the following:

$$\lim_{h\to 0}\frac{1}{h^2}\left(\frac{1}{2}-\sigma_\kappa^{(1/2)}(2h\theta)\right) = -\frac{1}{4}\kappa\theta^2.$$

□

We conclude this section with some remarks about (K,N)-convexity. The first property is immediate from the definition.

Lemma 2.9. *If S is (K,N)-convex, then for $\lambda > 0$ the functional $\lambda\cdot S$ is $(\lambda K,\lambda N)$-convex.*

Lemma 2.10. *Let $S^1 : X \to (-\infty,\infty]$ be a (K_1,N_1)-convex functional and $S^2 : X \to (-\infty,\infty]$ a strongly (K_2,N_2)-convex functional. Then the functional $S := S^1+S^2$ is (K_1+K_2,N_1+N_2)-convex. In particular, S is strongly (K_1+K_2,N_1+N_2)-convex if S^1 is strongly (K_1,N_1)-convex.*

Proof. Let us set $K = K_1+K_2$ and $N = N_1+N_2$ and given $x_0,x_1\in D(S) = D(S^1)\cap D(S^2)$ take a constant speed geodesic $\gamma : [0,1]\to X$ from x_0 to x_1 according to the convexity assumption of S^1. By the convexity assumption on S^1 and S^2 we have

$$\begin{aligned}\log U_N(\gamma_t) &= \frac{N_1}{N}\frac{(-1)}{N_1}S^1(\gamma_t)+\frac{N_2}{N}\frac{(-1)}{N_2}S^2(\gamma_t)\\ &\geq \frac{N_1}{N}G_t\left(\frac{(-1)}{N_1}S^1(\gamma_0),\frac{(-1)}{N_1}S^1(\gamma_1),\frac{K_1}{N_1}d(\gamma_0,\gamma_1)^2\right)\\ &\quad+\frac{N_2}{N}G_t\left(\frac{(-1)}{N_2}S^2(\gamma_0),\frac{(-1)}{N_2}S^2(\gamma_1),\frac{K_2}{N_2}d(\gamma_0,\gamma_1)^2\right),\end{aligned}$$

where the function G_t is given by (2.14). By Lemma 2.11 below, G_t is convex. Hence we obtain

$$\log U_N(\gamma_t) \geq G_t\left(\frac{(-1)}{N}S(\gamma_0),\frac{(-1)}{N}S(\gamma_1),\frac{K}{N}d(\gamma_0,\gamma_1)^2\right).$$

Taking the exponential on both sides yields the claim. The last assertion is obvious from the proof. □

Lemma 2.11. *For any fixed $t\in[0,1]$ the function $G_t:\mathbb{R}\times\mathbb{R}\times(-\infty,\pi^2)\to\mathbb{R}$ given by*

$$G_t(x,y,\kappa) = \log\left[\sigma_\kappa^{(1-t)}(1)e^x+\sigma_\kappa^{(t)}(1)e^y\right] \tag{2.14}$$

is convex.

Note that we have $\sigma_\kappa^{(s)}(\theta) = \sigma_{\kappa\theta^2}^{(s)}(1)$ for $s\in[0,1]$, $\theta\geq 0$ and $\kappa\in(-\infty,\pi^2/\theta^2)$. It is useful to apply this lemma.

Proof. We define the function $g^{(t)}:\kappa\mapsto\log\sigma_\kappa^{(t)}(1)$ on $(-\infty,\pi^2)$ and write

$$G_t(x,y,\kappa) = F\Big(x+g^{(1-t)}(\kappa),y+g^{(t)}(\kappa)\Big),$$

where $F(u,v) = \log\left(e^u+e^v\right)$. The claim then follows by noting that the function F is convex, $a\mapsto F(u+a,v+a)$ is increasing and that the functions $g^{(t)}$ are convex. □

Finally we remark that the notion of (K,N)-convexity is consistent in the parameters K and N.

Lemma 2.12. *If S is (K,N)-convex then it is also (K',N')-convex for all $K'\leq K$ and $N'\geq N$. Moreover, it is K-convex in the sense that for each pair $x_0,x_1\in D(S)$ there exist a constant speed geodesic $\gamma:[0,1]\to X$ connecting x_0 to x_1 such that for all $t\in[0,1]$:*

$$S(\gamma_t) \leq (1-t)S(\gamma_0)+tS(\gamma_1)-\frac{K}{2}t(1-t)d(\gamma_0,\gamma_1)^2. \tag{2.15}$$

Proof. Consistency in K is immediate from the fact that for any fixed t and θ the coefficient $\sigma^{(t)}_{K/N}(\theta)$ is increasing in K. Consistency in N is a consequence e.g. of Lemma 2.10 and the trivial observation that for any $N' > N$ the constant functional $S^0 \equiv 0$ is $(0, N' - N)$-convex.

Using the consistency in N we can derive (2.15) by subtracting 1 on both sides of (2.11), multiplying with N and passing to the limit $N \nearrow \infty$. Here we use the fact that $\sigma^{(t)}_{K/N}(\theta) = t + -K(t^3 - t)\theta^2/(6N) + o(1/N)$ and $U_N(x) = 1 - S(x)/N + o(1/N)$. □

2.3. **Evolution Variational Inequalities in metric spaces.** In this section we study the Evolution Variational Inequality with parameters K and N and the associated notion of gradient flow in a purely metric setting. In particular, we investigate the relation with geodesic convexity. Our approach extends the results obtained in [16, 4] where the case $N = \infty$ has been considered.

Let (X, d) be a complete separable geodesic metric space and $S : X \to (-\infty, \infty]$ a lower semi-continuous functional. Note that our framework is slightly more restrictive than that in the last section. We define the *descending slope* of S at $x \in D(S)$ as

$$|\nabla^- S|(x) \ := \ \limsup_{y \to x} \frac{\big(S(x) - S(y)\big)_+}{d(x,y)} \ .$$

For $x \notin D(S)$ we set $|\nabla^- S| = +\infty$. A curve $\gamma : I \to X$ defined on an interval $I \subset \mathbb{R}$ is called *absolutely continuous* if

$$d(\gamma_s, \gamma_t) \ \le \ \int_s^t g(r) \mathrm{d}r \quad \forall s, t \in I \ , s \le t \ , \tag{2.16}$$

for some $g \in L^1(I)$. For an absolutely continuous curve γ the *metric speed*, defined by

$$|\dot\gamma|(t) \ := \ \lim_{h \to 0} \frac{d(\gamma_{t+h}, \gamma_t)}{|h|} \ ,$$

exists for a.e. $t \in I$ and is the minimal g in (2.16) (see e.g. [3, Thm. 1.1.2]). The following is a classical notion of gradient flow in a metric space, see e.g. [3].

Definition 2.13 (Gradient flow)**.** *We say that a locally absolutely continuous curve $x : [0, \infty) \to X$ with $x_0 \in D(S)$ is a (downward) gradient flow of S starting in x_0 if the* Energy Dissipation Equality *holds:*

$$S(x_s) \ = \ S(x_t) + \frac{1}{2} \int_s^t |\dot x_r|^2 + |\nabla^- S|(x_r) \mathrm{d}r \quad \forall 0 \le s \le t \ . \tag{2.17}$$

We introduce here a more restrictive notion of gradient flow based on the Evolution Variational Inequality.

Definition 2.14 ($\mathrm{EVI}_{K,N}$ gradient flow)**.** *Let $K \in \mathbb{R}$, $N \in (0, \infty)$ and let $x : (0, \infty) \to D(S)$ be a locally absolutely continuous curve. We say that (x_t) is an* $\mathrm{EVI}_{K,N}$ gradient flow *of S starting in x_0 if $\lim_{t \to 0} x_t = x_0$ and if for all $z \in D(S)$ the* Evolution Variational Inequality

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathfrak{s}_{K/N}\left(\frac{1}{2} d(x_t, z)\right)^2 + K \cdot \mathfrak{s}_{K/N}\left(\frac{1}{2} d(x_t, z)\right)^2 \ \le \ \frac{N}{2}\left(1 - \frac{U_N(z)}{U_N(x_t)}\right) \tag{2.18}$$

holds for a.e. $t > 0$.

Lemma 2.15. *If $(x_t)_t$ is an $\mathrm{EVI}_{K,N}$ flow for S, then it is also an $\mathrm{EVI}_{K',N'}$ flow for S for any $K' \le K$ and $N' \ge N$. Moreover, (x_t) is an EVI_K flow for S, i.e. for all $z \in D(S)$ and a.e. $t > 0$:*

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} d(x_t, z)^2 + \frac{K}{2} d(x_t, z)^2 \ \le \ S(z) - S(x_t) \ . \tag{2.19}$$

Proof. Using the (2.9) one checks that (2.18) is equivalent to either of the following inequalities:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}d(x_t,z)^2 \;\le\; \frac{Nd}{\mathfrak{s}_{K/N}(d)}\left[\mathfrak{c}_{K/N}(d) - \frac{U_N(z)}{U_N(x_t)}\right] \tag{2.20}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}d(x_t,z)^2 \;\le\; \frac{d}{\mathfrak{s}_{K/N}(d)}N\left[1 - \frac{U_N(z)}{U_N(x_t)}\right] - 2Kd\frac{\mathfrak{s}_{K/N}\left(\frac{1}{2}d\right)^2}{\mathfrak{s}_{K/N}(d)}\;, \tag{2.21}$$

where we set $d = d(x_t,z)$. Consistency in K can be seen from (2.20) by noting that for any $\theta \ge 0$ we have that $\mathfrak{s}_{K/N}(\theta)$ and $\mathfrak{c}_{K/N}(\theta)/\mathfrak{s}_{K/N}(\theta)$ is decreasing in K. Consistency in N follows from (2.21) and the fact that for any $v \in \mathbb{R}$ and $\theta \ge 0$ both

$$N\left[1-\exp\Big(-\frac{1}{N}v\Big)\right]\frac{1}{\mathfrak{s}_{K/N}(\theta)} \qquad \text{and} \qquad -K\cdot\frac{\mathfrak{s}_{K/N}\left(\frac{1}{2}\theta\right)^2}{\mathfrak{s}_{K/N}(\theta)}$$

are increasing in N. (2.19) follows immediately from (2.21) by passing to the limit as $N \to \infty$. For this we note that

$$\lim_{N\to\infty}\mathfrak{s}_{K/N}(d) = d\;,\qquad \lim_{N\to\infty}N\left[1-\frac{U_N(z)}{U_N(x_t)}\right] = S(z) - S(x_t)\;.$$

□

Remark 2.16. This shows consistency with the theory of EVI_K gradient flows of geodesically K-convex functions. It can be thought of as the limiting case $N = \infty$. By taking the limit $N \to \infty$ in the estimates obtained in this section we recover the corresponding results for EVI_K flows established in [16, 4].

We summarize here some properties of $\mathrm{EVI}_{K,N}$ gradient flows. To this end we set for $\kappa \in \mathbb{R}$ and $t \ge 0$:

$$e_\kappa(t) \;=\; \int_0^t \mathrm{e}^{\kappa s}\mathrm{d}s\;.$$

Proposition 2.17. *Let (x_t) be an $\mathrm{EVI}_{K,N}$ gradient flow of S starting in x_0. Then the following statements hold:*

(i) *If $x_0 \in D(S)$ then (x_t) is also a metric gradient flow in the sense of Definition 2.13. In particular, the map $t \mapsto S(x_t)$ is non-increasing.*

(ii) *We have the uniform regularization bound*

$$\frac{U_N(z)}{U_N(x_t)} \;\le\; 1 + \frac{2}{Ne_K(t)}\mathfrak{s}_{K/N}\left(\frac{1}{2}d(x_0,z)\right)^2 \tag{2.22}$$

(iii) *If S is bounded below we have the uniform continuity estimate*

$$\mathfrak{s}_{K/N}\left(\frac{1}{2}d(x_{t_1},x_{t_0})\right)^2 \;\le\; \frac{N}{2e_{-K}(t_1-t_0)}\left[1-\frac{U_N(x_{t_0})}{U_N^{\max}}\right]\;. \tag{2.23}$$

Proof. By Lemma 2.15 (x_t) is an EVI_K flow of S and hence a metric gradient flow by [1, Prop. 3.9]. (2.22) follows immediately from (2.24) in Proposition 2.18 below by taking $t_0 = 0$. The uniform continuity estimate (2.23) is obtained similarly by taking $z = x_{t_0}$. □

The following result collects several equivalent reformulations of the definition of $\mathrm{EVI}_{K,N}$ gradient flows which will be useful in the sequel. For this we say that a subset $D \subset D(S)$ is *dense in energy*, if for any $z \in D(S)$ there exists a sequence $(z_n) \subset D$ such that $d(z_n,z) \to 0$ and $S(z_n) \to S(z)$ as $n \to \infty$. For a function $f : I \to \mathbb{R}$ on some interval I we use the notation

$$\frac{\mathrm{d}^+}{\mathrm{d}t}f(t) \;=\; \limsup_{h\searrow 0}\frac{f(t+h)-f(t)}{h}$$

to denote the right derivative.

Proposition 2.18. *Let $D \subset D(S)$ be dense in energy and let $x : (0,\infty) \to D(S)$ be a locally absolutely continuous curve with $\lim_{t\to 0} x_t = x_0$. Then (x_t) is an $\mathrm{EVI}_{K,N}$ gradient flow of S if and only if one of the following statements holds:*

(i) *The differential inequality* (2.18) *holds for all* $z \in D$ *and a.e.* $t > 0$.

(ii) *For all* $z \in D$ *and all* $0 \le t_0 \le t_1$*:*

$$e_K(t_1 - t_0)\frac{N}{2}\left(1 - \frac{U_N(z)}{U_N(x_{t_1})}\right) \ \ge\ \mathrm{e}^{K(t_1-t_0)}\mathfrak{s}_{K/N}\left(\frac{1}{2}d(x_{t_1}, z)\right) - \mathfrak{s}_{K/N}\left(\frac{1}{2}d(x_{t_0}, z)\right)^2 . \tag{2.24}$$

(iii) *For all* $z \in D$ *and* all $t > 0$*:*

$$\frac{\mathrm{d}^+}{\mathrm{d}t}\mathfrak{s}_{K/N}\left(\frac{1}{2}d(x_t, z)\right)^2 + K \cdot \mathfrak{s}_{K/N}\left(\frac{1}{2}d(x_t, z)\right)^2 \ \le\ \frac{N}{2}\left(1 - \frac{U_N(z)}{U_N(x_t)}\right) \tag{2.25}$$

Proof. We prove the equivalence of Definition 2.14 and (ii). Assume that (x_t) is an $\mathrm{EVI}_{K,N}$ flow and note that the right hand side of (2.18) can be rewritten as

$$\mathrm{e}^{-Kt}\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathrm{e}^{Kt}\mathfrak{s}_{K/N}\left(\frac{1}{2}d(x_t, z)\right)^2\right] .$$

Integrating from t_0 to t_1 and using that the map $t \mapsto U_N(x_t)$ is non-decreasing by (i) of Proposition 2.17 then yields (2.24) for all $z \in D(S)$. Conversely, differentiating (2.24) yields (2.18). The fact that (2.24) holds for all $z \in D(S)$ if and only if it holds for all $z \in D$ is obvious. Similar arguments show the equivalence of Definition 2.14 with (i) and (iii). □

An important property of $\mathrm{EVI}_{K,N}$ flows is the following expansion bound.

Theorem 2.19. *Let* $(x_t), (y_t)$ *be two* $\mathrm{EVI}_{K,N}$ *gradient flows of* S *starting from* x_0 *resp.* y_0*. Then for all* $s, t \ge 0$*:*

$$\mathfrak{s}_{K/N}\left(\frac{1}{2}d(x_t, y_s)\right)^2 \ \le\ \mathrm{e}^{-K(s+t)}\mathfrak{s}_{K/N}\left(\frac{1}{2}d(x_0, y_0)\right)^2 + \frac{N}{K}\left(1 - \mathrm{e}^{-K(s+t)}\right)\frac{\left(\sqrt{t} - \sqrt{s}\right)^2}{2(s+t)} . \tag{2.26}$$

Proof. Let us fix $s, t > 0$. Choose $\lambda, r > 0$ such that $\lambda r = t$ and $\lambda^{-1}r = s$, i.e. $\lambda = \sqrt{\frac{t}{s}}$ and $r = \sqrt{ts}$. From (2.24) applied to (x_t) with $z = y_{\lambda^{-1}r}$ and $t_0 = \lambda r, t_1 = \lambda(r + \varepsilon)$ for some $\varepsilon > 0$ we obtain

$$\begin{aligned}\frac{N}{2}\frac{U_N(y_{\lambda^{-1}r})}{U_N(x_{\lambda(r+\varepsilon)})} \ \le\ & \frac{N}{2} - \frac{1}{e_{-K}(\lambda\varepsilon)}\mathfrak{s}_{K/N}\left(\frac{1}{2}d(x_{\lambda(r+\varepsilon)}, y_{\lambda^{-1}r})\right)^2 \\ & + \frac{1}{e_K(\lambda\varepsilon)}\mathfrak{s}_{K/N}\left(\frac{1}{2}d(x_{\lambda r}, y_{\lambda^{-1}r})\right)^2 .\end{aligned} \tag{2.27}$$

Similarly, choosing $z = x_{\lambda(r+\varepsilon)}$ and $t_0 = \lambda^{-1}r, t_1 = \lambda^{-1}(r + \varepsilon)$ and applying (2.24) to (y_s) we obtain

$$\begin{aligned}\frac{N}{2}\frac{U_N(x_{\lambda(r+\varepsilon)})}{U_N(y_{\lambda^{-1}(r+\varepsilon)})} \ \le\ & \frac{N}{2} - \frac{1}{e_{-K}(\lambda^{-1}\varepsilon)}\mathfrak{s}_{K/N}\left(\frac{1}{2}d(y_{\lambda^{-1}(r+\varepsilon)}, x_{\lambda(r+\varepsilon)})\right)^2 \\ & + \frac{1}{e_K(\lambda^{-1}\varepsilon)}\mathfrak{s}_{K/N}\left(\frac{1}{2}d(y_{\lambda^{-1}r}, x_{\lambda(r+\varepsilon)})\right)^2 .\end{aligned} \tag{2.28}$$

Multiplying (2.27) and (2.28) after taking square roots and using Young's inequality, $2\sqrt{ab} \le \lambda a + \lambda^{-1}b$, we deduce the estimate

$$\begin{aligned} N\sqrt{\frac{U_N(y_{\lambda^{-1}r})}{U_N(y_{\lambda^{-1}(r+\varepsilon)})}} \ \le\ & \frac{N}{2}(\lambda^{-1} + \lambda) \\ & + \mathfrak{s}_{K/N}\left(\frac{1}{2}d(y_{\lambda^{-1}r}, x_{\lambda(r+\varepsilon)})\right)^2\left[\frac{\lambda^{-1}}{e_K(\lambda^{-1}\varepsilon)} - \frac{\lambda}{e_{-K}(\lambda\varepsilon)}\right] \\ & + \mathfrak{s}_{K/N}\left(\frac{1}{2}d(x_{\lambda r}, y_{\lambda^{-1}r})\right)^2\left[\frac{\lambda}{e_K(\lambda\varepsilon)} - \frac{\lambda^{-1}}{e_{-K}(\lambda^{-1}\varepsilon)}\right] \\ & - \frac{\lambda^{-1}\varepsilon}{e_{-K}(\lambda^{-1}\varepsilon)}\frac{1}{\varepsilon}\left[\mathfrak{s}_{K/N}\left(\frac{1}{2}d(y_{\lambda^{-1}(r+\varepsilon)}, x_{\lambda(r+\varepsilon)})\right)^2 - \mathfrak{s}_{K/N}\left(\frac{1}{2}d(x_{\lambda r}, y_{\lambda^{-1}r})\right)^2\right] .\end{aligned} \tag{2.29}$$

Note that as $\varepsilon \to 0$ we have

$$\frac{e_{-K}(\lambda^{-1}\varepsilon)}{\lambda^{-1}\varepsilon} \to 1 \quad \text{and} \quad \left[\frac{\lambda^{-1}}{e_K(\lambda^{-1}\varepsilon)} - \frac{\lambda}{e_{-K}(\lambda\varepsilon)}\right] \to -\frac{K}{2}(\lambda+\lambda^{-1}) .$$

Hence, if we consider the function $g : \mathbb{R}_+ \to \mathbb{R}$ given by

$$g(\tau) \;=\; \frac{2}{N}\mathfrak{s}_{K/N}\left(\frac{1}{2}d(x_{\lambda\tau}, y_{\lambda^{-1}\tau})\right)^2$$

and take the limit as $\varepsilon \searrow 0$ in (2.29) we obtain

$$\frac{\mathrm{d}^+}{\mathrm{d}\tau}\bigg|_{\tau=r} g(\tau) \;\le\; -K(\lambda+\lambda^{-1})g(r) + (\lambda+\lambda^{-1}-2) .$$

By an application of Gronwall's lemma we deduce that

$$g(r) \;\le\; e^{-K(\lambda+\lambda^{-1})r}\Big[g(0) + \frac{\lambda+\lambda^{-1}-2}{(\lambda+\lambda^{-1})}e_K\big((\lambda+\lambda^{-1})r\big)\Big] .$$

Rewriting r, λ in terms of s, t finally yields (2.26). □

Remark 2.20. In the limit $d(x_0, y_0) \to 0$ and $s \to t$ the contraction estimate (2.26) reads asymptotically as follows:

$$d(x_t, y_s)^2 \;\le\; e^{-2Kt}d(x_0, y_0)^2 + \frac{N}{K}\frac{1-e^{-2Kt}}{4t^2} \cdot |s-t|^2 + o\big(d(x_0, y_0)^2 + |t-s|^2\big) . \tag{2.30}$$

Corollary 2.21. *For each $x_0 \in \overline{D(S)}$ there exist at most one $\mathrm{EVI}_{K,N}$ gradient flow of S starting from x_0. The maps $P_t : x_0 \mapsto x_t$, where (x_t) is the unique gradient flow starting from x_0 constitute a continuous semigroup defined on a closed (possibly empty) subset of $\overline{D(S)}$.*

The previous expansion estimate in Theorem 2.26 implies a slightly weaker estimate directly for the distance d not involving the functions $\mathfrak{s}_{K/N}$. More precisely, we have the following:

Proposition 2.22. *The expansion bound* (2.26) *implies the following bound: For each $x_0, x_1 \in X$ and $s, t \ge 0$, $x_t := P_t x_0$ and $y_s := P_s y_0$ satisfies*

$$d(x_t, y_s)^2 \;\le\; \mathrm{e}^{-K\tau(s,t)}d(x_0, y_0)^2 + 2N\frac{1-\mathrm{e}^{-K\tau(s,t)}}{K\tau(s,t)}\left(\sqrt{t}-\sqrt{s}\right)^2 ,$$

where $\tau(s,t) = 2(t+\sqrt{ts}+s)/3$. In particular, setting $t = s$ yields the following estimate:

$$d(x_t, y_t) \le \mathrm{e}^{-Kt}d(x_0, y_0). \tag{2.31}$$

Proof. For $0 < s' < t'$, let $\Phi : [0,1] \to [s', t']$ be given by $\Phi(r) := (\sqrt{s'} + (\sqrt{t'}-\sqrt{s'})r)^2$. Let $(\gamma_u)_{u\in[0,1]}$ be a constant speed geodesic. By (2.26), there exists $C_1 > 0$ such that

$$d(P_r\gamma_u, P_{r'}\gamma_{u'}) \le C_1\left(|u-u'| + |\sqrt{r}-\sqrt{r'}|\right) \tag{2.32}$$

when $|u-u'|$ and $|\sqrt{r}-\sqrt{r'}|$ is sufficiently small. By the convexity of $z \mapsto z^2$ on $\mathbb{R}$, for $k \in \mathbb{N}$,

$$d(P_{t'}\gamma_1, P_{s'}\gamma_0)^2 \le \sum_{j=1}^k d\big(P_{\Phi((j-1)/k)}\gamma_{(j-1)/k}, P_{\Phi(j/k)}\gamma_{j/k}\big)^2 k.$$

By virtue of (2.32), we have

$$\begin{aligned}
&\lim_{k\to\infty}\sum_{j=1}^k d\big(P_{\Phi((j-1)/k)}\gamma_{(j-1)/k}, P_{\Phi(j/k)}\gamma_{j/k}\big)^2 k\\
&\quad\le 4\lim_{k\to\infty}\sum_{j=1}^k \mathfrak{s}_{K/N}\left(\frac{1}{2}d\big(P_{\Phi((j-1)/k)}\gamma_{(j-1)/k}, P_{\Phi(j/k)}\gamma_{j/k}\big)\right)^2 k\\
&\quad\le 4\lim_{k\to\infty}\left[\sum_{j=1}^k \mathrm{e}^{-K(\Phi(j/k)+\Phi((j-1)/k))}\mathfrak{s}_{K/N}\left(\frac{1}{2}d(\gamma_{(j-1)/k},\gamma_{j/k})\right)^2 k\right.\\
&\qquad\qquad\left.+\frac{N}{2}\sum_{j=1}^k\frac{1-\mathrm{e}^{-K(\Phi(j/k)+\Phi((j-1)/k))}}{K(\Phi(j/k)+\Phi((j-1)/k))}\frac{\big(\sqrt{t'}-\sqrt{s'}\big)^2}{k}\right]\\
&\quad=\int_0^1 \mathrm{e}^{-2K\Phi(r)}dr\,d(\gamma_0,\gamma_1)^2+2N\int_0^1\frac{1-\mathrm{e}^{-K\Phi(r)}}{K\Phi(r)}dr(\sqrt{t'}-\sqrt{s'})^2.
\end{aligned}$$

Let $\lambda\ge 1$, $\tau,h>0$, $s'=\lambda^{-1}(\tau+h)$, $t'=\lambda(\tau+h)$, $\gamma_0:=P_{\lambda^{-1}r}y_0$ and $\gamma_1:=P_{\lambda r}x_0$. Then the last inequality implies

$$\frac{\mathrm{d}^+}{\mathrm{d}\tau}d(x_{\lambda\tau},y_{\lambda^{-1}\tau})^2\le-\frac{2K}{3}(\lambda+\lambda^{-1}+1)d(x_{\lambda\tau},y_{\lambda^{-1}\tau})^2+2N(\sqrt{\lambda}-\sqrt{\lambda^{-1}})^2.$$

Thus the conclusion follows from this estimate as in the proof of Theorem 2.19. □

We now investigate the relation between the Evolution Variational Inequality and geodesic convexity of the functional S.

Theorem 2.23. *Assume that for every starting point $x_0\in\overline{D(S)}$ the $\mathrm{EVI}_{K,N}$ flow for S exists. Then S is strongly (K,N)-convex.*

Proof. Let P denote the $\mathrm{EVI}_{K,N}$ gradient flow semigroup of S. We treat the case $K\neq 0$ first. So let $(\gamma_s)_{s\in[0,1]}$ be a constant speed geodesic. Let us fix $s\in[0,1], t>0$ and set $\gamma_s^t:=P_t\gamma_s$. We can assume that $d:=d(\gamma_0,\gamma_1)\neq 0$. Using the identity (2.9) we see that (2.24) can be rewritten as

$$\frac{U_N(z)}{U_N(P_{t_1}x)}e_K(t_1-t_0)\ \le\ \frac{1}{K}\Big[\mathrm{e}^{K(t_1-t_0)}\mathfrak{c}_{K/N}\big(d(P_{t_1}x,z)\big)-\mathfrak{c}_{K/N}\big(d(P_{t_0}x,z)\big)\Big]\,. \tag{2.33}$$

Using (2.33) with $t_0=0, t_1=t, x=\gamma_s$ and $z=\gamma_0$ respectively $z=\gamma_1$ we immediately obtain

$$\begin{aligned}
&\sigma_{K/N}^{(1-s)}(d)\cdot U_N(\gamma_0)+\sigma_{K/N}^{(s)}(d)\cdot U_N(\gamma_1)\\
&\le\ \frac{U_N(\gamma_s^t)}{K\cdot e_K(t)}\Big[\sigma_{K/N}^{(1-s)}(d)\cdot\Big(\mathrm{e}^{Kt}\mathfrak{c}_{K/N}\big(d(\gamma_s^t,\gamma_0)\big)-\mathfrak{c}_{K/N}\big(d(\gamma_s,\gamma_0)\big)\Big)\\
&\qquad\qquad+\sigma_{K/N}^{(s)}(d)\cdot\Big(\mathrm{e}^{Kt}\mathfrak{c}_{K/N}\big(d(\gamma_s^t,\gamma_1)\big)-\mathfrak{c}_{K/N}\big(d(\gamma_s,\gamma_1)\big)\Big)\Big]\,.
\end{aligned}$$

Let A denote the term in square brackets in the last inequality. The claim follows if we show that for t small enough we have $A\le K\cdot e_K(t)=\mathrm{e}^{Kt}-1$ if $K>0$ and $A\ge\mathrm{e}^{Kt}-1$ if $K<0$. Using the fact that $d(\gamma_s,\gamma_{s'})=|s-s'|d$, we first find

$$\begin{aligned}
A\ &=\ \frac{\mathrm{e}^{Kt}}{\mathfrak{s}_{K/N}(d)}\Big[\mathfrak{s}_{K/N}\left((1-s)d\right)\cdot\mathfrak{c}_{K/N}\big(d(\gamma_s^t,\gamma_0)\big)+\mathfrak{s}_{K/N}\left(sd\right)\cdot\mathfrak{c}_{K/N}\big(d(\gamma_s^t,\gamma_1)\big)\Big]\\
&\quad-\frac{1}{\mathfrak{s}_{K/N}(d)}\Big[\mathfrak{s}_{K/N}\left((1-s)d\right)\cdot\mathfrak{c}_{K/N}\big(sd\big)+\mathfrak{s}_{K/N}\left(sd\right)\cdot\mathfrak{c}_{K/N}\big((1-s)d\big)\Big]\\
&:=A_1+A_2\,.
\end{aligned}$$

By the angle sum identity for sin (resp. sinh) we have $A_2=-1$. To see that $A_1\le\mathrm{e}^{Kt}$ (resp. $A_1\ge\mathrm{e}^{Kt}$), we observe the following fact, which is easily verified using the angle sum identities for trigonometric or hyperbolic functions: If $\alpha, \alpha' \geq 0$ and $\varepsilon, \varepsilon' \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ such that $\varepsilon + \varepsilon' \geq 0$, then, putting $\beta = \alpha + \varepsilon, \beta' = \alpha' + \varepsilon'$, we have that

$$\begin{aligned} \sin(\alpha)\cos(\beta') + \cos(\beta)\sin(\alpha') &\leq \sin(\alpha + \alpha') , \\ \sinh(\alpha)\cosh(\beta') + \cosh(\beta)\sinh(\alpha') &\geq \sinh(\alpha + \alpha') . \end{aligned}$$

To conclude, we apply this with $\alpha = (1 - s)d$, $\alpha' = sd$ and $\varepsilon = d(\gamma_s^t, \gamma_1) - (1 - s)d$, $\varepsilon' = d(\gamma_s^t, \gamma_0) - sd$ and note that $\varepsilon + \varepsilon' \geq 0$ by the triangle inequality.

Finally, we treat the case $K = 0$. By Lemma 2.15 P is a $\mathrm{EVI}_{K',N}$ flow for every $K' < 0$. Thus by the previous argument (2.11) holds with K' instead of K and we can pass to the limit as $K' \nearrow 0$. □

3. Entropic and Riemannian curvature-dimension conditions

3.1. **The entropic curvature-dimension condition.** In this section we introduce a new curvature-dimension condition for metric measure spaces based on (K, N)-convexity of the entropy on the Wasserstein space.

Let (X, d, m) be a metric measure space, i.e. (X, d) is a complete and separable metric space and m is a locally finite, σ-finite Borel measure on X. We denote by $\mathscr{P}_2(M, d)$ the L^2-Wasserstein space over (X, d), i.e. the set of all Borel probability measures μ satisfying

$$\int d(x_0, x)^2 \mu(\mathrm{d}x) \ < \infty$$

for some, hence any, $x_0 \in X$. The subspace of all measures absolutely continuous w.r.t. m is denoted by $\mathscr{P}_2(X, d, m)$. The L^2-Wasserstein distance between $\mu_0, \mu_1 \in \mathscr{P}_2(X, d)$ is defined by

$$W_2(\mu_0, \mu_1)^2 \ = \ \inf \int d(x, y)^2 \mathrm{d}q(x, y) ,$$

where the infimum is taken over all Borel probability measures q on $X \times X$ with marginals μ_0 and μ_1. Let us denote by $\mathrm{Geo}(X) = \{\gamma : [0, 1] \to X \mid \gamma \text{ const. speed geodesic}\}$ the space of constant speed geodesics in X equipped with the topology of uniform convergence. For any $t \in [0, 1]$ we denote by $e_t : \mathrm{Geo}(X) \to X$ the evaluation map $\gamma \mapsto \gamma_t$. Recall that a *dynamic optimal coupling* between $\mu_0, \mu_1 \in \mathscr{P}_2(X, d)$ is a probability measure $\pi \in \mathscr{P}(\mathrm{Geo}(X))$ such that $(e_0, e_1)_{\#}\pi$ is an optimal coupling of μ_0, μ_1. The curve $(\mu_t)_{t\in[0,1]}$ with $\mu_t = (e_t)_{\#}\pi$ is then a geodesic in $\mathscr{P}_2(X, d)$ connecting μ_0 to μ_1. Moreover, by [39, Lem. I.2.11], for each geodesic $\Gamma : [0, 1] \to \mathscr{P}_2(X, d)$, there exists a probability measure π on $\mathrm{Geo}(X)$ such that $\Gamma_t = (e_t)_{\#}\pi$ for all $t \in [0, 1]$.

Given a measure $\mu \in \mathscr{P}_2(X, d)$ we define its relative entropy by

$$\mathrm{Ent}(\mu) \ := \ \int \rho \log \rho \mathrm{d}m ,$$

if $\mu = \rho m$ is absolutely continuous w.r.t. m and $(\rho \log \rho)_+$ is integrable. Otherwise we set $\mathrm{Ent}(\mu) = +\infty$. The subset of probability measures with finite entropy will be denoted by $\mathscr{P}_2^*(X, d, m)$. Moreover, for a number $N \in (0, \infty)$ we introduce the functional $U_N : \mathscr{P}_2(X, d) \to [0, \infty]$ by

$$U_N(\mu) \ := \ \exp\left(-\frac{1}{N}\mathrm{Ent}(\mu)\right) .$$

Definition 3.1. *Given two numbers $K \in \mathbb{R}$, $N \in (0, \infty)$ we say that a metric measure space (X, d, m) satisfies the* entropic curvature-dimension condition $\mathrm{CD}^e(K, N)$ *if and only if for each pair $\mu_0, \mu_1 \in \mathscr{P}_2^*(X, d, m)$ there exists a constant speed geodesic $(\mu_t)_{t\in[0,1]}$ in $\mathscr{P}_2^*(X, d, m)$ connecting μ_0 to μ_1 such that for all $t \in [0, 1]$:*

$$U_N(\mu_t) \ \geq \ \sigma_{K/N}^{(1-t)}\big(W_2(\mu_0, \mu_1)\big)U_N(\mu_0) + \sigma_{K/N}^{(t)}\big(W_2(\mu_0, \mu_1)\big)U_N(\mu_1) . \tag{3.1}$$

If (3.1) holds for any constant speed geodesic $(\mu_t)_{t\in[0,1]}$ in $\mathscr{P}_2^*(X, d, m)$ we say that (X, d, m) is a strong $\mathrm{CD}^e(K, N)$ *space.*

In other words, the $\mathrm{CD}^e(K,N)$-condition means that the entropy is (K,N)-convex along Wasserstein geodesic. As an immediate consequence of Lemma 2.12 we obtain the following consistency result.

Lemma 3.2. *If (X,d,m) satisfies the $\mathrm{CD}^e(K,N)$ condition, then it also satisfies $\mathrm{CD}^e(K',N')$ for any $K' \le K$ and $N' \ge N$. Moreover, it satisfies the $\mathrm{CD}(K,\infty)$ condition.*

Using similar arguments as in the case of the $\mathrm{CD}(K,\infty)$ condition introduced in [39] it is immediate to check that $\mathrm{CD}^e(K,N)$ is invariant under isomorphisms of metric measure spaces. Moreover, adapting [39, Thm. I.4.20], one can check that it is stable under convergence of metric measure spaces in the transportation distance $\mathbb{D}$, also introduced in [39].

As an application of the additivity of (K,N)-convexity we note the following

Proposition 3.3 (Weighted spaces)**.** *Let (X,d,m) be a metric measure space satisfying $\mathrm{CD}^e(K,N)$ and let $V : X \to \mathbb{R}$ be a measurable function bounded from below that is strongly (K',N')-convex in the sense of Definition 2.7. Then $(X,d,\mathrm{e}^{-V}m)$ satisfies $\mathrm{CD}^e(K+K',N+N')$. In particular, if (X,d,m) satisfies strong $\mathrm{CD}^e(K,N)$, then $(X,d,\mathrm{e}^{-V}m)$ also satisfies strong $\mathrm{CD}^e(K+K',N+N')$.*

Proof. We will first show that the functional $\overline{V} : \mathscr{P}_2(X,d) \to (-\infty,\infty]$ defined by $\overline{V}(\mu) = \int V \mathrm{d}\mu$ is strongly (K',N')-convex on $\mathscr{P}_2(X,d)$. Let $\pi \in \mathscr{P}(\mathrm{Geo}(X))$ be an dynamic optimal coupling. and set $\mu_t = (e_t)_\# \pi$. From the (K',N')-convexity of V we have for any $\gamma \in \mathrm{Geo}(X)$ and $t \in [0,1]$:

$$\mathrm{e}^{-V(\gamma_t)/N'} \;\ge\; \sigma^{(1-t)}_{K'/N'}\big(d(\gamma_0,\gamma_1)\big)\cdot \mathrm{e}^{-V(\gamma_t)/N'} + \sigma^{(t)}_{K'/N'}\big(d(\gamma_0,\gamma_1)\big)\cdot \mathrm{e}^{-V(\gamma_t)/N'} \,. \tag{3.2}$$

Take the logarithm on both sides of (3.2). By virtue of Lemma 2.11, we can use Jensen's inequality when integrating it w.r.t. π to obtain

$$\begin{aligned} -\frac{1}{N'}\overline{V}(\Gamma_t) \;=\; -\frac{1}{N'}\int V(\gamma_t)\mathrm{d}\pi(\gamma) \;&\ge\; \int G_t\Big(-\frac{1}{N'}V(\gamma_0), -\frac{1}{N'}V(\gamma_1), \frac{K'}{N'}d(\gamma_0,\gamma_1)^2\Big)\mathrm{d}\pi(\gamma)\\ &\ge G_t\Big(-\frac{1}{N'}\overline{V}(\mu_0), -\frac{1}{N'}\overline{V}(\mu_1), \frac{K'}{N'}W_2(\mu_0,\mu_1)^2\Big)\,. \end{aligned}$$

Taking the exponential again then yields the claim. By the lower boundedness of V we have $\mathscr{P}_2(X,d,\mathrm{e}^{-V}m) \subset \mathscr{P}_2(X,d,m)$. Now the assertion of the proposition is a consequence of the observation

$$\mathrm{Ent}(\mu|\mathrm{e}^{-V}m) \;=\; \mathrm{Ent}(\mu|m) + \overline{V}(\mu)$$

and Lemma 2.10. The latter assertion is obvious from the proof. □

We will now derive some first geometric consequences of the entropic curvature-dimension condition.

Proposition 3.4 (Generalized Brunn–Minkowski inequality)**.** *Assume that (X,d,m) satisfies the condition $\mathrm{CD}^e(K,N)$ with $N \ge 1$. Then for all measurable sets $A_0, A_1 \subset X$ with $m(A_0), m(A_1) > 0$ and all $t \in [0,1]$ we have*

$$\bar{m}(A_t)^{1/N} \;\ge\; \sigma^{(1-t)}_{K/N}(\Theta)\cdot m(A_0)^{1/N} + \sigma^{(t)}_{K/N}(\Theta)\cdot m(A_1)^{1/N}\,, \tag{3.3}$$

where $\bar{m}$ is the completion of m, A_t denotes the set of t-midpoints and Θ the minimal/maximal distance between points in A_0 and A_1, i.e.

$$\begin{aligned} A_t \;&=\; \{\gamma_t : \gamma : [0,1] \to X \text{ geodesic s.t. } \gamma_0 \in A_0, \gamma_1 \in A_1\}\,,\\ \Theta \;&=\; \begin{cases} \inf_{x_0\in A_0, x_1\in A_1} d(x_0,x_1)\,, & K \ge 0\,,\\ \sup_{x_0\in A_0, x_1\in A_1} d(x_0,x_1)\,, & K < 0\,. \end{cases} \end{aligned}$$

Proof. We first prove the assertion under the assumption that $m(A_0), m(A_1) < \infty$, the general case then follows by approximating the sets A_0, A_1 by sets of finite volume. Applying the condition $\mathrm{CD}^e(K,N)$ to $\mu_i = m(A_i)^{-1}\mathbf{1}_{A_i}m$ for $i = 0,1$ yields

$$U_N(\Gamma_t) \;\ge\; \sigma^{(1-t)}_{K/N}\big(W_2(\mu_0,\mu_1)\big)\cdot m(A_0)^{1/N} + \sigma^{(t)}_{K/N}\big(W_2(\mu_0,\mu_1)\big)\cdot m(A_1)^{1/N}\,, \tag{3.4}$$

where $\mu_t = \rho_t m$ is the t-midpoint of a geodesic connecting μ_0 and μ_1. Since μ_t is concentrated on A_t, which is a Souslin set, a double application of Jensen's inequality gives that

$$\begin{aligned} U_N(\mu_t) &= \exp\Big(-\frac{1}{N}\int \log\rho_t \mathrm{d}\mu_t\Big) \leq \int \rho_t^{-1/N}\mathrm{d}\mu_t \\ &= \int_{A_t} \rho_t^{1-1/N}\mathrm{d}\bar{m} \leq \bar{m}(A_t)^{1/N} . \end{aligned}$$

Hence (3.3) follows by noting that $\theta \mapsto \sigma^{(t)}_{K/N}(\theta)$ is increasing if $K \geq 0$ and decreasing if $K < 0$ and that $W_2(\mu_0,\mu_1) \geq \Theta$ (resp. $\leq \Theta$). □

The Brunn–Minkowski inequality entails further geometric consequences like a Bishop–Gromov type volume growth estimate and a generalized Bonnet–Myers theorem. The following results can be deduced from Proposition 3.4 using similar arguments as in [39] and replacing the coefficients $\tau^{(t)}_{K/N}(\cdot)$ by $\sigma^{(t)}_{K/N}(\,\cdot\,)$.

Remark 3.5. The estimates presented below are not sharp, yet they provide necessary local compactness results for example. We will see below that under the assumption that (X,d,m) is non-branching the $\mathrm{CD}^e(K,N)$ condition is equivalent to the $\mathrm{CD}^*(K,N)$ condition. It has been proven by Cavaletti & Sturm [15] that under the same assumption $\mathrm{CD}^*(K,N)$ implies the measure contraction property $\mathrm{MCP}(K,N)$ from which a sharp Bishop–Gromov and Lichnerowicz inequality can be derived, see [39].

To state the volume growth estimate we introduce the following notation. Given a metric measure space (X,d,m) and a point $x_0 \in \mathrm{supp}[m]$ we denote by

$$v(r) := m(\overline{B_r(x_0)})$$

the volume of the closed ball of radius r around x_0. Moreover, we set

$$s(r) := \limsup_{\delta\to 0} \frac{1}{\delta} m(\overline{B_{r+\delta}(x_0)} \setminus B_r(x_0))$$

for the volume of the corresponding sphere.

Proposition 3.6 (Generalized Bishop–Gromov inequality)**.** *Assume that (X,d,m) satisfies the condition $\mathrm{CD}^e(K,N)$ with $N \geq 1$. Then each bounded closed set $M \subset \mathrm{supp}[m]$ is compact and has finite volume. More precisely, for each $x_0 \in \mathrm{supp}[m]$ and $0 < r < R \leq \pi\sqrt{N/(K\vee 0)}$,*

$$\frac{s(r)}{s(R)} \geq \left(\frac{\mathfrak{s}_{K/N}(r)}{\mathfrak{s}_{K/N}(R)}\right)^N \quad \text{and} \quad \frac{v(r)}{v(R)} \geq \frac{\int_0^r \mathfrak{s}_{K/N}(t)^N\,\mathrm{d}t}{\int_0^R \mathfrak{s}_{K/N}(t)^N\,\mathrm{d}t} . \tag{3.5}$$

Corollary 3.7 (Generalized Bonnet–Myers theorem)**.** *If (X,d,m) satisfies the condition $\mathrm{CD}^e(K,N)$ with $K > 0$ and $N \geq 1$, then the support of m is compact and its diameter L can be bounded as $L \leq \pi\sqrt{N/K}$.*

Remark 3.8. $\mathrm{CD}^e(K,N)$ or $\mathrm{CD}(K,\infty)$ yields that $\mathscr{P}(X,d)$ is a length space and hence so is $(\mathrm{supp}\, m, d)$ [39, Rem. I.4.6(iii), Prop. 2.11(iii)]. Thus, by the local compactness ensured in Proposition 3.6, if (X,d,m) is a $\mathrm{CD}^e(K,N)$ space then $(\mathrm{supp}\, m,d)$ and hence $\mathscr{P}_2(\mathrm{supp}\, m,d)$ is a geodesic space (see e.g. [14, Thm. 2.5.23]). In addition, the volume growth estimate (3.5) implies in particular that for any $x_0 \in X$ and $c > 0$:

$$\int_X \mathrm{e}^{-cd(x_0,x)^2}\mathrm{d}m(x) < \infty . \tag{3.6}$$

It is well known that the latter implies that Ent does not take the value $-\infty$ on $\mathscr{P}_2(X,d)$ and is lower semi-continuous w.r.t. W_2 (see e.g. [6, Sec. 7]). Thus, when $\mathrm{supp}\, m = X$, Definition 3.1 fits well into the setting of Section 2.3, where we assumed these additional regularity properties.

It turns out that under mild assumptions the modified curvature-dimension condition $\mathrm{CD}^e(K,N)$ is equivalent to the reduced curvature-dimension condition $\mathrm{CD}^*(K,N)$ introduced in [9]. We recall here the definition. Denote by $\mathscr{P}_\infty(X,d,m)$ the set of measures in $\mathscr{P}_2(X,d,m)$ with bounded support.

Definition 3.9. *We say that a metric measure space (X,d,m) satisfies the* reduced curvature-dimension condition $\mathrm{CD}^*(K,N)$ *if and only if for each pair $\mu_0=\rho_0 m, \mu_1=\rho_1 m\in\mathscr{P}_\infty(X,d,m)$ there exist an optimal coupling q of them and a geodesic $(\mu_t)_{t\in[0,1]}$ in $\mathscr{P}_\infty(X,d,m)$ connecting them such that for all $t\in[0,1]$ and $N'\geq N$:*

$$\int \rho_t^{-\frac{1}{N'}}\,\mathrm{d}\mu_t \;\geq\; \int\limits_{X\times X}\Big[\sigma^{(1-t)}_{K/N'}\big(d(x_0,x_1)\big)\rho_0(x_0)^{-\frac{1}{N'}} + \sigma^{(t)}_{K/N'}\big(d(x_0,x_1)\big)\rho_1(x_1)^{-\frac{1}{N'}}\Big]\mathrm{d}q(x_0,x_1)\;. \tag{3.7}$$

If (3.7) *holds for any geodesic $(\mu_t)_{t\in[0,1]}$ in $\mathscr{P}_\infty(X,d,m)$ we say that (X,d,m) is a* strong $\mathrm{CD}^*(K,N)$ *space.*

The assumption we need to prove equivalence of the different curvature-dimension conditions is the following weak form of non-branching.

Definition 3.10. *We say that a metric measure space (X,d,m) is* essentially non-branching *if any dynamic optimal coupling $\pi\in\mathscr{P}(\mathrm{Geo}(X))$ between two absolutely continuous measures is supported in a set of non-branching geodesics, i.e. there exists $A\subset\mathrm{Geo}(X)$ such that $\pi(A)=1$ and for all $\gamma,\widetilde{\gamma}\in A$:*

$$\gamma_t=\widetilde{\gamma}_t\quad\forall t\in[0,\varepsilon]\ \text{for some}\ \varepsilon>0\ \Rightarrow\ \gamma=\widetilde{\gamma}\;.$$

This condition has been introduced in [37] and it has been shown that strong $\mathrm{CD}(K,\infty)$ spaces are essentially non-branching. It has also been noted there that the essential non-branching condition is equivalent to the following apparently stronger condition: Every dynamic optimal coupling π between absolutely continuous measures is concentrated on a set of geodesics that do not meet at intermediate times, i.e. there is $A'\subset\mathrm{Geo}(X)$ such that $\pi(A')=1$ and for all $\gamma,\widetilde{\gamma}\in A'$:

$$\gamma_t=\widetilde{\gamma}_t\quad\text{for some } t\in(0,1)\ \Rightarrow\ \gamma=\widetilde{\gamma}\;.$$

Indeed, assuming the existence of a dynamic optimal coupling where such crossings happen with positive probability, one can reshuffle the geodesics before and after the crossing to produce a dynamic optimal coupling of the same marginals where branching happens with positive probability, contradicting the essentially non-branching assumption.

An immediate consequence of this observation is the following adaption of [9, Lem. 2.8].

Lemma 3.11. *Let (X,d,m) be an essentially non-branching metric measure space and let π be a dynamic optimal coupling. Assume that $\pi=\sum_{k=1}^n\alpha_k\pi^k$ for suitable $\alpha_k>0$ and dynamic optimal couplings π^k. For given $t\in(0,1)$ and $i\in\{0,t\}$ we set $\mu_i^k=(e_i)_\#\pi^k$. If the family $\{\mu_0^k\}_k$ is mutually singular, then also the family $\{\mu_t^k\}_k$ is mutually singular.*

Theorem 3.12. *Let (X,d,m) be an essentially non-branching metric measure space. Then the following assertions are equivalent:*

(i) *(X,d,m) satisfies $\mathrm{CD}^*(K,N)$,*
(ii) *For each pair $\mu_0,\mu_1\in\mathscr{P}_\infty(X,d,m)$ there is a dynamic optimal coupling π of them such that we have $(e_t)_\#\pi\ll m$ and*

$$\rho_t(\gamma_t)^{-\frac{1}{N}}\;\geq\sigma^{(1-t)}_{K/N}\big(d(\gamma_0,\gamma_1)\big)\rho_0(\gamma_0)^{-\frac{1}{N}}+\sigma^{(t)}_{K/N}\big(d(\gamma_0,\gamma_1)\big)\rho_1(\gamma_1)^{-\frac{1}{N}}\;, \tag{3.8}$$

for π-a.e. $\gamma\in\mathrm{Geo}(X)$, where ρ_t denotes the density of $(e_t)_\#\pi$ w.r.t. m.
(iii) *(X,d,m) satisfies $\mathrm{CD}^e(K,N)$.*

Proof. The equivalence of (i) and (ii) has already been proven in [9, Prop. 2.8] under the assumption that X is non-branching. Note that the statement (ii) is slightly different there but equivalent, since under the non-branching assumption m^2-a.e. pair of points is connected by a unique geodesic. Under the weaker essential non-branching condition the equivalence of (i) and (ii) follows by repeating almost verbatim the proof of [9, Prop. 2.8] substituting [9, Lem. 2.6] with Lemma 3.11. For details on the necessary modifications see also the implication (iii)⇒(ii) below which follows a similar argument.

(ii)⇒(iii): First note that by an approximation argument as in [9, Lem. 2.11] one can show that (3.8) also holds for $\mu_0, \mu_1 \in \mathscr{P}_2(X, d, m)$ not necessarily with bounded support. Now fix $\mu_0, \mu_1 \in \mathscr{P}_2(X, d, m)$ and a dynamic optimal coupling π of them satisfying (3.8). Taking logarithms on both sides of (3.8) we obtain

$$-\frac{1}{N}\log\rho_t(\gamma_t) \;\geq\; G_t\Big(-\frac{1}{N}\log\rho_0(\gamma_0), -\frac{1}{N}\log\rho_1(\gamma_1), \frac{K}{N}d(\gamma_0,\gamma_1)^2\Big)\,, \tag{3.9}$$

where the function G_t is given by (2.14). Integrating (3.9) w.r.t. π and using Jensen's inequality with the aid of Lemma 2.11 we obtain

$$-\frac{1}{N}\operatorname{Ent}(\mu_t) \;\geq\; G_t\Big(-\frac{1}{N}\operatorname{Ent}(\mu_0), -\frac{1}{N}\operatorname{Ent}(\mu_1), \frac{K}{N}W_2(\mu_0,\mu_1)^2\Big)\,.$$

Hence (3.1) follows by taking the exponential on both sides.

(iii)⇒(ii): Here we follow closely the arguments in the proof of [39, Prop. II.4.2]. Fix $\mu_0, \mu_1 \in \mathscr{P}_\infty(X, d, m)$ and a dynamic optimal coupling π of them. Let $\{M_n\}_{n\in\mathbb{N}}$ be a $\cap$-stable generator of the Borel σ-field of X with $m(\partial M_n) = 0$ for all n. For each n consider the disjoint covering of X given by the 2^n sets $L_1 = M_1 \cap \cdots \cap M_n$, $L_2 = M_1 \cap \cdots \cap M_n^c$, ..., $L_{2^n} = M_1^c \cap \cdots \cap M_n^c$. For fixed n and $i, j = 1, \ldots, 2^n$ we define sets $A_{i,j} = \{\gamma \in \mathrm{Geo}(X) \;:\; (\gamma_0, \gamma_1) \in L_i \times L_j\}$ and probability measures $\mu_0^{i,j}, \mu_1^{i,j}$ by

$$\mu_0^{i,j}(B) = \alpha_{i,j}^{-1}\pi\big(\{\gamma_0 \in B \cap L_i, \gamma_1 \in L_j\}\big)\,, \quad \mu_1^{i,j}(B) = \alpha_{i,j}^{-1}\pi\big(\{\gamma_0 \in L_i, \gamma_1 \in B \cap L_j\}\big)\,,$$

provided that $\alpha_{i,j} = \pi(A_{i,j}) > 0$. By (iii) we can choose dynamic optimal couplings $\pi^{i,j}$ of them such that

$$U_N(\mu_t^{i,j}) \;\geq \sigma_{K/N}^{(1-t)}\big(W_2(\mu_0^{i,j},\mu_1^{i,j})\big)\cdot U_N(\mu_0^{i,j}) + \sigma_{K/N}^{(t)}\big(W_2(\mu_0^{i,j},\mu_1^{i,j})\big)\cdot U_N(\mu_1^{i,j})\,, \tag{3.10}$$

where $\mu_t^{i,j} = (e_t)_\#\pi^{i,j}$. Define

$$\pi^{(n)} := \sum_{i,j=1}^{2^n}\alpha_{i,j}\pi^{i,j}\,, \qquad \mu_t^{(n)} = (e_t)_\#\pi^{(n)}\,.$$

Then $\pi^{(n)}$ is a dynamic optimal coupling of the measures μ_0, μ_1 and $(\mu_t^{(n)})_{t\in[0,1]}$ is a geodesic between them. Since the measures $\mu_0^{i,j} \otimes \mu_1^{i,j}$ are mutually singular and X is essentially non-branching, also the measures $\mu_t^{i,j}$ are mutually singular for each fixed t by Lemma 3.11. We conclude that $\rho_t^{(n)}(\gamma_t) = \alpha_{i,j}\rho_t^{i,j}(\gamma_t)$ on the set $A_{i,j}$. Plugging this into (3.10) and taking logarithms on both sides we find

$$\begin{aligned}
&-\frac{\alpha_{i,j}^{-1}}{N}\int_{A_{i,j}}\log\rho_t^{(n)}(\gamma_t)\mathrm{d}\pi^{(n)} \qquad\qquad (3.11)\\
&\geq\; G_t\Big(-\frac{\alpha_{i,j}^{-1}}{N}\int_{A_{i,j}}\log\rho_0(\gamma_0)\mathrm{d}\pi^{(n)}, -\frac{\alpha_{i,j}^{-1}}{N}\int_{A_{i,j}}\log\rho_1(\gamma_1)\mathrm{d}\pi^{(n)}, \alpha_{i,j}^{-1}\frac{K}{N}\int_{A_{i,j}}d^2(\gamma_0,\gamma_1)\mathrm{d}\pi^{(n)}\Big)\,.
\end{aligned}$$

Since μ_0, μ_1 have bounded support, all geodesic in the support of the measures $\pi^{(n)}$ stay within a single closed bounded set B. By Proposition 3.6 B is compact and has finite mass. Hence also the measures $\pi^{(n)}$ are supported in a single compact set and thus converge weakly, up to extraction of a subsequence, to a dynamic optimal coupling $\widetilde{\pi}$ of μ_0 and μ_1. Since $m(\partial M_i) = 0$ for all i we deduce that

$$\pi\big(\{\gamma_0 \in M_i, \gamma_1 \in M_j\}\big) \;=\; \lim_{n\to\infty} \pi^{(n)}\big(\{\gamma_0 \in M_i, \gamma_1 \in M_j\}\big) \;=\; \widetilde{\pi}\big(\{\gamma_0 \in M_i, \gamma_1 \in M_j\}\big)$$

for each i, j and hence $(e_0, e_1)_\# \pi = (e_0, e_1)_\# \widetilde{\pi}$. In particular $\widetilde{\pi}$ is a dynamic optimal coupling of μ_0 and μ_1. By weak lower semi-continuity of the entropy we can pass to the limit as $n \to \infty$ in the left hand side of (3.11). Invoking furthermore the convexity of G_t given by Lemma 2.11 and Jensen's inequality we see that

$$\begin{aligned} &-\frac{\alpha^{-1}}{N}\int_A \log \rho_t(\gamma_t) \mathrm{d}\widetilde{\pi} \qquad\qquad (3.12)\\ &\geq\; G_t\Big(-\frac{\alpha^{-1}}{N}\int_A \log \rho_0(\gamma_0)\mathrm{d}\widetilde{\pi}, -\frac{\alpha^{-1}}{N}\int_A \log \rho_1(\gamma_1)\mathrm{d}\widetilde{\pi}, \alpha^{-1}\frac{K}{N}\int_A d^2(\gamma_0,\gamma_1)\mathrm{d}\widetilde{\pi}\Big)\,, \end{aligned}$$

for any set A which is a union of a finite number of the sets $A_{i,j}$ and $\alpha = \widetilde{\pi}(A)$. This implies the $\widetilde{\pi}$-a.s. inequality (3.8). □

Corollary 3.13. *For a metric measure space (X, d, m) the following assertions are equivalent:*

(i) *(X, d, m) is a strong $\mathrm{CD}^*(K, N)$ space,*
(ii) *For each pair $\mu_0, \mu_1 \in \mathscr{P}_\infty(X, d, m)$, and each dynamic optimal coupling π of it (3.8) holds,*
(iii) *(X, d, m) is a strong $\mathrm{CD}^e(K, N)$ space.*

Proof. Note that both (i) and (iii) imply that (X, d, m) satisfies the strong $\mathrm{CD}(K, \infty)$ condition. [37, Thm. 1.1] gives that every strong $\mathrm{CD}(K, \infty)$ space is essentially non-branching. In addition, [37, Cor. 1.4] also states that on strong $\mathrm{CD}(K, \infty)$ spaces the dynamic optimal coupling of μ_0 and μ_1 is unique for each $\mu_0, \mu_1 \in \mathscr{P}_2(X, d, m)$. Hence the assertion follows from the same arguments as Theorem 3.12. Indeed, the dynamic optimal coupling $\widetilde{\pi}$ obtained in the proof of Theorem 3.12 (iii)⇒(ii) coincides with π. Note that the essentially non-branching assumption is not used in the implications (ii)⇒(i),(iii). □

We conclude this section with a globalization property of the strong entropic curvature-dimension condition. We say that a metric measure space (X, d, m) satisfies the *local* entropic curvature-dimension condition $\mathrm{CD}^e_{\mathrm{loc}}(K, N)$ if and only if every point $x \in \operatorname{supp} m$ has a neighborhood M such that for each pair $\mu_0, \mu_1 \in \mathscr{P}_2^*(X, d, m)$ supported in M there exists a geodesic $(\mu_t)_{t\in[0,1]}$ in $\mathscr{P}_2^*(X, d, m)$ satisfying (3.1). Similarly, we say that (X, d, m) is a *strong* $\mathrm{CD}^e_{\mathrm{loc}}(K, N)$ space if in addition (3.1) holds along *every* constant speed geodesic $(\mu_t)_{t\in[0,1]}$ in $\mathscr{P}_2^*(X, d, m)$ with μ_0, μ_1 supported in M. Note that (X, d, m) is essentially non-branching if it is $\mathrm{CD}^e_{\mathrm{loc}}(K, N)$ space. Indeed, we first localize the problem in the argument in [37] and hence the local condition is sufficient.

Theorem 3.14 (Local-global)**.** *Let (X, d, m) be a geodesic metric measure space. Then it satisfies the strong $\mathrm{CD}^e(K, N)$ condition if and only if it satisfies the strong $\mathrm{CD}^e_{loc}(K, N)$ condition.*

Proof. The only if part is obvious. For the if part, assume that (X, d, m) is a strong $\mathrm{CD}^e_{\mathrm{loc}}(K, N)$ space. First note that this implies that X is locally compact. Indeed, this can be seen by estimating the volume growth of balls in a small neighborhood around any point similarly as in Proposition 3.6. (X, d) being a length space, local compactness implies that bounded closed sets in X are compact, see [14, Prop. 2.5.22].

Now we first verify the $\mathrm{CD}^e(K, N)$ inequality (3.1) for a geodesic $(\mu_t)_{t\in[0,1]}$ in $\mathscr{P}_2^*(X, d, m)$ where the measures μ_t are jointly supported in a compact set K. By compactness and the strong $\mathrm{CD}^e_{\mathrm{loc}}(K, N)$ condition we can find $\epsilon > 0$ and a disjoint partition $(Y_i)_i$ of K such that the ε-neighborhoods U_i of Y_i have the following property: any geodesic $(\mu_t)_{t\in[0,1]}$ in $\mathscr{P}_2^*(X, d, m)$ with μ_0, μ_1 supported in U_i satisfies (3.1). Write $\mu_t = (e_t)_\# \pi$, where $\pi \in \mathscr{P}(\mathrm{Geo}(X))$ is the associated dynamic optimal coupling. Then there exists $L > 0$ such $d(\gamma_0, \gamma_1) \leq L$ for all γ in the support of π. We claim that for any $0 \le r \le t \le s \le 1$ with $|s-r| < \varepsilon/L$:

$$U_N(\mu_t) \;\ge\; \sigma_{K/N}^{(\frac{s-t}{s-r})}\big(W_2(\mu_r,\mu_s)\big)U_N(\mu_r) + \sigma_{K/N}^{(\frac{t-r}{s-r})}\big(W_2(\mu_r,\mu_s)\big)U_N(\mu_s)\;, \tag{3.13}$$

which suffices to show (3.1) by virtue of Lemma 2.8. Indeed, let us define the sets $A_i = \{\gamma \in \mathrm{Geo}(X) \;:\; \gamma_t \in Y_i\}$ and define the measures

$$\pi_i \;:=\; \alpha_i^{-1}\pi|_{A_i}\;,$$

provided that $\alpha_i := \pi(A_i) > 0$. Then for π_i-a.e. geodesic γ and $\tau \in [r,s]$ one has $\gamma_\tau \in U_i$. Setting $\mu_\tau^i = (e_\tau)_\#\pi_i$ we infer that the geodesic $(\mu_\tau^i)_{\tau\in[r,s]}$ is supported in U_i. From the construction of U_i we obtain for $\tau \in [r,s]$:

$$U_N(\mu_\tau^i) \;\ge\; \sigma_{K/N}^{(\frac{s-\tau}{s-r})}\big(W_2(\mu_r^i,\mu_s^i)\big)U_N(\mu_r^i) + \sigma_{K/N}^{(\frac{\tau-r}{s-r})}\big(W_2(\mu_r^i,\mu_s^i)\big)U_N(\mu_s^i)\;. \tag{3.14}$$

Note that $\mu_\tau = \sum_i \alpha_i\mu_\tau^i$. Hence we have that (see e.g. [39, Rem. I.4.2])

$$\mathrm{Ent}(\mu_\tau) \ge \sum_i \alpha_i\,\mathrm{Ent}(\mu_\tau^i) + \sum_i \alpha_i \log\alpha_i\;. \tag{3.15}$$

For $\tau = t$ we have equality in (3.15) since the family $(\mu_t^i)_i$ is mutually singular by construction. Taking logarithms in (3.14) and summing over i we obtain

$$\begin{aligned}
-\frac{1}{N}\,\mathrm{Ent}(\mu_t) \;&=\; -\frac{1}{N}\sum_i \alpha_i\big[\,\mathrm{Ent}(\mu_t^i) + \log\alpha_i\big]\\
&\ge\; \sum_i \alpha_i G_{\frac{t-r}{s-r}}\left(-\frac{1}{N}\Big[\,\mathrm{Ent}(\mu_r^i) + \log\alpha_i\Big], -\frac{1}{N}\Big[\,\mathrm{Ent}(\mu_s^i) + \log\alpha_i\Big], \frac{K}{N}W_2^2(\mu_r^i,\mu_s^i)\right)\\
&\ge\; G_{\frac{t-r}{s-r}}\left(-\frac{1}{N}\sum_i\alpha_i\Big[\,\mathrm{Ent}(\mu_r^i) + \log\alpha_i\Big], -\frac{1}{N}\sum_i\alpha_i\Big[\,\mathrm{Ent}(\mu_s^i) + \log\alpha_i\Big], \frac{K}{N}\sum_i\alpha_i W_2^2(\mu_r^i,\mu_s^i)\right)\\
&\ge\; G_{\frac{t-r}{s-r}}\left(-\frac{1}{N}\,\mathrm{Ent}(\mu_r), -\frac{1}{N}\,\mathrm{Ent}(\mu_s), \frac{K}{N}W_2^2(\mu_r,\mu_s)\right)\;,
\end{aligned}$$

where we have used (3.15) as well as the convexity of $G_{\frac{t-r}{s-r}}(x,y,\kappa)$ given by Lemma 2.11 and its monotonicity in x, y. Taking the exponential yields (3.13).

Finally, we establish the $\mathrm{CD}^e(K,N)$ inequality (3.1) for an arbitrary, not necessarily compactly supported geodesic $(\mu_t)_{t\in[0,1]}$ in $\mathscr{P}_2^*(X,d,m)$. Partition X in a disjoint collection of precompact sets K_i and let $\pi_{i,j}$ be dynamic optimal couplings obtained by conditioning the coupling π associated to $(\mu_t)_t$ to have starting point in K_i and endpoint in K_j. By the previous argument any compactly supported geodesic satisfies (3.1). Since $\mathrm{CD}^e_{\mathrm{loc}}(K,N)$ implies that (X,d,m) is essentially non-branching, the measures $(e_t)_\#\pi_{i,j}$ are mutually singular using Lemma 3.11. Thus arguing as before the inequality (3.1) for $(\mu_t)_t$ can be obtained by summing the corresponding inequalities valid along the geodesics $(\mu_t^{i,j})_t$ associated to $\pi_{i,j}$. □

3.2. **Calculus and heat flow on metric measure spaces.** Here we recapitulate briefly some of the results obtained by Ambrosio, Gigli and Savaré in a series of recent works, see [6, 4, 5, 19]. In particular, we introduce notation and concepts that we use in the sequel about the powerful machinery of calculus on metric measure spaces developed by these authors. We refer to [6, 4] for more details on the definitions and results.

Let (X,d,m) be a metric measure space. The basic object of study, introduced in [6] is the Cheeger energy. For a measurable function $f : X \to \mathbb{R}$ it can be defined by

$$\mathrm{Ch}(f) = \frac{1}{2}\int |\nabla f|_w^2 \mathrm{d}m\;,$$

where $|\nabla f|_w : X \to [0,\infty]$ denotes the so called minimal weak upper gradient of f. An important approximation result [6, Thm. 6.2] states that for $f \in L^2(X,m)$ the Cheeger energy can also be obtained by a relaxation procedure:

$$\mathrm{Ch}(f) = \inf\left\{\liminf_{n\to\infty}\frac{1}{2}\int|\nabla f_n|^2\mathrm{d}m\right\}\ ,$$

where the infimum is taken over all sequences of Lipschitz functions (f_n) converging to f in $L^2(X,m)$ and where $|\nabla f_n|$ denotes the local Lipschitz constant. In particular, Lipschitz functions are dense in in the domain of Ch in $L^2(X,m)$ denoted by $D(\mathrm{Ch}) = W^{1,2}(X,d,m)$ in the following sense: For each $f\in D(\mathrm{Ch})$ there exist a sequence $(f_n)_{n\in\mathbb{N}}$ of Lipschitz functions such that $f_n\to f$ in L^2 and $|\nabla f_n|\to|\nabla f|_w$ in L^2 [6, Lem. 4.3(c)]. For a Lipschitz function f the slope, or local Lipschitz constant, is an upper gradient. Thus

$$|\nabla f|_w \ \le\ |\nabla f| \qquad \text{a.e.} \tag{3.16}$$

It turns out that Ch is a convex and lower semi-continuous functional on $L^2(X,m)$. It allows to define the Laplacian $-\Delta f\in L^2(X,m)$ of a function $f\in W^{1,2}(X,d,m)$ as the element of minimal L^2-norm in the subdifferential $\partial^-\mathrm{Ch}(f)$ provided the latter is non-empty. In this generality, Ch is not necessarily a quadratic form and consequently Δ need not be a linear operator.

The classical theory of gradient flows of convex functionals in Hilbert-spaces allows to study the gradient flow of Ch in $L^2(X,m)$: For any $f\in L^2(X,m)$ there exists a unique continuous curve $(f_t)_{t\in[0,\infty)}$ in $L^2(X,m)$, locally absolutely continuous in $(0,\infty)$ with $f_0=f$ such that $\frac{\mathrm{d}}{\mathrm{d}t}f_t\ \in\ \partial^-\mathrm{Ch}(f_t)$ for a.e. $t>0$. In fact, we have $f_t\in D(\Delta)$ and

$$\frac{\mathrm{d}^+}{\mathrm{d}t}f_t\ =\ \Delta f_t$$

for all $t>0$. This gives rise to a semigroup $(\mathrm{H}_t)_{t\ge0}$ on $L^2(X,m)$ defined by $\mathrm{H}_t f = f_t$, where f_t is the unique L^2-gradient flow of Ch.

On the other hand, one can study the metric gradient flow of the relative entropy Ent in $\mathscr{P}_2(X,d)$. Under the assumption that (X,d,m) satisfies $\mathrm{CD}(K,\infty)$ it has been proven in [20] and more generally in [6, Thm. 9.3(ii)] that for any $\mu\in D(\mathrm{Ent})$ there exist a unique gradient flow of Ent starting from μ in the sense of Definition 2.13. This gives rise to a semigroup $(\mathscr{H}_t)_{t\ge0}$ on $\mathscr{P}_2(X,d)$ defined by $\mathscr{H}_t\mu=\mu_t$ where μ_t is the unique gradient flow of Ent starting from μ.

One of the main result of [6] is the identification of the two gradient flows, which allows to consistently define the heat flow on $\mathrm{CD}(K,\infty)$ spaces.

Theorem 3.15 ([6, Thm. 9.3])**.** *Let (X,d,m) be a $\mathrm{CD}(K,\infty)$ space and let $f\in L^2(X,d,m)$ such that $\mu=fm\in\mathscr{P}_2(X,d)$. Then we have*

$$\mathscr{H}_t\mu = (\mathrm{H}_t f)m \quad \forall t\ge0\ .$$

A byproduct of this result is a representation of the slope of the entropy.

$$|\nabla^-\mathrm{Ent}|(\rho m)\ =\ 4\int|\nabla\sqrt{\rho}|_w^2\mathrm{d}m \tag{3.17}$$

for all probability densities ρ with $\sqrt{\rho}\in D(\mathrm{Ch})$. Note that the minimal weak upper gradient satisfies a chain rule, [6, Prop. 5.16]: for $\varphi: I\to\mathbb{R}$ non-decreasing and locally Lipschitz we have

$$|\nabla\varphi(f)|_w\ =\ \varphi'(f)|\nabla f|_w\ . \tag{3.18}$$

A basic property of the heat flow is the maximum principle, see [6, Thm. 4.16]: If $f\in L^2(X,m)$ satisfies $f\le C$ m-a.e. then also $\mathrm{H}_t f\le C$ m-a.e. for all $t\ge0$.

If Ch is assumed to be a quadratic form, and without any curvature assumption, the notion of weak upper gradient gives rise to a powerful calculus, in which not only the norm of the gradient, but also scalar products between gradients are defined. For details we refer to [4, Sec. 4.3] and [19, Sec. 4.3], where this calculus has been developed in larger generality. We note briefly that given $f,g\in D(\mathrm{Ch})$, the limit

$$\langle\nabla f,\nabla g\rangle\ :=\ \lim_{\varepsilon\searrow0}\frac{1}{2\varepsilon}\left(|\nabla(f+\varepsilon g)|_w^2-|\nabla f|_w^2\right) \tag{3.19}$$

can be shown to exists in $L^1(X,m)$. Moreover, the map $D(\mathrm{Ch})^2 \ni (f,g) \mapsto \langle \nabla f, \nabla g\rangle \in L^1(X,m)$ is bilinear, symmetric and satisfies

$$|\langle \nabla f, \nabla g\rangle| \;\le\; |\nabla f|_w |\nabla g|_w \;.$$

For all $f,g,h \in D(\mathrm{Ch}) \cap L^\infty(X,m)$ we have the Leibniz rule:

$$\int \langle \nabla f, \nabla (gh)\rangle \mathrm{d}m \;=\; \int h \langle \nabla f, \nabla g\rangle \mathrm{d}m + \int g\langle \nabla f, \nabla h\rangle \mathrm{d}m \;. \tag{3.20}$$

A quadratic Cheeger energy gives rise to a strongly local Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(X,m)$ by setting $\mathcal{E}(f,f) = \mathrm{Ch}(f)$ and $D(\mathcal{E}) = W^{1,2}(X,d,m)$. In particular, $W^{1,2}(X,d,m)$ is a Hilbert space and L^2-Lipschitz functions are dense in the usual sense [4, Prop. 4.10]. In this case H_t is a semigroup of self-adjoined linear operators on $L^2(X,m)$ with the Laplacian Δ as its generator. The previous result implies that for $f,g \in W^{1,2}(X,d,m)$

$$\mathcal{E}(f,g) \;=\; \int \langle \nabla f, \nabla g\rangle \mathrm{d}m \;,$$

i.e. the energy measure of $\mathcal{E}$ has a density given by (3.19). Moreover, for $f \in W^{1,2}$ and $g \in D(\Delta)$ we have the integration by parts formula

$$\int \langle \nabla f, \nabla g\rangle \mathrm{d}m \;=\; -\int f \Delta g \mathrm{d}m \;. \tag{3.21}$$

3.3. **The Riemannian curvature-dimension condition.** In this section we introduce the notion of Riemannian curvature-dimension bounds. This notion can be seen as a generalization of the Riemannian Ricci curvature bounds for metric measure spaces introduced in [4] for mms with finite reference measure and later generalized in [2] to σ-finite reference measures. We will rely on the powerful machinery of calculus on metric measure spaces already developed by Ambrosio, Gigli, Savaré and co-authors in a series of recent works. Following their nomenclature, we make the following

Definition 3.16. *We say that a metric measure space* (X,d,m) *is* infinitesimally Hilbertian *if the associated Cheeger energy is quadratic. Moreover, we say that it satisfies the* Riemannian curvature-dimension condition $\mathrm{RCD}^*(K,N)$ *if it satisfies any of the equivalent properties of Theorem 3.17 below.*

Theorem 3.17. *Let* (X,d,m) *be a metric measure space with* $\mathrm{supp}\, m = X$*. The following properties are equivalent:*

(i) (X,d,m) *is infinitesimally Hilbertian and satisfies the* $\mathrm{CD}^*(K,N)$ *condition.*
(ii) (X,d,m) *is infinitesimally Hilbertian and satisfies the* $\mathrm{CD}^e(K,N)$ *condition.*
(iii) (X,d,m) *is a length space satisfying the exponential integrability condition* (3.6) *and any* $\mu \in \mathscr{P}_2(X,d)$ *is the starting point of an* $\mathrm{EVI}_{K,N}$ *gradient flow of* Ent.

Remark 3.18. Note that according to Theorem 2.23, (iii) even implies that (X,d,m) is a strong $\mathrm{CD}^e(K,N)$ space and a geodesic space.

Remark 3.19. Since both $\mathrm{CD}^*(K,N)$ and $\mathrm{CD}^e(K,N)$ imply the $\mathrm{CD}(K,\infty)$ condition, [4, Thm. 5.1], resp. [2, Thm. 6.1] show that the requirement that the Cheeger energy Ch is quadratic can equivalently be replaced in (i) and (ii) by additivity of the semigroup $\mathscr{H}_t$, in the sense that $\mathscr{H}_t\big(\lambda\mu + (1-\lambda)\nu\big) = \lambda \mathscr{H}_t\mu + (1-\lambda)\mathscr{H}_t\nu$ for any $\mu,\nu \in \mathscr{P}_2(X,d)$ and $\lambda \in [0,1]$.

Proof. (i)⇔(ii): Both $\mathrm{CD}^*(K,N)$ and $\mathrm{CD}^e(K,N)$ imply the $\mathrm{CD}(K,\infty)$ condition. Thus [2, Thm. 6.1] yields that under either (i) or (ii) the EVI_K gradient flow of Ent exists for every starting point. This implies that (X,d,m) is a strong $\mathrm{CD}(K,\infty)$ space and hence essentially non-branching by [37, Thm. 1.1]. In this setting, Theorem 3.12 yields equivalence of $\mathrm{CD}^*(K,N)$ and $\mathrm{CD}^e(K,N)$.

(ii)⇒(iii): By Remark 3.8, (X,d) is a geodesic space and satisfies (3.6). Taking Theorem 2.19 into account it is sufficient to show that $\mathscr{H}_t(\mu)$ is an $\mathrm{EVI}_{K,N}$-gradient flow of Ent for every $\mu \in \mathscr{P}_2(X,d,m)$ of the form $\mu = fm$ with f bounded and $\mathrm{Ch}(\sqrt{f}) < \infty$. Set $\mu_t := \mathscr{H}_t(\mu) = f_t m$ and note that f_t is still bounded with $\mathrm{Ch}(\sqrt{f_t}) < \infty$ for all $t > 0$. By Proposition 2.18 it is sufficient to take reference measures in (2.18) of the form $\sigma = gm$ where g is bounded and has bounded support. Taking into account (2.20) we have to show that for a.e. $t > 0$:

$$\frac{U_N(\sigma)}{U_N(\mu_t)} \;\le\; \mathfrak{c}_{K/N}\big(W_2(\mu_t,\sigma)\big) - \frac{\mathfrak{s}_{K/N}\left(W_2(\mu_t,\sigma)\right)}{N\cdot W_2(\mu_t,\sigma)}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}W_2(\mu_t,\sigma)^2\;. \tag{3.22}$$

This will follow from essentially the same arguments as in the proof of [2, Thm. 6.1]. Let us briefly sketch these arguments, indicating the modifications that are necessary.

First, [2, Thm. 6.3] yields that for a.e. $t > 0$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}W_2(\mu_t,\sigma)^2 \;=\; -\mathcal{E}_{\mu_t}(\varphi_t,\log f_t)\;, \tag{3.23}$$

where φ_t is a suitable Kantorovich potential for the optimal transport from μ_t to σ and $\mathcal{E}_{\mu_t}(\cdot,\cdot)$ is the bilinear form associated to the weighted Cheeger energy $\mathrm{Ch}_{\mu_t}(f) = \frac{1}{2}\int |\nabla f|_{w,\mu_t}\mathrm{d}\mu_t$ (see [2, Sec. 3]). We claim that also

$$\mathcal{E}_{\mu_t}(\varphi_t,\log f_t) \;\ge\; \frac{N\cdot W_2(\mu_t,\sigma)}{\mathfrak{s}_{K/N}\left(W_2(\mu_t,\sigma)\right)}\Big[-\mathfrak{c}_{K/N}\big(W_2(\mu_t,\sigma)\big) + \frac{U_N(\sigma)}{U_N(\mu_t)}\Big]\;. \tag{3.24}$$

Combining then (3.23) and (3.24) yields the desired inequality (3.22).

To prove (3.24) one argues similar as in [2, Thm. 6.5]. First f_t is approximated by suitable truncated probability densities f_t^δ. Then, by successively minimizing the entropy of midpoints, a particularly nice geodesic $(\Gamma_s^{\delta,t})_{s\in[0,1]}$ connecting $\mu_t^\delta = f_t^\delta m$ to σ is constructed which satisfies the $\mathrm{CD}(K,\infty)$ condition and has density bounds. From the construction it is immediate that in our setting this geodesic also satisfies the $\mathrm{CD}^e(K,N)$ condition. Thus on one hand, we have by Lemma 3.20 below the inequality

$$\liminf_{s\searrow 0}\frac{U_N(\Gamma_s^{\delta,t})-U_N(\mu_t^\delta)}{s} \;\ge\; \frac{W_2(\mu_t^\delta,\sigma)}{\mathfrak{s}_{K/N}\left(W_2(\mu_t^\delta,\sigma)\right)}\Big[-U_N(\mu_t^\delta)\cdot\mathfrak{c}_{K/N}\big(W_2(\mu_t^\delta,\sigma)\big)+U_N(\sigma)\Big]\;. \tag{3.25}$$

On the other hand, [2, Prop. 6.6] yields that

$$-\mathcal{E}_{\mu_t^\delta}(\varphi_t^\delta,\log f_t^\delta) \;\le\; \liminf_{s\searrow 0}\frac{\mathrm{Ent}(\Gamma_s^{\delta,t})-\mathrm{Ent}(\mu_t^\delta)}{s}\;, \tag{3.26}$$

where φ_t^δ is a Kantorovich potential relative to μ_t^δ and σ. By K-convexity of Ent along the geodesic $\Gamma^{\delta,t}$ we have

$$\limsup_{s\searrow 0}\frac{\mathrm{Ent}(\Gamma_s^{\delta,t})-\mathrm{Ent}(\mu_t^\delta)}{s} \;\le\; \mathrm{Ent}(\sigma)-\mathrm{Ent}(\mu_t^\delta)-\frac{K}{2}W_2(\mu_t^\delta,\sigma)^2$$

and thus $\big(\mathrm{Ent}(\Gamma_s^{\delta,t})-\mathrm{Ent}(\mu_t^\delta)\big)^2 = o(s)$ as $s\to 0$. Now (3.25) and (3.26) together with a Taylor expansion of $x\mapsto e^{-x/N}$ yield

$$\mathcal{E}_{\mu_t^\delta}(\varphi_t^\delta,\log f_t^\delta) \;\ge\; \frac{N\cdot W_2(\mu_t^\delta,\sigma)}{\mathfrak{s}_{K/N}\left(W_2(\mu_t^\delta,\sigma)\right)}\Big[-\mathfrak{c}_{K/N}\big(W_2(\mu_t^\delta,\sigma)\big)+\frac{U_N(\sigma)}{U_N(\mu_t^\delta)}\Big]\;. \tag{3.27}$$

Finally (3.24) is obtained by lifting the truncation and passing to the limit $\delta\to 0$ in (3.27). Passage to the limit in the RHS is obvious, for the LHS a delicate argument is needed which is given in the proof of [2, Thm. 6.5].

(iii)⇒(ii). Since by Lemma 2.15 an $\mathrm{EVI}_{K,N}$ flow is in particular an EVI_K flow, [4, Thm. 5.1] or [2, Thm. 6.1] already gives that (X,d,m) is infinitesimally Hilbertian. Let us now show that (X,d,m) is a strong $\mathrm{CD}^e(K,N)$ space. The same argument as in the proof of [4, Lem. 5.2] yields for any pair $\mu_0,\mu_1\in D(\mathrm{Ent})\subset\mathscr{P}_2(X,d,m)$ the existence of a geodesic $\Gamma:[0,1]\to D(\mathrm{Ent})$ connecting μ_0 to μ_1. Hence $D(\mathrm{Ent})$ is a geodesic space and Theorem 2.23 shows that (3.1) holds along any geodesic in $D(\mathrm{Ent})$. □

Lemma 3.20. *Let (X, d, m) satisfy the $\mathrm{CD}^e(K, N)$ condition and let $\mu_0, \mu_1 \in \mathscr{P}_2(X, d, m)$. Then there exists a geodesic $(\mu_t)_{t\in[0,1]}$ in $\mathscr{P}_2(X, d, m)$ connecting μ_0 and μ_1 such that, with $\theta = W_2(\mu_0, \mu_1)$,*

$$U_N(\mu_1) \ \le\ \mathfrak{c}_{K/N}(\theta) \cdot U_N(\mu_0) + \frac{\mathfrak{s}_{K/N}(\theta)}{\theta} \cdot \liminf_{t\searrow 0} \frac{U_N(\mu_t) - U_N(\mu_0)}{t}\ . \tag{3.28}$$

Proof. Let $(\mu_t)_{t\in[0,1]}$ be the geodesic connecting μ_0 and μ_1 given by the $\mathrm{CD}^e(K, N)$ condition. We immediately obtain that for every $t \in [0, 1]$:

$$U_N(\mu_t) - U_N(\mu_0) \ \ge\ \left[\sigma^{(1-t)}_{K/N}(\theta) - 1\right] \cdot U_N(\mu_0) + \sigma^{(t)}_{K/N}(\theta) \cdot U_N(\mu_1)\ .$$

Dividing by t on both sides and passing to the limit $t \searrow 0$ the assertion follows from the fact that

$$\frac{\mathrm{d}}{\mathrm{d}t}\sigma^{(t)}_{K/N}(\theta) = +\frac{\theta \cdot \mathfrak{c}_{K/N}(t\theta)}{\mathfrak{s}_{K/N}(\theta)}\ , \quad \sigma^{(0)}_{K/N}(\theta) = 0\ , \quad \sigma^{(1)}_{K/N}(\theta) = 1\ .$$

□

Proposition 3.21 (Weighted spaces)**.** *Let (X, d, m) be a $\mathrm{RCD}^*(K, N)$ space and let $V : X \to \mathbb{R}$ be continuous, bounded below and strongly (K', N')-convex function in the sense of Definition 2.7 with $\int \exp(-V)\mathrm{d}m < \infty$. Then $(X, d, \mathrm{e}^{-V}m)$ is a $\mathrm{RCD}^*(K + K', N + N')$ space.*

Proof. By Proposition 3.3, $(X, d, \mathrm{e}^{-V}m)$ is a $\mathrm{CD}^e(K+K', N+N')$ space. Invariance of the weak upper gradient under multiplicative changes of the reference measure by [6, Lem. 4.11] together with the Leibniz rule (3.20) give that the Cheeger energy associated to $\mathrm{e}^{-V}m$ is again quadratic. See also [4, Prop. 6.19]. Thus the assertion follows from Theorem 3.17 (ii). □

The Riemannian curvature-dimension condition has a number of natural properties that we collect here. The first one is the stability under convergence of metric measure spaces in the transportation distance $\mathbb{D}$. We refer to [39, Sec. I.3] for the definition and properties of the transportation distance.

Theorem 3.22 (Stability)**.** *Let $((X_n, d_n, m_n))_{n\in\mathbb{N}}$ be a sequence of $\mathrm{RCD}^*(K, N)$ spaces with $m_n \in \mathscr{P}_2(X_n, d_n)$. If $\mathbb{D}\big((X_n, d_n, m_n), (X, d, m)\big) \to 0$ for some metric measure space (X, d, m) then (X, d, m) is also a $\mathrm{RCD}^*(K, N)$ space.*

Note that this in particular implies stability of the $\mathrm{RCD}^*(K, N)$-condition under *measured Gromov-Hausdorff convergence* (mGH-convergence for short). Indeed, for compact mms – and only for such spaces the concept of mGH-convergence is well-established – mGH-convergence implies $\mathbb{D}$-convergence [39, Lemma 3.18].

Proof. We follow essentially the arguments of Ambrosio, Gigli and Savaré in [4, Thm. 6.10] where stability of the $\mathrm{RCD}(K, \infty)$ condition has been established.

We show stability of characterization (iii) in Theorem 3.17. By Proposition 2.18 and Corollary 2.21 it is sufficient to show that for any $\mu = fm \in \mathscr{P}_2(X, d, m)$ with $f \in L^\infty(X, m)$ there exists a continuous curve $(\mu_t)_{t\in[0,\infty)}$ in $\mathscr{P}_2(X, d)$, locally absolutely continuous in $(0, \infty)$ and starting in μ such that for any $\nu = \sigma m \in \mathscr{P}_2(X, d)$ with $\sigma \in L^\infty(X, d, m)$ and any $s \le t$:

$$e_K(t-s)\frac{N}{2}\left(1 - \frac{U_N(\nu)}{U_N(\mu_t)}\right) \ \ge\ \mathrm{e}^{K(t-s)}\mathfrak{s}_{K/N}\left(\frac{1}{2}W_2(\mu_t, \nu)\right)^2 - \ \mathfrak{s}_{K/N}\left(\frac{1}{2}W_2(\mu_s, \nu)\right)^2\ . \tag{3.29}$$

Choose optimal couplings $(\hat{d}_n, q_n)$ of (X_n, d_n, m_n) and (X, d, m). Given $\mu = fm \in \mathscr{P}_2(X, d, m)$ we set

$$Q_n\mu(\mathrm{d}x) \ =\ \int f(y)q_n(\mathrm{d}x, \mathrm{d}y) \ \in\ \mathscr{P}_2(X_n, d_n, m_n)\ .$$

Similarly we obtain an operator $Q'_n : \mathscr{P}_2(X_n, d_n, m_n) \to \mathscr{P}_2(X, d, m)$, see [39, Lem. I.4.19] and also [4, Prop. 2.2,2.3].

Now set $\mu^n = Q_n\mu$. By assumption there exists a curve $(\mu^n_t)_{t\in[0,\infty)}$ in $\mathscr{P}_2(X_n,d_n)$ starting from μ^n such that for all $s\le t$:

$$e_K(t-s)\frac{N}{2}\left(1-\frac{U^n_N(\nu^n)}{U^n_N(\mu^n_t)}\right) \;\ge\; \mathrm{e}^{K(t-s)}\mathfrak{s}_{K/N}\left(\frac{1}{2}W_2(\mu^n_t,\nu^n)\right)^2 - \quad \mathfrak{s}_{K/N}\left(\frac{1}{2}W_2(\mu^n_s,\nu^n)\right)^2 , \tag{3.30}$$

where $\nu^n = Q_n\nu$ and U^n_N corresponds to the relative entropy functional in (X_n,d_n,m_n). By the maximum principle we have $\mu^n_t \le Cm_n$ with $C=\|\rho\|_{L^\infty(X,m)}$. For each $t\ge 0$ set $\widetilde{\mu}^n_t := Q'_n\mu^n_t \in \mathscr{P}_2(X,d)$. We claim that, after extraction of a subsequence, we have that $\widetilde{\mu}^n_t\to\mu_t$ in $\mathscr{P}_2(X,d)$ as $n\to\infty$ for a curve (μ_t) in $\mathscr{P}_2(X,d)$.

Indeed, note that $\widetilde{\mu}^n_t \le Cm$ for all n and t. From the Energy Dissipation Equality (2.17) we conclude that

$$\int_s^t |\dot{\mu}^n_r|^2 \mathrm{d}r \;\le\; \mathrm{Ent}(\mu^n|m^n) \;\le\; C\log C$$

and hence the curves (μ^n_t) are equi-absolutely continuous. Since $m\in\mathscr{P}_2(X,d)$, the set of measures $\{\mu\in\mathscr{P}_2(X,d,m)) : \mu\le Cm\}$ is relatively compact w.r.t W_2-convergence. Hence, by a diagonal argument, we conclude that up to extraction of a subsequence $\widetilde{\mu}^n_t\to\mu_t$ for all $t\in\mathbb{Q}_+$ and some $\mu_t\in\mathscr{P}_2(X,d)$. Using the equi-absolute continuity of the curves (μ^n_t) and the equi-continuity of the map Q'_n we obtain convergence for all times $t\in[0,\infty)$ for the same subsequence and a curve (μ_t) in $\mathscr{P}_2(X,d)$ which is again absolutely continuous.

Finally, we observe that since the operators Q_n, Q'_n do not increase the entropy we have $U^n_N(\nu^n)\ge U_N(\nu)$ and by lower semi-continuity of the entropy also $\mathrm{Ent}(\mu_t)\le\liminf_n \mathrm{Ent}(\widetilde{\mu}^n_t)\le \liminf_n \mathrm{Ent}(\mu^n_t|m^n)$. Moreover, we have $W_2(\mu^n_t,\nu^n)\to W_2(\mu_t,\nu)$. This allows to pass to the limit in (3.30) to obtain (3.29). □

Theorem 3.23 (Tensorization)**.** *For $i=1,2$ let (X_i,d_i,m_i) be* $\mathrm{RCD}^*(K,N_i)$ *spaces. Then the product space $(X_1\times X_2, d, m_1\otimes m_2)$, defined by*

$$d\big((x,y),(x',y')\big)^2 = d_1(x,x')^2 + d_2(y,y')^2 ,$$

also satisfies $\mathrm{RCD}^*(K,N_1+N_2)$.

Proof. The result will follow indirectly: According to Theorem 4.3 below, the $\mathrm{RCD}^*(K,N_i)$-conditions will imply the Bakry–Ledoux conditions $\mathrm{BL}(K,N_i)$ on the first and second factor. According to [5, Thm. 5.2], this implies that the product space satisfies $\mathrm{BL}(K,N_1+N_2)$. Now Theorems 4.19 and 3.17 imply that the $\mathrm{RCD}^*(K,N_1+N_2)$ condition holds on the product space. □

Remark 3.24. Let us also briefly sketch an alternative more direct argument using characterization (i) of Theorem 3.17: First, [4, Thm. 6.17] yields that the Cheeger energy on the product space is again quadratic. Since (X_i,d_i,m_i) are in particular strong $\mathrm{CD}(K,\infty)$ spaces, they are essentially non-branching according to Definition 3.10 by [37, Thm. 1.1]. This implies that also the product space is essentially non-branching. The latter can be seen using the fact that if $\gamma=(\gamma_1,\gamma_2)$ is a geodesic in $X_1\times X_2$, then γ_i are geodesics in X_i. Finally, the reduced curvature-dimension condition tensorizes under the essentially non-branching assumption. This follows from the same arguments as in [9, Thm. 4.1], where tensorization has been proven under the slightly stronger assumption that the full space is non-branching.

We conclude with a globalization property of the $\mathrm{RCD}^*(K,N)$ condition.

Theorem 3.25 (Local-to-global)**.** *Let (X,d,m) be a strong* $\mathrm{CD}^e_{loc}(K,N)$ *space with $m\in\mathscr{P}_2(X,d)$ and assume that it is locally infinitesimally Hilbertian in the following sense: there exists a countable covering $\{Y_i\}_{i\in I}$ by closed sets with $m(Y_i)>0$ such that the spaces (Y_i,d,m_i) are infinitesimally Hilbertian, where $m_i = m(Y_i)^{-1}m|_{Y_i}$. Then (X,d,m) satisfies the* $\mathrm{RCD}^*(K,N)$ *condition.*

Proof. Using characterization (ii) in Theorem 3.17, the assertion is a direct consequence of the fact that both infinitesimal Hilbertianity and the *strong* $\mathrm{CD}^e(K,N)$ condition by themselves have the local-to-global property. Indeed, by [4, Thm. 6.20] the mms (X, d, m) is again infinitesimally Hilbertian, i.e. the associated Cheeger energy is quadratic. By Theorem 3.14 it also satisfies the strong $\mathrm{CD}^e(K, N)$ condition. □

Remark 3.26. It is also possible to establish local–to–global property by passing through the corresponding result for $\mathrm{CD}^*(K, N)$ with the aid of Theorem 3.17. This requires to check that the (quite complicated) proof of globalization for $\mathrm{CD}^*(K, N)$ in [9, Thm. 5.1] also works under the slightly weaker ess. non-branching assumption. Thus, we prefer to give an independent and, to our knowledge, novel argument in the preceding proof.

3.4. **Dimension dependent functional inequalities.** Here we present dimensional versions of classical transport inequalities. Namely, we show that the new entropic curvature-dimension condition entails improvements of the HWI inequality, the logarithmic Sobolev inequality and the Talagrand inequality taking into account the dimension bound. These results can be seen as finite dimensional analogues of the famous results by Bakry–Émery [10] and Otto–Villani [33].

Given a probability measure $\mu \in \mathscr{P}_2(X, d)$ we define the *Fisher information* by

$$I(\mu) \;=\; 4\int |\nabla\sqrt{f}|_w^2 \mathrm{d}m\;,$$

provided that $\mu = fm$ is absolutely continuous with a density f such that $\sqrt{f} \in D(\mathrm{Ch})$. Otherwise we set $I(\mu) = +\infty$. With this notation, the equality (3.17), which is valid on $\mathrm{RCD}(K, \infty)$ spaces, means $|\nabla^- \mathrm{Ent}|(fm) \;=\; I(fm)$.

Theorem 3.27 (N-HWI inequality)**.** *Assume that the mms (X, d, m) satisfies the $\mathrm{CD}^e(K, N)$ condition. Then for all $\mu_0, \mu_1 \in \mathscr{P}_2(X, d, m)$,*

$$\frac{U_N(\mu_1)}{U_N(\mu_0)} \;\le\; \mathfrak{c}_{K/N}\big(W_2(\mu_0, \mu_1)\big) + \frac{1}{N}\mathfrak{s}_{K/N}\left(W_2(\mu_0, \mu_1)\right)\sqrt{I(\mu_0)}\;. \tag{3.31}$$

Proof. We can assume that $I(\mu_0) \;=\; |\nabla^- \mathrm{Ent}|(\mu_0)$ is finite, as otherwise there is nothing to prove. Let $(\mu_t)_{t\in[0,1]}$ be the constant speed geodesic connecting μ_0 to μ_1 given by the $\mathrm{CD}^e(K, N)$ condition. Since (K, N)-convexity of Ent along the geodesic (μ_t) implies usual K-convexity along the same geodesic we have

$$\limsup_{t\searrow 0}\frac{\mathrm{Ent}(\mu_t) - \mathrm{Ent}(\mu_0)}{t} \;\le\; \mathrm{Ent}(\mu_1) - \mathrm{Ent}(\mu_0) - \frac{K}{2}W_2(\mu_0, \mu_1)^2\;.$$

On the other hand, we have

$$\begin{aligned}\liminf_{t\searrow 0}\frac{\mathrm{Ent}(\mu_t) - \mathrm{Ent}(\mu_0)}{t} \;&\ge\; -\limsup_{t\searrow 0}\frac{\max\{\mathrm{Ent}(\mu_0) - \mathrm{Ent}(\mu_t), 0\}}{t}\\ &\ge\; -|\nabla^- \mathrm{Ent}|(\mu_0)\cdot W_2(\mu_0, \mu_1)\;.\end{aligned} \tag{3.32}$$

Thus $\big(\mathrm{Ent}(\mu_t) - \mathrm{Ent}(\mu_0)\big)^2 \;=\; o(t)$ as $t \to 0$. By Lemma 3.20 and a Taylor expansion of $x \mapsto e^{-x/N}$ we obtain

$$\begin{aligned}\frac{U_N(\mu_1)}{U_N(\mu_0)} \;&\le\; \mathfrak{c}_{K/N}(\theta) + \frac{\mathfrak{s}_{K/N}(\theta)}{\theta\cdot U_N(\mu_0)}\cdot\liminf_{t\searrow 0}\frac{U_N(\mu_t) - U_N(\mu_0)}{t}\\ &=\; \mathfrak{c}_{K/N}(\theta) - \frac{\mathfrak{s}_{K/N}(\theta)}{\theta\cdot N}\cdot\limsup_{t\searrow 0}\frac{\mathrm{Ent}(\mu_t) - \mathrm{Ent}(\mu_0)}{t}\;,\end{aligned}$$

where we set $\theta = W_2(\mu_0, \mu_1)$. Applying the estimate (3.32) again yields the claim. □

Corollary 3.28 (N-LogSobolev inequality)**.** *Assume that (X, d, m) is a $\mathrm{CD}^e(K, N)$ space with $K > 0$ and that $m \in \mathscr{P}_2(X, d)$. Then for all $\mu \in \mathscr{P}_2(X, d, m)$,*

$$KN\left[\exp\left(\frac{2}{N}\mathrm{Ent}(\mu)\right) - 1\right] \;\le\; I(\mu)\;. \tag{3.33}$$

The LHS obviously is bounded from below by $2K\cdot\mathrm{Ent}(\mu)$.

Proof. We apply the N-HWI inequality from Theorem 3.27 to the measures $\mu_0 = \mu$ and $\mu_1 = m$. Noting that $U_N(m) = 1$ and setting $\theta = W_2(\mu, m)$ we obtain

$$\exp\left(\frac{1}{N}\operatorname{Ent}(\mu)\right) \ \le\ \mathfrak{c}_{K/N}(\theta) + \frac{1}{N}\mathfrak{s}_{K/N}(\theta)\sqrt{I(\mu)}\ .$$

Taking the square and using Young's inequality $2ab \le Ka^2 + K^{-1}b^2$ we obtain

$$\begin{aligned}\exp\left(\frac{2}{N}\operatorname{Ent}(\mu)\right) \ &\le\ \mathfrak{c}_{K/N}(\theta)^2 + \frac{2}{N}\mathfrak{s}_{K/N}(\theta)\,\mathfrak{c}_{K/N}(\theta)\sqrt{I(\mu)} + \frac{1}{N^2}\mathfrak{s}_{K/N}(\theta)^2\, I(\mu)\\ &\le\ \left(\mathfrak{c}_{K/N}(\theta)^2 + \frac{K}{N}\mathfrak{s}_{K/N}(\theta)^2\right)\left[1 + \frac{1}{KN}I(\mu)\right]\ .\end{aligned}$$

Since $\mathfrak{c}_{K/N}(\,\cdot\,)^2 + \frac{K}{N}\mathfrak{s}_{K/N}(\cdot)^2 = 1$, this yields the claim. □

Corollary 3.29 (N-Talagrand inequality)**.** *Assume that (X, d, m) is a $\mathrm{CD}^e(K, N)$ space with $K > 0$ and that $m \in \mathscr{P}_2(X, d)$. Then $W_2(\mu, m) \le \sqrt{\frac{N}{K}}\frac{\pi}{2}$ for any $\mu \in \mathscr{P}_2(X, d, m)$ and*

$$\operatorname{Ent}(\mu) \ \ge\ -N\log\cos\left(\sqrt{\frac{K}{N}}W_2(\mu, m)\right)\ . \tag{3.34}$$

Note that under the given upper bound on $W_2(\mu, m)$, the RHS in the above estimate is bounded from below by $\frac{K}{2}\, W_2(\mu, m)^2$.

Proof. The claims follow immediately by applying the N-HWI inequality (3.31) from Theorem 3.27 to the measures $\mu_0 = m$ and $\mu_1 = \mu$ and noting that $U_N(m) = 1$ as well as $I(m) = 0$. □

It is interesting to note that in the spirit of Otto–Villani a slightly weaker Talagrand-like inequality can also be derived from the N-LogSobolev inequality.

Proposition 3.30. *Let (X, d, m) be a $\mathrm{CD}(K', \infty)$ space for some $K' \in \mathbb{R}$ such that $m \in \mathscr{P}_2(X, d)$. Assume that the N-LogSobolev inequality (3.33) holds for some $K > 0$. Then for any $\mu \in \mathscr{P}_2(X, d, m)$,*

$$W_2(\mu, m) \ \le\ \sqrt{\frac{N}{K}\left[\exp\left(\frac{2}{N}\operatorname{Ent}(\mu)\right) - 1\right]}\ . \tag{3.35}$$

Proof. We fix $\mu \in \mathscr{P}_2(X, d, m)$ and introduce the function $A : [0, \infty) \to \mathbb{R}_+$ defined by

$$A(t) \ =\ W_2(\mathscr{H}_t\mu, \mu) + \sqrt{\frac{N}{K}\left[\exp\left(\frac{2}{N}\operatorname{Ent}(\mathscr{H}_t\mu)\right) - 1\right]}\ .$$

Obviously, $A(0)$ equals the right hand side of (3.35), while $A(t) \to W_2(\mu, m)$ as $t \to \infty$. Thus it is sufficient to prove that A is non-increasing. First note that under the $\mathrm{CD}(K', \infty)$ condition we have the estimate

$$\frac{\mathrm{d}^+}{\mathrm{d}t}W_2(\mathscr{H}_t\mu, \mu) \ \le\ \sqrt{I(\mathscr{H}_t\mu)}\ . \tag{3.36}$$

Indeed, using triangle inequality we find

$$\limsup_{h\searrow 0}\frac{1}{h}\Big(W_2(\mathscr{H}_{t+h}\mu, \mu) - W_2(\mathscr{H}_t\mu, \mu)\Big) \ \le\ \limsup_{h\searrow 0}\frac{1}{h}W_2(\mathscr{H}_{t+h}\mu, \mathscr{H}_t\mu) \ =\ |(\dot{\mathscr{H}_t\mu})|\ .$$

Now (3.36) follows from the fact that $\mathscr{H}_t\mu$ is a metric gradient flow of Ent by virtue of the Energy Dissipation Equality (2.17) and (3.17). Moreover, we calculate

$$\begin{aligned}\frac{\mathrm{d}^+}{\mathrm{d}t}\sqrt{\frac{N}{K}\left[\exp\left(\frac{2}{N}\operatorname{Ent}(\mathscr{H}_t\mu)\right) - 1\right]} \ &=\ \left(NK\left[\exp\left(\frac{2}{N}\operatorname{Ent}(\mathscr{H}_t\mu)\right) - 1\right]\right)^{-\frac{1}{2}}\frac{\mathrm{d}^+}{\mathrm{d}t}\operatorname{Ent}(\mathscr{H}_t\mu)\\ &=\ -\left(NK\left[\exp\left(\frac{2}{N}\operatorname{Ent}(\mathscr{H}_t\mu)\right) - 1\right]\right)^{-\frac{1}{2}} I(\mathscr{H}_t\mu)\\ &\le\ -\sqrt{I(\mathscr{H}_t\mu)}\ ,\end{aligned}$$

where we have used (3.33) in the last step. Thus we have shown that $\frac{\mathrm{d}^+}{\mathrm{d}t}A(t) \leq 0$ which yields the claim. □

Remark 3.31. Note that the arguments in the proofs above are of a purely metric nature. The preceding results can be formulated and proven verbatim in the setting of Section 2.3 by replacing Ent with a (K,N)-convex function S on a metric space, the Fisher information I with the slope $|\nabla^- S|$ and $\mathscr{H}_t\mu$ with the gradient flow of S. However, for concreteness we choose to work in the Wasserstein framework.

4. Equivalence of $\mathrm{CD}^e(K,N)$ and the Bochner Inequality $\mathrm{BE}(K,N)$

In this section we will study properties of the gradient flow $H_t f$ of the (quadratic) Cheeger energy Ch in $L^2(X,m)$. We refer to Section 3.2 and references therein for notations and basic properties of them.

4.1. From $\mathrm{CD}^e(K,N)$ to $\mathrm{BL}(K,N)$ and $\mathrm{BE}(K,N)$.

In this section we study the analytic consequences of the Riemannian curvature-dimension condition. In particular, we show that it implies a pointwise gradient estimate in the spirit of Bakry–Ledoux. This in turn allows us to establish the full Bochner inequality.

As an immediate consequence of Definition 3.16 and Theorem 2.19 we obtain the following Wasserstein expansion bound. Recall from Proposition 2.22 that this bound in turn implies a slightly weaker and simpler bound not involving the function $\mathfrak{s}_{K/N}(\cdot)$.

Theorem 4.1 (W_2-expansion bound)**.** *Let (X,d,m) be a $\mathrm{RCD}^*(K,N)$ space. For any $\mu,\nu \in \mathscr{P}_2(X,d)$ and $0 < s,t$ we have*

$$\mathfrak{s}_{K/N}\left(\frac{1}{2}W_2(\mathscr{H}_t\mu,\mathscr{H}_s\nu)\right)^2 \leq e^{-K(s+t)}\mathfrak{s}_{K/N}\left(\frac{1}{2}W_2(\mu,\nu)\right)^2 + \frac{N}{K}\Big(1-e^{-K(s+t)}\Big)\frac{\big(\sqrt{t}-\sqrt{s}\big)^2}{2(s+t)}\,. \tag{4.1}$$

In particular, in the limit $s \to t$ and $\nu \to \mu$ we have

$$W_2(\mathscr{H}_t\mu,\mathscr{H}_s\nu)^2 \leq e^{-2Kt}W_2(\mu,\nu)^2 + \frac{N}{K}\frac{1-e^{-2Kt}}{4t^2}\cdot|s-t|^2 + o\big(W_2(\mu,\nu)^2 + |t-s|^2\big)\,. \tag{4.2}$$

Next we will show that (4.1) implies Bakry–Ledoux's gradient estimate. To do it with minimal a priori regularity assumptions, we will introduce another condition, which is satisfied for each $\mathrm{RCD}(K',\infty)$ space (see Remark 4.5 below).

Assumption 4.2. *(X,d,m) is a length metric measure space satisfying* $\operatorname{supp} m = X$ *and* (3.6)*. In addition, every $f \in D(\mathrm{Ch})$ with $|\nabla f|_w \leq 1$ has a 1-Lipschitz representative.*

Theorem 4.3 (Bakry–Ledoux gradient estimate)**.** *Let (X,d,m) be an infinitesimally Hilbertian metric measure space satisfying Assumption 4.2. Assume that* (4.1) *with $K \in \mathbb{R}$, $N \in (0,\infty)$ holds for the measures $(\mathrm{H}_t\,\eta)m$ and $(\mathrm{H}_s\,\sigma)m$ instead of $\mathscr{H}_t\mu$ and $\mathscr{H}_s\nu$ for each $\mu = \eta m$ and $\nu = \sigma m$ in $\mathscr{P}_2(X,d,m)$ and $t,s \geq 0$. Then*

$$|\nabla \mathrm{H}_t f|_w^2 + \frac{4Kt^2}{N\big(e^{2Kt}-1\big)}|\Delta \mathrm{H}_t f|^2 \leq e^{-2Kt}\,\mathrm{H}_t\big(|\nabla f|_w^2\big)\,. \tag{4.3}$$

m-a.e. in X for any $f \in D(\mathrm{Ch})$ and $t > 0$.

Before giving the proof we note the following result, which gives a stronger version of the gradient estimate involving the Lipschitz constant under more restrictions on f.

Proposition 4.4. *Let (X,d,m) be an infinitesimally Hilbertian metric measure space satisfying Assumption 4.2. If* (4.3) *holds and $|\nabla f|_w \in L^\infty(X,m)$ then $\mathrm{H}_t f$, $\mathrm{H}_t(|\nabla f|_w^2)$ and $\Delta \mathrm{H}_t f$ have continuous representatives satisfying everywhere in X:*

$$|\nabla \mathrm{H}_t f|^2 + \frac{4Kt^2}{N\big(e^{2Kt}-1\big)}|\Delta \mathrm{H}_t f|^2 \ \le\ e^{-2Kt}\,\mathrm{H}_t\left(|\nabla f|_w^2\right) . \tag{4.4}$$

Remark 4.5. Under $\mathrm{RCD}(K',\infty)$, Assumption 4.2 is always satisfied (see [2, 4, 5]). Moreover, with the aid of Theorem 3.15, the other assumption in Theorem 4.3 easily yields (4.1) in this case. Conversely, the assumptions in Theorem 4.3 implies $\mathrm{RCD}(K,\infty)$. Indeed, by Proposition 2.22, (4.1) yields the W_2-contraction estimate, which corresponds to (2.31). Under Assumption 4.2, such an estimate yields Bakry–Émery's L^2-gradient estimate (see [5, Cor. 3.18], [27, Thm. 2.2]). Then $\mathrm{RCD}(K,\infty)$ follows from [5, Thm 4.18] under Assumption 4.2 again.

Note that $\mathrm{RCD}(K',\infty)$ ensures some regularization property of H_t. For instance, $\mathrm{H}_t f(x) = \int_X f\,\mathrm{d}\mathscr{H}_t\delta_x$ holds m-a.e. for every $f \in L^2(X,m)$. Moreover, this representative of $\mathrm{H}_t f$ satisfies the strong Feller property, that is, $x \mapsto \int_X f\,\mathrm{d}\mathscr{H}_t\delta_x$ is bounded and continuous for any bounded measurable f (see [4, Thm. 6.1], [2, Thm. 7.1]).

Proof of Theorem 4.3. For simplicity of presentation, we give a proof when (X,d) is a geodesic space. One can easily extend the argument to the length space case. We first consider the case that f is bounded and Lipschitz with bounded support. Let us denote $\widetilde{\mathrm{H}}_t f(x) := \int_X f\,\mathrm{d}\mathscr{H}_t\delta_x$, which is a representative of $\mathrm{H}_t f$, see Remark 4.5. For $x,y\in X$, $x\ne y$ and $t,s\ge 0$ and any coupling $\pi_{s,t}$ of $\mathscr{H}_s(\delta_x)$ and $\mathscr{H}_t(\delta_y)$, we have

$$\widetilde{\mathrm{H}}_s f(x) - \widetilde{\mathrm{H}}_t f(y) \le \int_{X\times X} |f(z)-f(w)|\pi_{s,t}(\mathrm{d}z\mathrm{d}w) . \tag{4.5}$$

Since $|f(z)-f(w)| \le \mathrm{Lip}(f)d(x,y)$, (4.5) and (4.1) yield

$$\begin{aligned}
\mathfrak{s}_{K/N}\left(\frac{1}{2\,\mathrm{Lip}(f)}(\widetilde{\mathrm{H}}_s f(x) - \widetilde{\mathrm{H}}_t f(y))\right)^2 &\le \mathfrak{s}_{K/N}\left(\frac{1}{2}W_1(\mathscr{H}_s(\delta_x),\mathscr{H}_t(\delta_y))\right)^2\\
&\le \mathfrak{s}_{K/N}\left(\frac{1}{2}W_2(\mathscr{H}_s(\delta_x),\mathscr{H}_t(\delta_y))\right)^2\\
\le \mathrm{e}^{-K(s+t)}\mathfrak{s}_{K/N}\left(\frac{1}{2}d(x,y)\right)^2 &+ \frac{N(1-\mathrm{e}^{-K(s+t)})}{2K(s+t)}(\sqrt{t}-\sqrt{s})^2 .
\end{aligned}$$

It implies that the map $(u,z)\mapsto \widetilde{\mathrm{H}}_u f(z)$ is locally Lipschitz on $(0,1)\times X$ and hence $u\mapsto \widetilde{\mathrm{H}}_u f(z)$ is differentiable $\mathcal{L}^1$-a.e. for each fixed $z\in X$, where $\mathcal{L}^1$ is the one-dimensional Lebesgue measure.

The first step is to show the following inequality:

$$|\nabla \widetilde{\mathrm{H}}_t f|(x)^2 + \frac{4Kt^2}{N\big(e^{2Kt}-1\big)}\left(\frac{\partial}{\partial t}\widetilde{\mathrm{H}}_t f(x)\right)^2 \le \mathrm{e}^{-2Kt}\widetilde{\mathrm{H}}_t(|\nabla f|^2)(x) \tag{4.6}$$

for each $x\in X$ and $t>0$ such that $u\mapsto \mathrm{H}_u f(x)$ is differentiable at t. Let $y\in X$ and $s\ge 0$. let us define $r=r(x,y;s,t)>0$ and $G_r f : X\to\mathbb{R}$ by

$$\begin{aligned}
r &:= \begin{cases} W_2(\mathscr{H}_s(\delta_x),\mathscr{H}_t(\delta_y))^{1/2} & \text{if } W_2(\mathscr{H}_s(\delta_x),\mathscr{H}_t(\delta_y))>0,\\ d(x,y) & \text{otherwise.}\end{cases}\\
G_r f(z) &:= \sup_{z';\, d(z,z')\in(0,r)} \frac{|f(z)-f(z')|}{d(z,z')} .
\end{aligned}$$

Then by taking a coupling $\pi_{s,t}$ as a minimizer of $W_2(\mathscr{H}_s(\delta_x), \mathscr{H}_t(\delta_y))$ in (4.5),

$$\begin{aligned}
\int_{X\times X} & |f(z) - f(w)| \pi_{s,t}(\mathrm{d}z\mathrm{d}w) \\
&= \int_{X\times X} |f(z) - f(w)| 1_{\{d(z,w)\le r\}} \pi_{s,t}(\mathrm{d}z\mathrm{d}w) \\
&\quad + \int_{X\times X} |f(z) - f(w)| 1_{\{d(z,w)> r\}} \pi_{s,t}(\mathrm{d}z\mathrm{d}w) \\
&\le \int_{X\times X} G_r f(z) d(z,w) \pi_{s,t}(\mathrm{d}z\mathrm{d}w) + 2\|f\|_\infty \pi_{s,t}(d > r) \\
&\le \left(\int_X (G_r f)^2 d\mathscr{H}_s(\delta_x) \right)^{1/2} W_2(\mathscr{H}_s(\delta_x), \mathscr{H}_t(\delta_y)) \\
&\qquad + \frac{2\|f\|_\infty}{r^2} W_2(\mathscr{H}_s(\delta_x), \mathscr{H}_t(\delta_y))^2 \, . \qquad (4.7)
\end{aligned}$$

After substituting (4.7) into (4.5), we apply (4.1) with $\mu = \delta_y$ and $\nu = \delta_x$ to obtain

$$\begin{aligned}
\widetilde{\mathrm{H}}_s & f(x) - \widetilde{\mathrm{H}}_t f(y) \\
&\le \widetilde{\mathrm{H}}_s((G_r f)^2)(x)^{1/2} \\
&\quad \times 2 s_{K/N}^{-1} \left(\sqrt{ \mathrm{e}^{-K(s+t)} s_{K/N} \left(\frac{1}{2} d(x,y) \right)^2 + \frac{N(1-\mathrm{e}^{-K(s+t)})}{2K(s+t)} (\sqrt{t} - \sqrt{s})^2 } \right) \\
&\quad + 2\|f\|_\infty W_2(\mathscr{H}_s(\delta_x), \mathscr{H}_t(\delta_y)) \qquad (4.8)
\end{aligned}$$

by using our choice of r. Since the inequality (4.6) is quadratic w.r.t. scalar multiplication of f, we may assume without loss of generality that

$$|\nabla \widetilde{\mathrm{H}}_t f|(x) = \limsup_{y\to x} \frac{[\widetilde{\mathrm{H}}_t f(x) - \widetilde{\mathrm{H}}_t f(y)]_+}{d(x,y)} \, .$$

Take a sequence $(y_n)_{n\in\mathbb{N}}$ in X such that $\lim_{n\to\infty} \dfrac{\widetilde{\mathrm{H}}_t f(x) - \widetilde{\mathrm{H}}_t f(y_n)}{d(x,y_n)} = |\nabla \widetilde{\mathrm{H}}_t f|(x)$ holds. Take $\alpha \in \mathbb{R} \setminus \{0\}$, which will be specified later. For each $n \in \mathbb{N}$, let us take $s_n = t + \alpha d(x,y_n)$ and $r_n = r(x,y_n;s_n,t)$. Then we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{\widetilde{\mathrm{H}}_{s_n} f(x) - \widetilde{\mathrm{H}}_t f(y_n)}{d(x,y_n)} &= \lim_{n\to\infty} \left(\alpha \frac{\widetilde{\mathrm{H}}_{s_n} f(x) - \widetilde{\mathrm{H}}_t f(x)}{s_n - t} + \frac{\widetilde{\mathrm{H}}_t f(x) - \widetilde{\mathrm{H}}_t f(y_n)}{d(x,y_n)} \right) \\
&= \alpha \frac{\partial}{\partial t} \widetilde{\mathrm{H}}_t f(x) + |\nabla \widetilde{\mathrm{H}}_t f|(x) \, .
\end{aligned}$$

Take $\varepsilon > 0$ arbitrary. Since $G_r f$ is non-decreasing in r, by substituting $s = s_n$, $y = y_n$ into (4.8), dividing both sides by $d(x,y_n)$ and letting $n \to \infty$, we obtain

$$\begin{aligned}
\alpha \frac{\partial}{\partial t} \widetilde{\mathrm{H}}_t f(x) + |\nabla \widetilde{\mathrm{H}}_t f|(x) &\le \widetilde{\mathrm{H}}_t(|G_\varepsilon f|^2)(x)^{1/2} \\
&\qquad \times \sqrt{ \mathrm{e}^{-2Kt} + \alpha^2 \frac{N(1-\mathrm{e}^{-2Kt})}{4Kt^2} } \, .
\end{aligned}$$

Here we used the fact that $\widetilde{\mathrm{H}}_u(|G_\varepsilon f|^2)$ is continuous in u (see Remark 4.5). Let v_α be a unit vector in $\mathbb{R}^2$ of the form $\lambda(1, \alpha\sqrt{N(\mathrm{e}^{2Kt}-1)/(4Kt^2)})$ with $\lambda > 0$. Then, by rewriting the last inequality after $\varepsilon \downarrow 0$, we obtain

$$v_\alpha \cdot \left(|\nabla \widetilde{\mathrm{H}}_t f|(x), \sqrt{\frac{4Kt}{N(\mathrm{e}^{2Kt}-1)}} \frac{\partial}{\partial t} \widetilde{\mathrm{H}}_t f(x) \right) \le \mathrm{e}^{-Kt} \widetilde{\mathrm{H}}_t(|\nabla f|^2)(x)^{1/2} \, .$$

By optimizing this inequality in α, we obtain (4.6).

The second step is to show the following for any bounded and Lipschitz $f \in D(\mathrm{Ch})$: For each $t > 0$ and m-a.e. $x \in X$,

$$|\nabla \widetilde{\mathrm{H}}_t f|(x)^2 + \frac{4Kt^2}{N\big(\mathrm{e}^{2Kt}-1\big)} |\Delta \mathrm{H}_t f(x)|^2 \le \mathrm{e}^{-2Kt} \widetilde{\mathrm{H}}_t\big(|\nabla f|^2\big)(x) \, . \tag{4.9}$$

For each $x \in X$, we already know that $t \mapsto \widetilde{\mathrm{H}}_t f(x)$ is differentiable for $\mathcal{L}^1$-a.e. $t \in [0,\infty)$. Thus the Fubini theorem yields that the set $I \subset (0,\infty)$ given by

$$I := \left\{ t \in (0,\infty) \;\middle|\; t \mapsto \widetilde{\mathrm{H}}_t f(x) \text{ is differentiable for } m\text{-a.e. } x \in X \right\}$$

is of full $\mathcal{L}^1$-measure. Take $t \in I$. Then we have $\dfrac{\partial}{\partial t} \widetilde{\mathrm{H}}_t f(x) = \Delta \mathrm{H}_t f(x)$ m-a.e. and hence (4.6) yields (4.9). Thus it suffices to show $I = (0,\infty)$ to prove (4.9). Indeed, for any $t \in (0,\infty)$, there is $s \in I$ with $s < t$. Since $(u,z) \mapsto \widetilde{\mathrm{H}}_u f(z)$ is locally Lipschitz, the dominated convergence theorem implies

$$\widetilde{\mathrm{H}}_{t-s}\big(\frac{\partial}{\partial s} \widetilde{\mathrm{H}}_s f\big)(x) = \widetilde{\mathrm{H}}_{t-s}\left(\lim_{u\to 0} \frac{\widetilde{\mathrm{H}}_{s+u} f - \widetilde{\mathrm{H}}_s f}{u} \right)(x) = \frac{\partial}{\partial t} \widetilde{\mathrm{H}}_t f(x)$$

and hence $u \mapsto \widetilde{\mathrm{H}}_u f(x)$ is differentiable at t for any $x \in X$.

Finally we prove the assertion for $f \in D(\mathrm{Ch})$. Let $f_n \in D(\mathrm{Ch})$ be a sequence of bounded Lipschitz functions on X converging to f in $W^{1,2}$ strongly and $|\nabla f_n| \to |\nabla f|_w$ in L^2. Then $\Delta \mathrm{H}_t f_n \to \Delta \mathrm{H}_t f$ in L^2 and hence the conclusion follows (cf. [4, Thm. 6.2]). □

Proof of Proposition 4.4. Note first that (4.3) implies $\mathrm{RCD}(K,\infty)$ as in Remark 4.5. Take $\mu_0, \mu_1 \in \mathscr{P}_2(X,d,m)$ with bounded densities and bounded supports and π be a dynamic optimal coupling satisfying $(e_i)_\# \pi = \mu_i$ for $i = 0, 1$. Note that $(e_t)_\# \pi \ll m$ holds since $\mathrm{RCD}(K,\infty)$ holds. Let $f_n \in D(\mathrm{Ch})$ be an approximating sequence of f as above. We may assume that $(|\nabla f_n|)_{n\in\mathbb{N}}$ is uniformly bounded without loss of generality since $|\nabla f|_w \in L^\infty(X,m)$. Then $(\Delta \mathrm{H}_t f_n)_{n\in\mathbb{N}}$ is uniformly bounded in $L^\infty(X,m)$ by (4.9). We may assume also that $\mathrm{H}_t(|\nabla f_n|^2)$ and $\Delta \mathrm{H}_t f_n$ converges m-a.e. by taking a subsequence if necessary. We apply (4.9) to f_n to obtain

$$\begin{aligned} \left| \int_X \mathrm{H}_t f_n \, \mathrm{d}\mu_1 - \int_X \mathrm{H}_t f_n \, \mathrm{d}\mu_0 \right| &\le \int_{\mathrm{Geo}(X)} \int_0^1 |\nabla \widetilde{\mathrm{H}}_t f_n|(\gamma_t) |\dot{\gamma}_t| \, \mathrm{d}t \, \pi(\mathrm{d}\gamma) \\ &\le W_2(\mu_0,\mu_1) \int_0^1 \int_{\mathrm{Geo}(X)} \left(\mathrm{e}^{-2Kt} \mathrm{H}_t(|\nabla f_n|^2)(\gamma_t) - \frac{4Kt^2}{N(\mathrm{e}^{-2Kt}-1)} |\Delta \mathrm{H}_t f_n(\gamma_t)|^2 \right) \pi(\mathrm{d}\gamma) \, \mathrm{d}t. \end{aligned}$$

Then, as $n \to \infty$, the dominated convergence theorem yields

$$\begin{aligned} &\left| \int_X \mathrm{H}_t f \, \mathrm{d}\mu_1 - \int_X \mathrm{H}_t f \, \mathrm{d}\mu_0 \right| \\ &\quad \le W_2(\mu_0,\mu_1) \int_0^1 \int_{\mathrm{Geo}(X)} \left(\mathrm{e}^{-2Kt} \mathrm{H}_t(|\nabla f|_w^2)(\gamma_t) - \frac{4Kt^2}{N(\mathrm{e}^{-2Kt}-1)} |\Delta \mathrm{H}_t f(\gamma_t)|^2 \right) \pi(\mathrm{d}\gamma) \, \mathrm{d}t. \end{aligned}$$

By the strong Feller property, $\mathrm{H}_t(|\nabla f|_w^2)$ has a continuous representative. Since $\Delta \mathrm{H}_{t/2} f \in L^\infty(X,m)$ by (4.3) with $t/2$ instead of t, the strong Feller property again implies that $\Delta \mathrm{H}_t f = \mathrm{H}_{t/2} \Delta \mathrm{H}_{t/2} f$ has a continuous representative. Thus by taking μ_0 and μ_1 as a uniform distribution on $B_r(x_0)$ and $B_r(x_1)$ respectively and letting $r \to 0$, we obtain

$$\begin{aligned} &|\mathrm{H}_t f(x_0) - \mathrm{H}_t f(x_1)| \\ &\qquad \le d(x_0,x_1) \sup_{z \in B_{2d(x_0,x_1)}(x_0)} \left[\mathrm{e}^{-2Kt} \mathrm{H}_t(|\nabla f|_w^2)(z) - \frac{4Kt^2}{N(\mathrm{e}^{-2Kt}-1)} |\Delta \mathrm{H}_t f(z)|^2 \right] \end{aligned}$$

for m-a.e. x_0, x_1. Thus $\mathrm{H}_t f$ has a Lipschitz representative and (4.4) holds. □

Definition 4.6. *We say that* (X, d, m) *satisfies the* Bakry–Ledoux gradient estimate $\mathrm{BL}(K, N)$ *with* $K \in \mathbb{R}$, $N \in (0, \infty)$ *if for any* $f \in D(\mathrm{Ch})$ *and* $t > 0$

$$|\nabla \mathrm{H}_t f|_w^2 + \frac{2t}{N} C(t) |\Delta \mathrm{H}_t f|^2 \ \leq \ e^{-2Kt} \mathrm{H}_t \left(|\nabla f|_w^2 \right) \quad m\text{-a.e. in } X \ , \tag{4.10}$$

where $C > 0$ *is a function satisfying* $C(t) = 1 + O(t)$ *as* $t \to 0$.

Now Theorem 4.3 can be reformulated as follows: For an infinitesimally Hilbertian metric measure space, the W_2-expansion bound (4.1) implies the $\mathrm{BL}(K, N)$ condition under Assumption 4.2. Indeed, (4.3) states that (4.10) holds with $C(t) = 2Kt/(\mathrm{e}^{2Kt} - 1)$. The Bakry–Ledoux gradient estimate $\mathrm{BL}(K, N)$ will allow us to establish the full Bochner inequality including the dimension term in $\mathrm{RCD}^*(K, N)$ spaces. This extends the result in [4], where a Bochner inequality without dimension term has been established on $\mathrm{RCD}(K, \infty)$ spaces. Let us also make precise what we mean by Bochner's inequality, or the Bakry–Émery condition.

Definition 4.7. *We say that an infinitesimally Hilbertian metric measure space* (X, d, m) *satisfies the* Bakry–Émery condition $\mathrm{BE}(K, N)$*, or* Bochner inequality*, with* $K \in \mathbb{R}$, $N \in (0, \infty)$ *if for all* $f \in D(\Delta)$ *with* $\Delta f \in W^{1,2}(X, d, m)$ *and all* $g \in D(\Delta) \cap L^\infty(X, m)$ *with* $g \geq 0$ *and* $\Delta g \in L^\infty(X, m)$ *we have*

$$\frac{1}{2} \int \Delta g |\nabla f|_w^2 \mathrm{d}m - \int g \langle \nabla(\Delta f), \nabla f \rangle \mathrm{d}m \ \geq \ K \int g |\nabla f|_w^2 \mathrm{d}m + \frac{1}{N} \int g \big(\Delta f\big)^2 \mathrm{d}m \ . \tag{4.11}$$

To investigate the relation between Bochner's inequality and the Bakry-Ledoux gradient estimate, we introduce a mollification of the semigroup h^ε given by

$$h^\varepsilon f \ = \ \int_0^\infty \frac{1}{\varepsilon} \eta \left(\frac{t}{\varepsilon} \right) \mathrm{H}_t f \, \mathrm{d}t \ , \tag{4.12}$$

with a non-negative kernel $\eta \in C_c^\infty(0, \infty)$ satisfying $\int_0^\infty \eta(t) \mathrm{d}t = 1$ for $f \in L^p(X, m)$, $1 \leq p \leq \infty$. Note that $h^\varepsilon f \in D(\Delta)$ and

$$\Delta h^\varepsilon f = - \int_0^1 \frac{1}{\varepsilon} \eta' \left(\frac{t}{\varepsilon} \right) \mathrm{H}_t f \, \mathrm{d}t \tag{4.13}$$

for any $f \in L^p(X, m)$, $1 \leq p < \infty$.

Theorem 4.8 (Bochner inequality)**.** *Let* (X, d, m) *be an infinitesimally Hilbertian metric measure space satisfying* $\mathrm{BL}(K, N)$*. Then the Bochner inequality* $\mathrm{BE}(K, N)$ *holds.*

Proof. In the language of Dirichlet forms, this is proven in [5, Cor. 2.3, (vi)⇒(i)]. We sketch here an argument following basically the ideas developed in [21] in the setting of Alexandrov spaces.

We will first prove (4.11) for $f \in D(\Delta) \cap L^\infty(X, m)$ with $\Delta f \in D(\Delta) \cap L^\infty(X, m)$ and for g satisfying $\Delta g \in D(\mathrm{Ch})$ additionally. From (4.3) we obtain immediately

$$\int g |\nabla \mathrm{H}_t f|_w^2 \mathrm{d}m + \frac{2t}{N} C(t) \int g |\Delta \mathrm{H}_t f|^2 \mathrm{d}m \leq \ e^{-2Kt} \int g \, \mathrm{H}_t \left(|\nabla f|_w^2 \right) \mathrm{d}m \ . \tag{4.14}$$

This will yield (4.11) by subtracting $\int g |\nabla f|_w^2 \mathrm{d}m$ on both sides, dividing by t and taking the limit $t \searrow 0$. Indeed, for the left hand side of (4.14), we can argue exactly as in the proof of [21, Thm. 4.6], using the Leibniz rule 3.20, and note in addition that

$$\lim_{t \to 0} \frac{2}{N} C(t) \int g |\Delta \mathrm{H}_t f|^2 \mathrm{d}m \ = \ \frac{2}{N} \int g \big(\Delta f\big)^2 \mathrm{d}m \ .$$

For the right hand side of (4.14), by a similar calculation, we obtain

$$\begin{aligned} &\frac{1}{t} \left(\int g \, \mathrm{H}_t \left(|\nabla f|_w^2 \right) \mathrm{d}m - \int g |\nabla f|_w^2 \, \mathrm{d}m \right) \\ &\quad = -\frac{1}{t} \left(\int \mathrm{H}_t \, g f \Delta f \, \mathrm{d}m - \int g f \Delta f \, \mathrm{d}m \right) + \frac{1}{2t} \left(\int \Delta \, \mathrm{H}_t \, g \cdot f^2 \, \mathrm{d}m - \int \Delta g \cdot f^2 \, \mathrm{d}m \right) \ . \end{aligned} \tag{4.15}$$

Since $\Delta g, f^2, f \Delta f \in D(\mathrm{Ch})$, it converges to $\int \Delta g |\nabla f|_w^2 \, \mathrm{d}m$ as $t \to 0$ and thus we obtain (4.11). To obtain the estimate (4.11) for general f, we approximate f by $h^\varepsilon(f \wedge R)$ and g by $T_{\varepsilon'} g$. By

(4.13), these functions have the expected regularity. First we take $\varepsilon' \to 0$. Since $|\nabla f|_w, |\nabla \Delta f|_w \in L^1(X,m) \cap L^\infty(X,m)$ by virtue of (4.10) and (4.13), it goes well. Next we take $R \to \infty$. Since $\lim_{R\to\infty} \mathrm{Ch}(f \wedge R - f) = 0$ and $\mathrm{Ch}(f \wedge R) \le \mathrm{Ch}(f)$, we can show $|\nabla h^\varepsilon (f \wedge R)|_w^2 \to |\nabla h^\varepsilon f|_w^2$ weakly in $L^1(X,m)$ similarly as in the proof of [21, Thm. 4.6]. The same argument also works for $\langle \nabla \Delta h^\varepsilon (f \wedge R), \nabla h^\varepsilon (f \wedge R)\rangle$ with the aid of (4.13). Again (4.13) helps the convergence of the term involving N. Finally we take $\varepsilon \to 0$. we can employ the approximation argument in [21, Thm. 4.6] again when arguing this limit to conclude the convergence of the same kind. The additional dimension term posing no difficulty at this moment. □

Also the converse implication holds. Originally, this was proven by Bakry and Ledoux in [11] in the setting of Gamma calculus. See also the work of Wang [41], where the equivalence of gradient estimates and Bochner's inequality has been rediscovered in the setting of smooth Riemannian manifolds. Note that the function C in the next proposition gives a stronger estimate than (4.3) for large t.

Proposition 4.9. *Let (X, d, m) be an infinitesimally Hilbertian mms satisfying the Bakry–Émery condition* $\mathrm{BE}(K,N)$*. Then the* $\mathrm{BL}(K,N)$ *condition holds with* $C(t) = (1 - \mathrm{e}^{-2Kt})/2Kt$.

Proof. In the language of Dirichlet forms, this is basically proven in [5, Cor. 2.3, (i)⇒(vi)]. Let us sketch the argument.

As in the proof of Theorem 4.8, we first assume $f \in D(\Delta) \cap L^\infty(X,m)$ with $\Delta f \in D(\Delta) \cap L^\infty(X,m)$. Fix $g \ge 0$ with $g \in D(\Delta) \cap L^\infty(X,m)$ and $\Delta g \in L^\infty(X,m) \cap D(\mathrm{Ch})$ and consider the function

$$h(s) \ := \ \mathrm{e}^{-2Ks} \int H_s g |\nabla H_{t-s} f|_w^2 \,\mathrm{d}m \,.$$

One estimates the derivative of h as:

$$\begin{aligned} h'(s) \ &= \ -2K\mathrm{e}^{-2Ks} \int H_s g |\nabla H_{t-s} f|_w^2 \,\mathrm{d}m \\ &\quad + \mathrm{e}^{-2Ks} \int \Delta H_s g |\nabla H_{t-s} f|_w^2 \,\mathrm{d}m \\ &\quad - 2\mathrm{e}^{-2Ks} \int H_s g \langle \nabla H_{t-s} f, \nabla \Delta H_{t-s} f\rangle \,\mathrm{d}m \\ &\ge \ \frac{2}{N} \mathrm{e}^{-2Ks} \int H_s g \big(\Delta H_{t-s} f\big)^2 \,\mathrm{d}m \\ &\ge \ \frac{2}{N} \mathrm{e}^{-2Ks} \int g \big(\Delta H_t f\big)^2 \,\mathrm{d}m \,, \end{aligned}$$

where we have used (4.11) in the first and Jensen's inequality in the second inequality. A computation similar to the first equality in (4.15), deduces that h is continuous at 0 and t since $g, f \in L^\infty$. Thus, integrating from 0 to t we obtain:

$$\int g |\nabla H_t f|_w^2 \,\mathrm{d}m + \frac{1 - \mathrm{e}^{-2Kt}}{NK} \int g \big(\Delta H_t f\big)^2 \,\mathrm{d}m \le \mathrm{e}^{-2Kt} \int H_t g |\nabla f|_w^2 \,\mathrm{d}m \,.$$

For the general case, we approximate $f \in D(\mathrm{Ch})$ and $g \in L^2(X,m) \cap L^\infty(X,m)$ by $h^\varepsilon(f \wedge R)$ and $h^{\varepsilon'} g$ respectively. As we did in the proof of Theorem 4.8, We can take $R \to \infty$, $\varepsilon \to 0$ to obtain the last inequality for f and $h^{\varepsilon'} g$. Since $h^{\varepsilon'} g$ converges to g with respect to weak* topology in $L^\infty(X,m)$ as $\varepsilon' \to 0$, the last inequality holds for general f and g. This is sufficient to complete the proof. □

4.2. **From** $\mathrm{BL}(K,N)$ **to** $\mathrm{CD}^e(K,N)$**.** In the following section, we will always assume that (X, d, m) is an infinitesimally Hilbertian metric measure space and that Assumption 4.2 holds. We will show that the Bakry–Ledoux gradient estimate $\mathrm{BL}(K,N)$ implies the entropic curvature-dimension condition $\mathrm{CD}^e(K,N)$ and thus the $\mathrm{RCD}^*(K,N)$ condition.

Our approach is strongly inspired by the recent work [5] of Ambrosio, Gigli and Savaré. We follow their presentation and adopt to a large extent their notation. Under Assumption 4.2 we can rely on the results in [5], since the condition $\mathrm{BL}(K,N)$ is more restrictive than the classical

Bakry–Émery gradient estimate $\mathrm{BL}(K,\infty)$. In particular, we already know that the Riemannian curvature condition $\mathrm{RCD}(K,\infty)$ holds true, c.f. Remark 4.5, [5, Cor. 4.18]. Moreover, we also know that the semigroup H_t coincides with the gradient flow $\mathscr{H}_t$ of the entropy in $\mathscr{P}_2(X,d)$ in the sense of Theorem 3.15.

The crucial ingredient in our argument is the action estimate Proposition 4.16. This result calls for an extensive regularization procedure that was already used in [5], both for curves in $\mathscr{P}_2(X)$ and for the entropy functional, which we will discuss below. The main difference of our approach compared to [5] is that our argument now relies on the analysis of the (nonlinear) gradient flow $(\nu_t)_{t\geq 0}$ for the functional $-U_N$ instead of the analysis of the (linear) heat flow which is the gradient flow $(\mu_t)_{t\geq 0}$ for Ent. Both flows are related to each other via time change:

$$\nu_t = \mu_{\tau_t}, \qquad \partial_t \tau_t = \frac{1}{N} U_N(\mu_{\tau_t}) \;.$$

More precisely, the following lemma yields that this time change is well-defined.

Lemma 4.10. *Let $\rho \in D(\mathrm{Ent}) \subset \mathscr{P}_2(X,d,m)$. Then there exist constants $a,c>0$ depending only on $|\mathrm{Ent}(\rho)|$ and the second moment of ρ such that a map $\tau : [0,a] \to [0,\infty)$ can be defined implicitly by*

$$\int_0^{\tau_t} \exp\left(\frac{1}{N}\,\mathrm{Ent}(\mathscr{H}_r\rho)\right) \mathrm{d}r \;=\; t \tag{4.16}$$

and for any $t\in[0,a]$ we have $\tau_t \leq ct$. Moreover, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\tau_t \;=\; \frac{1}{N} U_N(\mathscr{H}_{\tau_t}\rho) \;. \tag{4.17}$$

Proof. We first derive a lower bound on $\mathrm{Ent}(\mathscr{H}_r\rho)$. Let us set $V(x) = d(x_0,x)$ for some $x_0\in X$. By (3.6) we have that $z = \int \mathrm{e}^{-V^2}\mathrm{d}m < \infty$ and $\tilde{m} = z^{-1}\mathrm{e}^{-V^2} m$ is a probability measure. Now [6, Thm. 4.20] (together with a trivial truncation argument) yields that

$$\int V^2 \mathrm{d}(\mathscr{H}_r\rho) \;\leq\; \mathrm{e}^{4r}\Big(\mathrm{Ent}(\rho) + 2\int V^2\mathrm{d}\rho\Big) \;=:\; \mathrm{e}^{4r}c' \;.$$

Hence we obtain

$$\mathrm{Ent}(\rho) \;\geq\; \mathrm{Ent}(\mathscr{H}_r\rho) \;=\; \mathrm{Ent}(\mathscr{H}_r\rho|\tilde{m}) - \int V^2\mathrm{d}(\mathscr{H}_r\rho) - \log z \;\geq\; -\mathrm{e}^{4r}c' - \log z \;.$$

Now fix some $R>0$ and put $a = z^{-1}\int_0^R \exp(-\mathrm{e}^{4r}c'/N)\mathrm{d}r$, $c = z\exp(\mathrm{e}^{4R}c'/N)$. Then define the function $F : [0,R] \to [0,F(R)]$ via $F(u) = \int_0^u \exp\big(\mathrm{Ent}(\mathscr{H}_r\rho)/N\big)\mathrm{d}r$. Since F is strictly increasing with $F(0)=0$ and $F(R)\geq a$ by the preceding estimate we can define $\tau_t = F^{-1}(t)$ for any $t\in[0,a]$. Moreover, we have $F(u)\geq c^{-1}u$ for any $u\leq R$ which implies $\tau_t\leq ct$. Finally (4.17) follows immediately from the differentiability of F. □

More generally, given a continuous curve $(\rho_s)_{s\in[0,1]}$ in $\mathscr{P}_2(X,d,m)$ such that $\max_s|\mathrm{Ent}(\rho_s)|<\infty$ we define a time change $\tau_{s,t}$ implicitly via

$$\int_0^{\tau_{s,t}} \exp\left(\frac{1}{N}\,\mathrm{Ent}(\mathscr{H}_r\rho_s)\right)\mathrm{d}r \;=\; st \tag{4.18}$$

for $s\in[0,1]$ and $t\in[0,a]$ satisfying

$$\tau_{s,t} \;\leq\; c\cdot st \tag{4.19}$$

for suitable constants $a,c>0$ depending only on a uniform bound on the entropy and second moments of $(\rho_s)_{s\in[0,1]}$ and moreover

$$\partial_t\tau_{s,t} \;=\; s\cdot U_N(\mathscr{H}_{\tau_{s,t}}\rho_s) \;. \tag{4.20}$$

We will now describe the regularization procedure needed in the sequel. We will use the notion of *regular curve* as introduced in [5, Def. 4.10]. Briefly, a curve $(\rho_s)_{s\in[0,1]}$ with $\rho_s = f_s m$ is called regular if the following are satisfied:

- (ρ_s) is 2-absolutely continuous in $\mathscr{P}_2(X,d)$,
- $\mathrm{Ent}(\rho_s)$ and $I(\mathrm{H}_t f_s)$ are bounded for $s\in[0,1], t\in[0,T]$,

- $f \in C^1\big([0,1], L^1(X,m)\big)$ and $\Delta^{(1)} f \in C\big([0,1], L^1(X,m)\big)$,
- $f_s = h^\varepsilon \widetilde{f}_s$ for some $\widetilde{f}_s \in L^1(X,m)$ and $\varepsilon > 0$.

Here $I(f) = 4\,\mathrm{Ch}(\sqrt{f})$ denotes the Fisher information, $\Delta^{(1)}$ denotes the generator of the semigroup H_t in $L^1(X,m)$ and h^ε is the mollification of the semigroup given in (4.12). In the sequel we will denote by $\dot{f}_s$ the derivative of $[0,1] \ni s \mapsto f_s \in L^1(X,m)$. We will mostly denote both the generator in L^1 and in L^2 by Δ. In the following we will need an approximation result which is a reinforcement of [5, Prop. 4.11].

Lemma 4.11 (Approximation by regular curves)**.** *Let $(\rho_s)_{s\in[0,1]}$ be an AC^2-curve in $\mathscr{P}_2(X,d,m)$ such that $s \mapsto \mathrm{Ent}(\rho_s)$ is bounded and continuous. Then there exists a sequence of regular curves (ρ_s^n) with the following properties. As $n \to \infty$ we have for any $s \in [0,1]$:*

$$W_2(\rho_s^n, \rho_s) \ \to\ 0\ , \tag{4.21}$$

$$\limsup |\dot{\rho}_s^n| \ \le\ |\dot{\rho}_s| \quad a.e.\ in\ [0,1]\ , \tag{4.22}$$

$$\mathrm{Ent}(\mathscr{H}_r \rho_s^n) \ \to\ \mathrm{Ent}(\mathscr{H}_r \rho_s) \quad \forall r > 0\ , \tag{4.23}$$

$$\tau_{s,t}^n \ \to\ \tau_{s,t}\ , \tag{4.24}$$

where τ^n and τ denote the time changes defined via the curves (ρ_s^n) and (ρ_s) respectively on $[0,1] \times [0,a]$ for suitable $a > 0$. Moreover, for any $\delta > 0$ there are $n_0, r_0 > 0$ such that for any $n > n_0$ and $r < r_0$ and all $s \in [0,1]$ we have:

$$|\mathrm{Ent}(\rho_s) - \mathrm{Ent}(\mathscr{H}_r \rho_s^n)| \ <\ \delta\ . \tag{4.25}$$

Proof. Following [5, Prop. 4.11] we employ a threefold regularization procedure. We trivially extend $(\rho_s)_s$ to $\mathbb{R}$ with value ρ_0 in $(-\infty, 0)$ and ρ_1 in $(1,\infty)$. Given n, we first define $\rho_s^{n,1} = \mathscr{H}_{1/n}\rho_s = f_s^{n,1} m$. The second step consists in a convolution in the time parameter. We set

$$\rho_s^{n,2} = f^{n,2} m\ , \qquad f_s^{n,2} \ =\ \int_{\mathbb{R}} f_{s-s'}^{n,1} \psi_n(s')\mathrm{d}s'\ ,$$

where $\psi_n(s) = n \cdot \psi(ns)$ for some smooth kernel $\psi : \mathbb{R} \to \mathbb{R}_+$ with $\int \psi(s)\mathrm{d}s = 1$. Finally, we set

$$\rho_s^n \ =\ f_s^n m\ , \qquad f_s^n \ =\ h^{1/n} f_s^{n,2}\ ,$$

where h^ε denotes a mollification of the semigroup given by (4.12). It has been proven in [5, Prop. 4.11] that $(\rho_s^n)_{s\in[0,1]}$ constructed in this way is a regular curve and that (4.21) holds. (4.22) follows from the convexity properties of W_2^2 and the K-contractivity of the heat flow. Let us now prove (4.23). Note that on the level of measures the semigroup commutes with the regularization, i.e. $H_r \rho_s^n = \widetilde{\rho}_s^n$ where $\widetilde{\rho}_s := \mathscr{H}_r \rho_s$. Thus it is sufficient to prove (4.23) for $r = 0$. By (4.21) and lower semicontinuity of the entropy we have $\mathrm{Ent}(\rho_s) \le \liminf_{n\to\infty} \mathrm{Ent}(\rho_s^n)$. On the other hand, using the convexity properties of the entropy and the fact that $\mathscr{H}_r$ and thus also $h^{1/n}$ decreases the entropy we estimate

$$\begin{aligned}\mathrm{Ent}(\rho_s^n) \ &\le\ \mathrm{Ent}(\rho_s^{n,2}) \ \le\ \int \psi_n(s')\,\mathrm{Ent}(\mathscr{H}_{1/n}\rho_{s-s'})\mathrm{d}s' \ \le\ \int \psi_n(s')\,\mathrm{Ent}(\rho_{s-s'})\mathrm{d}s' \\ &\le\ \mathrm{Ent}(\rho_s) + \int \psi_n(s')|\mathrm{Ent}(\rho_{s-s'}) - \mathrm{Ent}(\rho_s)|\mathrm{d}s'\ .\end{aligned} \tag{4.26}$$

The last term vanishes as $n \to \infty$ since $s \mapsto \mathrm{Ent}(\rho_s)$ is uniformly continuous by compactness. Thus we obtain $\limsup_{n\to\infty} \mathrm{Ent}(\rho_s^n) \le \mathrm{Ent}(\rho_s)$ and hence (4.23). To prove (4.24) define the functions

$$F_n(u) \ =\ \int_0^u \exp\left(\frac{1}{N}\,\mathrm{Ent}(\mathscr{H}_r \rho_s^n)\right)\mathrm{d}r\ , \qquad F(u) \ =\ \int_0^u \exp\left(\frac{1}{N}\,\mathrm{Ent}(\mathscr{H}_r \rho_s)\right)\mathrm{d}r\ .$$

Arguing as in Lemma 4.10 we see that $\tau_{s,t}^n = F_n^{-1}(st)$ and $\tau_{s,t} = F^{-1}(st)$ can be defined simultaneously on $[0,1] \times [0,a]$ and satisfy $|F_n(u) - F_n(v)| \ge c^{-1}|u - v|$ for suitable constants $a, c > 0$ independent of n. Since moreover, by (4.23) and dominated convergence we have $F_n \to F$ pointwise as $n \to \infty$ we conclude the convergence (4.24).

We now prove the last statement of the lemma. To conclude the proof we proceed by contradiction. Assume the contrary, i.e. that there exists $\delta > 0$ and a sequences $n_k \to \infty, r_k \to 0$ and $(s_k) \subset [0,1]$ such that $|\mathrm{Ent}(\rho_{s_k}) - \mathrm{Ent}(\mathscr{H}_{r_k}\rho_{s_k}^{n_k})| \geq \delta$ for all k. Taking into account (4.26) and the fact that $\mathscr{H}_r$ decreases entropy we must have that for all k sufficiently large

$$\mathrm{Ent}(\rho_{s_k}) - \mathrm{Ent}(\mathscr{H}_{r_k}\rho_{s_k}^{n_k}) \;\geq\; \delta\,. \tag{4.27}$$

By compactness we can assume $s_k \to s_0$ as $k \to \infty$ for some $s_0 \in [0,1]$. We claim that as $k \to \infty$ we have $\mathscr{H}_{r_k}\rho_{s_k}^{n_k} \to \rho_{s_0}$ in W_2. Indeed, since $\mathscr{H}_r$ satisfies a Wasserstein contraction and by the convexity properties of W_2 the regularizing procedure increases distances at most an exponential factor (see also [5, Prop. 4.11]). Hence, the triangle inequality yields

$$\begin{aligned} W_2(\rho_{s_0}, \mathscr{H}_{r_k}\rho_{s_k}^{n_k}) \;&\leq\; W_2(\rho_{s_0}, \mathscr{H}_{r_k}\rho_{s_0}) + W_2(\mathscr{H}_{r_k}\rho_{s_0}, \mathscr{H}_{r_k}\rho_{s_0}^{n_k}) + W_2(\mathscr{H}_{r_k}\rho_{s_0}^{n_k}, \mathscr{H}_{r_k}\rho_{s_k}^{n_k}) \\ &\leq\; W_2(\rho_{s_0}, \mathscr{H}_{r_k}\rho_{s_0}) + \mathrm{e}^{-Kr_k}W_2(\rho_{s_0}, \rho_{s_0}^{n_k}) + \mathrm{e}^{-Kr_k}W_2(\rho_{s_0}, \rho_{s_k}) + o(1)\,, \end{aligned}$$

and the claim follows from the continuity of $\mathscr{H}_r$ at $r = 0$, (4.21) and the continuity of the curve (ρ_s). Letting now $k \to \infty$ in (4.27), using continuity of $s \mapsto \mathrm{Ent}(\rho_s)$ and lower semicontinuity of Ent, we obtain the following contradiction:

$$0 \;=\; \mathrm{Ent}(\rho_{s_0}) - \mathrm{Ent}(\rho_{s_0}) \;\geq\; \limsup_{k\to\infty}\Big(\mathrm{Ent}(\rho_{s_k}) - \mathrm{Ent}(\mathscr{H}_{r_k}\rho_{s_k}^{n_k})\Big) \;\geq\; \delta\,.$$

□

The following calculations will be a crucial ingredient in our argument. For a detailed justification see [5, Lem. 4.13, 4.15]. The only difference here is the additional time change in the semigroup. For the following lemmas let $(\rho_s)_{s\in[0,1]}$ be a regular curve and let $\varphi : X \to \mathbb{R}$ be Lipschitz with bounded support. Let $\theta : [0,1] \to [0,\infty)$ be an increasing C^1 function with $\theta(0) = 0$ and set $\rho_{s,\theta} = \mathscr{H}_{\theta_s}\rho_s = f_{s,\theta}m$. Moreover, we set $\varphi_s = Q_s\varphi$ for $s \in [0,1]$, where

$$Q_s\varphi(x) \;:=\; \inf_{y\in X}\left[f(y) + \frac{d^2(x,y)}{2s}\right]$$

denotes the Hopf-Lax semigroup. We refer to [6, Sec. 3] for a detailed discussion. We recall that since (X,d) is a length space, Q provides a solution to the Hamilton–Jacobi equation, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}s}Q_s\varphi \;=\; -|\nabla Q_s\varphi|$$

for a.e. $s \in [0,1]$, see [6, Prop. 3.6]. Moreover, we have the a priori Lipschitz bound ([6, Prop. 3.4])

$$\mathrm{Lip}(Q_s\varphi) \;\leq\; 2\,\mathrm{Lip}(\varphi)\,. \tag{4.28}$$

Lemma 4.12. *The map $s \mapsto \int \varphi_s \mathrm{d}\rho_{s,\theta}$ is absolutely continuous and we have for a.e. $s \in [0,1]$:*

$$\frac{\mathrm{d}}{\mathrm{d}s}\int \varphi_s \mathrm{d}\rho_{s,\theta} \;=\; \int \left(-\frac{1}{2}|\nabla\varphi_s|^2 f_{s,\theta} + \dot{f}_s\,\mathrm{H}_{\theta_s}\varphi_s + \dot{\theta}_s\Delta f_{s,\theta}\cdot\varphi_s\right)\mathrm{d}m\,. \tag{4.29}$$

We use a regularization E_ε of the entropy functional where the singularities of the logarithm a truncated. Let us define $e_\varepsilon : [0,\infty) \to \mathbb{R}$ by setting $e'_\varepsilon(r) = \log(\varepsilon + r\wedge\varepsilon^{-1}) + 1$ and $e_\varepsilon(0) = 0$. Then for any $\rho = fm \in \mathscr{P}_2(X,d,m)$ we define

$$E_\varepsilon(\rho) := \int e_\varepsilon(f)\mathrm{d}m\,, \qquad U_N^\varepsilon(\rho) = \exp\left(-\frac{1}{N}E_\varepsilon(\rho)\right)\,.$$

Moreover we set $p_\varepsilon(r) = e'_\varepsilon(r^2) - \log\varepsilon - 1$. Note that for any $\rho \in D(\mathrm{Ent})$ we have $E_\varepsilon(\rho) \to \mathrm{Ent}(\rho)$ as $\varepsilon \to 0$.

Lemma 4.13. *The map $s \mapsto E_\varepsilon(\rho_{s,\theta})$ is absolutely continuous and we have for all $s \in [0,1]$:*

$$\frac{\mathrm{d}}{\mathrm{d}s}E_\varepsilon(\rho_{s,\theta}) \;=\; \int \Big(\dot{f}_s\,\mathrm{H}_{\theta_s}g^\varepsilon_{s,\theta} + \dot{\theta}_s\Delta f_{s,\theta}\cdot g^\varepsilon_{s,\theta}\Big)\,\mathrm{d}m\,, \tag{4.30}$$

where we put $g^\varepsilon_{s,r} = p_\varepsilon(\sqrt{f_{s,r}})$.

We also need to introduce the time change related to the regularized entropy. For fixed $\varepsilon > 0$ and let us define $\tau^\varepsilon_{s,t}$ implicitly by

$$\int_0^{\tau^\varepsilon_{s,t}} \exp\left(\frac{1}{N} E_\varepsilon(\mathscr{H}_r\rho_s)\right) \mathrm{d}r \;=\; st\;. \tag{4.31}$$

Lemma 4.14. *τ^ε is well defined on $[0,1]\times[0,a]$ and satisfies $\tau^\varepsilon_{s,t} \le c\cdot st$ for constants $a,c>0$ depending only on $\max_s|\mathrm{Ent}(\rho_s)|$ and the second moments of $(\rho_s)_{s\in[0,1]}$. For fixed t the map $s\mapsto\tau^\varepsilon_{s,t}$ is C^1 on $[0,1]$ and we have:*

$$\partial_s\tau^\varepsilon_{s,t} \;=\; t\cdot U_N^\varepsilon(\mathscr{H}_{\tau^\varepsilon}\rho_s) - \frac{1}{N}\int_0^{\tau^\varepsilon} \frac{U_N^\varepsilon(\mathscr{H}_{\tau^\varepsilon}\rho_s)}{U_N^\varepsilon(\mathscr{H}_r\rho_s)}\int_X \dot f_s H_r g^\varepsilon_{s,r}\,\mathrm{d}m\,\mathrm{d}r\;. \tag{4.32}$$

Moreover, as $\varepsilon\to 0$ we have $\tau^\varepsilon_{s,t}\to\tau_{s,t}$, where τ is the time change defined by (4.31).

Proof. Define the function $F_\varepsilon(s,u) = \int_0^u \exp\left(E_\varepsilon(\mathscr{H}_r\rho_s)/N\right)\mathrm{d}r$. Note that a uniform bound on $|\mathrm{Ent}(\rho_s)|$ implies a uniform bound on $|E_\varepsilon(\rho_s)|$ independent of ε. Thus we can argue as in Lemma 4.10 to find a,c such that $\tau^\varepsilon_{s,t}$ is well-defined on $[0,1]\times[0,a]$ by $F_\varepsilon(s,\tau^\varepsilon_{s,t}) = st$ and satisfies $\tau^\varepsilon_{s,t}\le c\cdot st$. Using Lemma 4.13 and the fact that $s\mapsto \dot f_s$ is continuous in $L^1(X,m)$, since $(\rho_s)_s$ is a regular curve, we see that $s\mapsto E_\varepsilon(\mathscr{H}_r\rho_s)$ is C^1 for fixed $r\ge 0$. Moreover, using the boundedness of $E_\varepsilon(\mathscr{H}_r\rho_s)$ we obtain that $F_\varepsilon(\cdot,\cdot)$ is C^1. Thus the differentiability of $s\mapsto\tau^\varepsilon_{s,t}$ follows from the implicit function theorem and (4.32) is obtained by differentiating (4.31) w.r.t. s. The last statement about convergence follows as for (4.24) using that $E_\varepsilon(\rho_s)\to\mathrm{Ent}(\rho_s)$ as $\varepsilon\to 0$. □

We need the following integrations by parts and estimates for the integrals appearing in (4.29), (4.30). Recall that $I(f) = 4\int|\nabla\sqrt{f}|_w^2\mathrm{d}m$ denotes the Fisher information of a measure $\rho = fm$.

Lemma 4.15. *Let $f = h^\varepsilon\tilde f$ for some $\tilde f\in L^1_+(X,m)$ with $\tilde f m\in\mathscr{P}_2(X,m)$. Then for any Lipschitz function φ with bounded support we have*

$$\int\langle\nabla\varphi,\nabla g^\varepsilon\rangle f\mathrm{d}m + \int q_\varepsilon(f)\langle\nabla\sqrt{f},\nabla\varphi\rangle\mathrm{d}m \;=\; -\int\varphi\Delta f\mathrm{d}m \;\le\; 2\,\mathrm{Lip}(\varphi)\cdot\sqrt{I(f)}\;, \tag{4.33}$$

where $q_\varepsilon(r) = \sqrt{r}\big(2-\sqrt{r}p'_\varepsilon(\sqrt{r})\big)$ and $g^\varepsilon = p_\varepsilon(\sqrt{f})$. Moreover we have

$$\int|\nabla g^\varepsilon|_w^2 f\mathrm{d}m \;\le\; -\int g^\varepsilon\Delta f\mathrm{d}m \;\le\; I(f)\;. \tag{4.34}$$

Proof. We first obtain from [5, Thm. 4.4]

$$-\int\varphi\Delta f\mathrm{d}m \;=\; 2\int\sqrt{f}\langle\nabla\varphi,\nabla\sqrt{f}\rangle\,\mathrm{d}m\;.$$

Now the first equality in (4.33) is immediate from the chain rule (3.18) for minimal weak upper gradients and integration by parts while the second inequality follows readily using Hölder's inequality. To prove (4.34) we use that by [5, Lem. 4.9] for any bounded non-decreasing Lipschitz function $\omega:[0,\infty)\to\mathbb{R}$ with $\sup_r r\omega'(r)<\infty$:

$$-\int\omega(f)\Delta^{(1)}f\mathrm{d}m \;\ge\; 4\int f\omega'(f)|\nabla\sqrt{f}|_w^2\mathrm{d}m\;. \tag{4.35}$$

Further note that $r\cdot e''_\varepsilon(r)\le 1$ and hence $4r\cdot e''_\varepsilon(r)\ge 4r^2\big(e''_\varepsilon(r)\big)^2 = r\big(p'_\varepsilon(\sqrt{r})\big)^2$. Hence we get by the chain rule:

$$f|\nabla g^\varepsilon|_w^2 \;=\; f\big(p'_\varepsilon(\sqrt{f})\big)^2|\nabla\sqrt{f}|_w^2 \;\le\; 4fe''_\varepsilon(f)|\nabla\sqrt{f}|_w^2\;. \tag{4.36}$$

Combining this with (4.35) yields the first inequality in (4.34). For the second inequality note that, since we already now that $\mathrm{RCD}(K,\infty)$ holds, $\tilde{\mathrm{H}}_\delta g^\varepsilon$ is bounded and Lipschitz for all $\delta>0$

by [4, Thm. 6.8]. Hence [5, Thm. 4.4] and Hölder's inequality yield

$$-\int \Delta f \,\mathrm{H}_\delta\, g^\varepsilon \mathrm{d}m \;=\; 2\int \sqrt{f}\langle \nabla \mathrm{H}_\delta\, g^\varepsilon, \nabla\sqrt{f}\rangle\, \mathrm{d}m \;\le\; 2\,\mathrm{Ch}(\sqrt{f})^{\frac12}\cdot\Big(\int f|\nabla \mathrm{H}_\delta\, g^\varepsilon|_w \mathrm{d}m\Big)^{\frac12}$$
$$\le\; 4\mathrm{e}^{-K\delta}\,\mathrm{Ch}(\sqrt{f})\;,$$

where we have used again (4.36) and $\mathrm{BL}(K,\infty)$ in the last step. Letting $\delta\to 0$ yields the second inequality in (4.34). □

We will often use the following estimate (see [5, Lem. 4.12]). For any AC^2 curve $(\rho_s)_{s\in[0,1]}$ with $\rho_s = f_s m$ and $f \in C^1\big((0,1), L^1(X,m)\big)$ and any Lipschitz function φ we have

$$\Big|\int \dot f_s \varphi \mathrm{d}m\Big| \;\le\; |\dot\rho_s|\cdot\sqrt{\int |\nabla\varphi|^2 f_s \mathrm{d}m}\;. \tag{4.37}$$

The following result is the crucial ingredient in our argument.

Proposition 4.16 (Action estimate)**.** *Assume that* (X,d,m) *satisfies* $\mathrm{BL}(K,N_0)$. *Let* $(\rho_s)_{s\in[0,1]}$ *be a regular curve and* φ *a Lipschitz function with bounded support and denote by* $\varphi_s = Q_s\varphi$ *the Hamilton–Jacobi flow for* $s\in[0,1]$. *Then for any* $N>N_0$ *and* $t\in[0,a]$*:*

$$\begin{aligned}&\int \varphi_1 \mathrm{d}\rho_{1,\tau} - \int \varphi_0 \mathrm{d}\rho_0 - \frac12\int_0^1 |\dot\rho_s|^2 \mathrm{e}^{-2K\tau}\mathrm{d}s + Nt\cdot[U_N(\rho_0) - U_N(\rho_{1,\tau})]\\ &\le C_1\int_0^1 \frac{\tau}{4}\left[\left(\frac{U_N(\rho_{s,\tau})}{U_N(\rho_s)}\right)^2 - 1 - 4\left(\frac{N}{N_0}-1\right) + C_2\tau\right]\mathrm{d}s\;,\end{aligned} \tag{4.38}$$

The constant C_2 *depends only on* K *and* $\max_{s\in[0,1]}|\mathrm{Ent}(\rho_s)|$, *the constant* C_1 *depends in addition on* $\max_{s\in[0,1]} I(\rho_s)$ *and* φ.

Proof. For simplicity we assume that $\mathrm{BL}(K,N_0)$ holds with $C\equiv 1$. We use the abbreviations $\alpha_r = \alpha_{s,r} = -\int g^\varepsilon_{s,r}\Delta f_{s,r}\,\mathrm{d}m$ and $\beta_r = \beta_{s,r} = \int \varphi_s \Delta f_{s,r}\mathrm{d}m$. Moreover, we put $u_r = u_{s,r} = U^\varepsilon_N(\rho_{s,r})$. We will also write $\alpha = \alpha_{s,\tau}$, $\beta=\beta_{s,\tau}$, $u = u^\varepsilon_{s,\tau}$.

Using Lemmas 4.12, 4.14 and (3.16), we obtain

$$\begin{aligned}(A) \;&:=\; \int\varphi_1 d\rho_{1,\tau} - \int \varphi_0 d\rho_0 - \frac12\int_0^1 |\dot\rho_s|^2\mathrm{e}^{-2K\tau}\mathrm{d}s\\ &=\; \int_0^1\left[-\frac12|\dot\rho_s|^2\mathrm{e}^{-2K\tau} + \int\left(-\frac12|\nabla\varphi_s|^2 f_{s,\tau} + \dot f_s H_\tau\varphi_s + \dot\tau\Delta H_\tau f_s\cdot\varphi_s\right)\mathrm{d}m\right]\mathrm{d}s\\ &\le\; \int_0^1\left[-\frac12|\dot\rho_s|^2\mathrm{e}^{-2K\tau} - \frac12\int|\nabla\varphi_s|^2_w f_{s,\tau}\mathrm{d}m\right.\\ &\qquad\left. + \int \dot f_s\cdot H_\tau\varphi_s\,\mathrm{d}m + \beta t u - \beta\frac1N\int_0^\tau \frac{u}{u_r}\int \dot f_s\cdot H_r g^\varepsilon_{s,r}\,\mathrm{d}m\,\mathrm{d}r\right]\mathrm{d}s.\end{aligned}$$

Moreover, by Lemma 4.13, we have

$$\begin{aligned}(B) \;&:=\; Nt\cdot[U^\varepsilon_N(\rho_0) - U^\varepsilon_N(\rho_{1,\tau})] = t\int_0^1 U^\varepsilon_N(\rho_{s,\tau})\partial_s E_\varepsilon(\rho_{s,\tau})\mathrm{d}s\\ &=\; t\int_0^1 U^\varepsilon_N(\rho_{s,\tau})\cdot\int g^\varepsilon_{s,\tau}\cdot\left[H_\tau\dot f_s + \dot\tau\Delta H_\tau f_s\right]\mathrm{d}m\,\mathrm{d}s\\ &=\; \int_0^1\left[tu\cdot\int\dot f_s\cdot H_\tau g^\varepsilon_{s,\tau}\,\mathrm{d}m - t^2u^2\alpha\right.\\ &\qquad\left. + tu\alpha\frac1N\int_0^\tau\frac{u}{u_r}\int \dot f_s\cdot H_r g^\varepsilon_{s,r}\,\mathrm{d}m\,\mathrm{d}r\right]\mathrm{d}s.\end{aligned}$$

Adding up

$$\begin{aligned}
(A)+(B) \quad \leq & \int_0^1 \Big[-\frac{1}{2}|\dot\rho_s|^2 \mathrm{e}^{-2K\tau} - \frac{1}{2}\int |\nabla\varphi_s|_w^2\, f_{s,\tau}\mathrm{d}m + tu(\beta - tu\alpha) \\
& + \frac{1}{\tau}\int_0^\tau \int \dot f_s\, \mathrm{e}^{-K\tau}\cdot \Big[H_\tau\left(\varphi_s + tug^\varepsilon_{s,\tau}\right) - \frac{\tau}{N}(\beta - tu\alpha)\frac{u}{u_r}H_r g^\varepsilon_{s,r}\Big]\, \mathrm{d}m\, \mathrm{e}^{K\tau}\,\mathrm{d}r\Big]\,\mathrm{d}s \\
\leq & \int_0^1 \Big[-\frac{1}{2}\int |\nabla\varphi_s|_w^2\, f_{s,\tau}\mathrm{d}m + tu(\beta - tu\alpha) \\
& + \frac{1}{\tau}\int_0^\tau \frac{1}{2}\int \Big|\nabla\Big[H_\tau\left(\varphi_s + tug^\varepsilon_{s,\tau}\right) - \frac{\tau}{N}(\beta - tu\alpha)\frac{u}{u_r}H_r g^\varepsilon_{s,r}\Big]\Big|^2 f_s\,\mathrm{d}m\,\mathrm{e}^{2K\tau}\,\mathrm{d}r\Big]\,\mathrm{d}s \\
\leq & \int_0^1 \Big[-\frac{1}{2}\int |\nabla\varphi_s|_w^2\, f_{s,\tau}\mathrm{d}m + tu(\beta - tu\alpha) \\
& + \frac{1}{\tau}\int_0^\tau \frac{1}{2}\int \Big|\nabla\Big[H_{\tau-r}\left(\varphi_s + tug^\varepsilon_{s,\tau}\right) - \frac{\tau}{N}(\beta - tu\alpha)\frac{u}{u_r}g^\varepsilon_{s,r}\Big]\Big|_w^2 f_{s,r}\,\mathrm{d}m\,\mathrm{e}^{2K(\tau-r)}\,\mathrm{d}r\Big]\,\mathrm{d}s \\
& - \frac{1}{\tau}\int_0^\tau \frac{r}{N_0}\int \Big|\Delta\Big[H_\tau\left(\varphi_s + tug^\varepsilon_{s,\tau}\right) - \frac{\tau}{N}(\beta - tu\alpha)\frac{u}{u_r}H_r g^\varepsilon_{s,r}\Big]\Big|^2 f_s\,\mathrm{d}m\,\mathrm{e}^{2K\tau}\,\mathrm{d}r\Big]\,\mathrm{d}s \\
=: & \; (C) + ([D+E]^2) + (F)\,.
\end{aligned}$$

Here we have used (4.37) in the second inequality and in the last inequality the Bakry–Ledoux gradient estimate $\mathrm{BL}(K, N_0)$ applied to the semigroup H_r in the strong form given by Proposition 4.4. The last term will be estimated as follows

$$\begin{aligned}
(F) \;\leq & \int_0^1 \Big[-\frac{1}{\tau}\int_0^\tau \frac{r}{N_0}\Big|\int \Delta\Big[H_\tau\left(\varphi_s + tug^\varepsilon_{s,\tau}\right) - \frac{\tau}{N}(\beta - tu\alpha)\frac{u}{u_r}H_r g^\varepsilon_{s,r}\Big] f_s\,\mathrm{d}m\Big|^2 \mathrm{e}^{2K\tau}\,\mathrm{d}r\Big]\,\mathrm{d}s \\
= & \int_0^1 \Big[-\frac{1}{\tau}\int_0^\tau \frac{r}{N_0}\Big|\beta - tu\alpha + \frac{\tau}{N}(\beta - tu\alpha)\frac{u}{u_r}\alpha_r\Big|^2 \mathrm{e}^{2K\tau}\,\mathrm{d}r\Big]\,\mathrm{d}s \\
= & \int_0^1 \Big[-\frac{1}{\tau}\int_0^\tau \frac{r}{N_0}\,|\beta - tu\alpha|^2\cdot\Big|1 + \frac{\tau}{N}\frac{u}{u_r}\alpha_r\Big|^2 \mathrm{e}^{2K\tau}\,\mathrm{d}r\Big]\,\mathrm{d}s.
\end{aligned}$$

By virtue of Lemma 4.15, the second last term $([D+E]^2)$ can be decomposed into

$$\begin{aligned}
(E^2) \;=& \int_0^1 \Big[\frac{1}{\tau}\int_0^\tau \frac{1}{2}\frac{\tau^2}{N^2}\left(\frac{u}{u_r}\right)^2 (\beta - tu\alpha)^2\,\mathrm{e}^{2K(\tau-r)} \int |\nabla g^\varepsilon_{s,r}|_w^2 f_{s,r}\,\mathrm{d}m\,\mathrm{d}r\Big]\,\mathrm{d}s \\
\leq & \int_0^1 \Big[\frac{1}{\tau}\int_0^\tau \frac{1}{2}\frac{\tau^2}{N^2}\left(\frac{u}{u_r}\right)^2 \alpha_r\cdot(\beta - tu\alpha)^2\,\mathrm{e}^{2K(\tau-r)}\mathrm{d}r\Big]\,\mathrm{d}s\,,
\end{aligned}$$

$$\begin{aligned}
(2DE) \;=& \int_0^1 -\frac{1}{\tau}\int_0^\tau (\beta - tu\alpha)\frac{u}{u_r}\frac{\tau}{N}\mathrm{e}^{2K(\tau-r)}\int \langle \nabla H_{\tau-r}\left(\varphi_s + tug^\varepsilon_{s,\tau}\right), \nabla g^\varepsilon_{s,r}\rangle f_{s,r}\,\mathrm{d}m\,\mathrm{d}r\mathrm{d}s \\
= & \int_0^1 \Big[\frac{1}{\tau}\int_0^\tau \frac{\tau}{N}\frac{u}{u_r}(\beta - tu\alpha)^2\,\mathrm{e}^{2K(\tau-r)} + \frac{\tau}{N}\frac{u}{u_r}(\beta - tu\alpha)\gamma^{(1)}\,\mathrm{e}^{2K(\tau-r)}\mathrm{d}r\Big]\,\mathrm{d}s\,,
\end{aligned}$$

where $\gamma^{(1)} = \int q_\varepsilon(f_{s,r})\langle \nabla H_{\tau-r}\left(\varphi_s + tug^\varepsilon_{s,\tau}\right), \nabla\sqrt{f_{s,r}}\rangle\,\mathrm{d}m$, and finally

$$\begin{aligned}
(D^2) &= \int_0^1 \left[\frac{1}{\tau}\int_0^\tau \frac{1}{2}\int \left|\nabla H_{\tau-r}\left(\varphi_s + tug^\varepsilon_{s,\tau}\right)\right|_w^2 f_{s,r}\,\mathrm{d}m\,\mathrm{e}^{2K(\tau-r)}\,\mathrm{d}r\right]\mathrm{d}s\\
&\le \int_0^1 \left[\frac{1}{\tau}\int_0^\tau \frac{1}{2}\int \left|\nabla\left(\varphi_s + tug^\varepsilon_{s,\tau}\right)\right|_w^2 f_{s,\tau}\,\mathrm{d}m\,\mathrm{d}r\right.\\
&\qquad \left. -\frac{1}{\tau}\int_0^\tau \frac{\tau-r}{N_0}\int \left|\Delta H_{\tau-r}\left(\varphi_s + tug^\varepsilon_{s,\tau}\right)\right|^2 f_{s,r}\,\mathrm{d}m\,\mathrm{e}^{2K(\tau-r)}\,\mathrm{d}r\right]\mathrm{d}s\\
&\le \int_0^1 \left[\frac{1}{\tau}\int_0^\tau \frac{1}{2}\int \left|\nabla\left(\varphi_s + tug^\varepsilon_{s,\tau}\right)\right|_w^2 f_{s,\tau}\,\mathrm{d}m\,\mathrm{d}r\right.\\
&\qquad \left. -\frac{1}{\tau}\int_0^\tau \frac{\tau-r}{N_0}\left|\int \Delta H_{\tau-r}\left(\varphi_s + tug^\varepsilon_{s,\tau}\right) f_{s,r}\,\mathrm{d}m\right|^2 \mathrm{e}^{2K(\tau-r)}\,\mathrm{d}r\right]\mathrm{d}s\\
&\le \int_0^1 \left[\frac{1}{2}\int |\nabla\varphi_s|_w^2 f_{s,\tau}\mathrm{d}m - tu\beta - tu\gamma^{(2)} + \frac{1}{2}t^2u^2\alpha - \frac{1}{\tau}\int_0^\tau \frac{\tau-r}{N_0}(\beta - tu\alpha)^2\,\mathrm{e}^{2K(\tau-r)}\mathrm{d}r\right]\mathrm{d}s
\end{aligned}$$

where $\gamma^{(2)} = \int q_\varepsilon(f_{s,\tau})\langle\nabla\varphi_s, \nabla\sqrt{f_{s,\tau}}\rangle\,\mathrm{d}m$ and where we applied again the Bakry–Ledoux estimate $\mathrm{BL}(K, N_0)$, now to the semigroup $H_{\tau-r}$. Summing up everything yields

$$(A) + (B) \le \int_0^1 \left[-\frac{1}{2}t^2u^2\alpha + \frac{1}{N}(\beta - tu\alpha)^2\cdot(G) + (H)\right]\mathrm{d}s$$

where

$$(H) := -tu\gamma^{(2)} + \int_0^\tau \frac{1}{N}\frac{u}{u_r}(\beta - tu\alpha)\gamma^{(1)}\,\mathrm{e}^{2K(\tau-r)}\mathrm{d}r\,,$$

and

$$\begin{aligned}
(G) &:= \int_0^\tau \left[-\frac{N}{N_0}\frac{r}{\tau}\left(1 + \frac{\tau}{N}\frac{u}{u_r}\alpha_r\right)^2 \mathrm{e}^{2K\tau} + \frac{\tau}{2N}\left(\frac{u}{u_r}\right)^2 \alpha_r\,\mathrm{e}^{2K(\tau-r)}\right.\\
&\qquad \left. + \frac{u}{u_r}\,\mathrm{e}^{2K(\tau-r)} - \frac{N}{N_0}\frac{\tau-r}{\tau}\,\mathrm{e}^{2K(\tau-r)}\right]\mathrm{d}r\\
&\le \int_0^\tau \left[\frac{N}{N_0}\frac{r}{\tau}\left(\mathrm{e}^{2|K|\tau} - \mathrm{e}^{-2|K|\tau}\right) - \frac{r}{N}\frac{u}{u_r}\alpha_r\mathrm{e}^{-2|K|\tau}\right.\\
&\qquad \left. + \frac{\tau}{2N}\left(\frac{u}{u_r}\right)^2\alpha_r\mathrm{e}^{2|K|\tau} + \frac{u}{u_r}\mathrm{e}^{2|K|\tau} - \frac{N}{N_0}\mathrm{e}^{-2|K|\tau}\right]\mathrm{d}r\\
&= \frac{\tau N}{2N_0}\left(\mathrm{e}^{2|K|\tau} - \mathrm{e}^{-2|K|\tau}\right) + \frac{\tau}{4}\left[\left(\frac{u}{u_0}\right)^2 - 1\right]\mathrm{e}^{2|K|\tau} + \tau\mathrm{e}^{-2|K|\tau}\left(1 - \frac{N}{N_0}\right)\\
&\qquad + \left(\mathrm{e}^{2|K|\tau} - \mathrm{e}^{-2|K|\tau}\right)\int_0^\tau \frac{u}{u_r}\mathrm{d}r\,.
\end{aligned}$$

Here we used that by Lemma 4.15 $\alpha_r \ge 0$, by Lemma 4.13 $\partial_r \frac{1}{u_r} = -\frac{1}{N\,u_r}\alpha_r$ and thus

$$0 > -\int_0^\tau \frac{r}{N}\frac{u}{u_r}\alpha_r\,\mathrm{d}r = \tau - \int_0^\tau \frac{u}{u_r}\,\mathrm{d}r$$

and

$$\frac{1}{N}\int_0^\tau \left(\frac{u}{u_r}\right)^2 \alpha_r\,\mathrm{d}r = \frac{1}{2}\left[\left(\frac{u_\tau}{u_0}\right)^2 - 1\right].$$

Since (ρ_s) is regular, $|\mathrm{Ent}(\rho_s)|$ and the second moments of $(\rho_s)_{s\in[0,1]}$ are uniformly bounded. Arguing as in the proof of Lemma 4.10 and using that $\tau_{s,t} \le c\cdot st$ we find that $\frac{u}{u_r}$ is bounded.

Taylor expansion of the exponentials in the estimate above thus yields, that for some constant C_2, depending only on K and the $\max_{s\in[0,1]}|\mathrm{Ent}(\rho_s)|$,

$$(G) \quad \leq \quad \frac{\tau}{4}\left[\left(\frac{u_\tau}{u_0}\right)^2 - 1 - 4\left(\frac{N}{N_0}-1\right)\right] + C_2\tau^2 \ .$$

To control (H) we estimate using Young inequality for any $\delta > 0$:

$$\begin{aligned}\gamma^{(2)} \quad &\leq \quad \frac{\delta}{8}I(\rho_{s,\tau}) + \frac{1}{2\delta}\int q_\varepsilon^2(f_{s,\tau})|\nabla\varphi_s|_w^2 \mathrm{d}m \ ,\\ \gamma^{(1)} \quad &\leq \quad \frac{\delta}{8}I(\rho_{s,r}) + \frac{1}{\delta}\int q_\varepsilon^2(f_{s,r})\Big(|\nabla H_{\tau-r}\varphi_s|_w^2 + t^2u^2|\nabla H_{\tau-r}g^\varepsilon_{s,\tau}|_w^2\Big)\mathrm{d}m \ .\end{aligned}$$

Note that $q_\varepsilon^2(r) \leq 4r$, $q_\varepsilon^2(r) \to 0$ as $\varepsilon \to 0$. Using the gradient estimate $\mathrm{BL}(K,\infty)$, (4.34) and (4.28) we estimate

$$\begin{aligned}\int f_{s,r}\Big(|\nabla H_{\tau-r}\varphi_s|_w^2 + t^2u^2|\nabla H_{\tau-r}g^\varepsilon_{s,\tau}|_w^2\Big)\mathrm{d}m \ &\leq \ \mathrm{e}^{-2K(\tau-r)}\int f_{s,\tau}\Big(|\nabla\varphi_s|_w^2 + t^2u^2|\nabla g^\varepsilon_{s,\tau}|_w^2\Big)\mathrm{d}m\\ &\leq \ \mathrm{e}^{-2K(\tau-r)}\Big(4\,\mathrm{Lip}(\varphi)^2 + t^2u^2I(\rho_{s,\tau})\Big) \ < \ \infty \ .\end{aligned}$$

Thus, dominated convergence yields that $\gamma^{(1)} \leq (\delta/8)I(\rho_{s,r}) + O(\varepsilon)$ and $\gamma^{(2)} \leq (\delta/8)I(\rho_{s,\tau}) + O(\varepsilon)$. It remains to estimate α,β. By Lemma 4.15 and (4.28) we have $\alpha \leq I(\rho_{s,\tau})$ and $\beta \leq 2\,\mathrm{Lip}(\varphi)\sqrt{I(\rho_{s,\tau})}$. Note that combining (2.17), (3.17) and K-contractivity of the heat flow we have $I(\rho_{s,r}) \leq \mathrm{e}^{-Kr}I(\rho_s)$ for any $r \geq 0$.

Putting everything together we conclude that there exist constants C_1, C_3 depending on K, $\max_{s\in[0,1]}|\mathrm{Ent}(\rho_s)|$, $\max_{s\in[0,1]}I(\rho_s)$ and φ such that

$$\begin{aligned}&\int\varphi_1\mathrm{d}\rho_{1,\tau^\varepsilon} - \int\varphi_0\mathrm{d}\rho_0 - \frac{1}{2}\int_0^1|\dot\rho_s|^2\mathrm{e}^{-2K\tau^\varepsilon}\mathrm{d}s + Nt\cdot[U_N^\varepsilon(\rho_0) - U_N^\varepsilon(\rho_{1,\tau^\varepsilon})]\\ \leq \ &\int_0^1 C_1\frac{\tau^\varepsilon}{4}\left[\left(\frac{U_N^\varepsilon(\rho_{s,\tau^\varepsilon})}{U_N^\varepsilon(\rho_s)}\right)^2 - 1 - 4\left(\frac{N}{N_0}-1\right) + C_2\tau^\varepsilon\right]\mathrm{d}s + C_3\delta + O(\varepsilon) \ ,\end{aligned}$$

where we have made the dependence of τ and u on ε explicit. Finally, passing to the limit first as $\varepsilon \to 0$ and then as $\delta \to 0$ yields (4.38). □

Proposition 4.17. *Assume that (X,d,m) satisfies $\mathrm{BL}(K,N)$. Then for each geodesic $(\rho_s)_{s\in[0,2]}$ in $\mathscr{P}_2(X,d,m)$ with $\rho_0,\rho_2 \in D(\mathrm{Ent})$ and $r \in [0,2]$ we have*

$$U_N(\rho_r) \ \geq \ \frac{2-r}{2}U_N(\rho_0) + \frac{r}{2}U_N(\rho_2) + \frac{K}{N}|\dot\rho|^2\cdot\int_0^2 g(s,r)U_N(\rho_s)\,\mathrm{d}s \tag{4.39}$$

where $g(s,r) = \frac{1}{2}\min\{s(2-r), r(2-s)\}$ denotes the Green function on the interval $[0,2]$.

Proof. We will only prove (4.39) for $r = 1$ the general argument being very similar. Obviously, it is sufficient to prove that the inequality (4.39) is satisfied with N replaced by N' for any $N' > N$ and then let $N' \to N$. So let us fix $N' > N$ and a geodesic $(\rho_s)_{s\in[0,2]}$ in $\mathscr{P}_2(X,d,m)$. Since we already know that (X,d,m) is a strong $\mathrm{CD}(K,\infty)$ space we have that $s \mapsto \mathrm{Ent}(\rho_s)$ is K-convex and thus continuous.

Using Lemma 4.11 we approximate the geodesic $(\rho_s)_{s\in[0,2]}$ by regular curves $(\rho_s^n)_{s\in[0,2]}$. Given $t > 0$, the estimate (4.38) from Proposition 4.16, with N_0, N replaced by N, N', holds true for each of the regular curves $(\rho_s^n)_{s\in[0,1]}$ and $(\rho_{2-s}^n)_{s\in[0,1]}$ and any Lipschitz function φ with bounded support. From the uniform convergence (4.25) in Lemma 4.11 and (4.19) we conclude that for all n large enough and t sufficiently small and all $s \in [0,1]$:

$$\left[\left(\frac{U_{N'}(\rho^n_{s,\tau^n})}{U_{N'}(\rho^n_s)}\right)^2 - 1 + C_2\tau^n\right] \ \leq \ 4\left(\frac{N'}{N} - 1\right) \ ,$$

i.e. the right hand side of (4.38) is non-positive. Hence we obtain

$$\int \varphi_1 \mathrm{d}\rho^n_{1,\tau^n} - \int \varphi_0 \mathrm{d}\rho^n_0 - \frac{1}{2}\int_0^1 |\dot{\rho^n_s}|^2 \mathrm{e}^{-2K\tau^n}\,\mathrm{d}s \;\le\; N't \cdot \left[U_{N'}(\rho^n_{1,\tau^n}) - U_{N'}(\rho^n_0)\right] \;,$$

for all such n and t. Taking the supremum over φ yields by Kantorovich duality

$$\frac{1}{2}W_2^2(\rho^n_0,\rho^n_{1,\tau^n}) - \frac{1}{2}\int_0^1 |\dot{\rho^n_s}|^2 \mathrm{e}^{-2K\tau^n}\,\mathrm{d}s \;\le\; N't \cdot \left[U_{N'}(\rho^n_{1,\tau^n}) - U_{N'}(\rho^n_0)\right] \;,$$

As $n \to \infty$, using the continuity properties (4.21)-(4.24) we obtain the same estimate for the geodesic $(\rho_s)_{s\in[0,1]}$.

$$\frac{1}{2}W_2^2(\rho_0,\rho_{1,\tau}) - \frac{1}{2}W_2^2(\rho_0,\rho_1)\cdot\int_0^1 e^{-2K\tau}\mathrm{d}s \;\le\; N't\cdot\left[U_{N'}(\rho_{1,\tau}) - U_{N'}(\rho_0)\right]\mathrm{d}s \;.$$

An analogous estimate holds true for the geodesic $(\rho_{2-s})_{s\in[0,1]}$

$$\frac{1}{2}W_2^2(\rho_2,\rho_{1,\tau}) - \frac{1}{2}W_2^2(\rho_2,\rho_1)\cdot\int_1^2 e^{-2K\tau}\mathrm{d}s \;\le\; N't\cdot\left[U_{N'}(\rho_{1,\tau}) - U_{N'}(\rho_2)\right]\mathrm{d}s \;.$$

Moreover, since $(\rho_s)_{s\in[0,2]}$ is a geodesic

$$\frac{1}{2}W_2^2(\rho_0,\rho_1) + \frac{1}{2}W_2^2(\rho_2,\rho_1) - \frac{1}{2}W_2^2(\rho_0,\rho_{1,\tau}) - \frac{1}{2}W_2^2(\rho_2,\rho_{1,\tau}) \;\le\; 0 \;.$$

Adding up the last three inequalities (and dividing by t) yields

$$\frac{1}{8}W_2^2(\rho_0,\rho_2)\cdot\frac{1}{t}\left[2 - \int_0^1 e^{-2K\tau}\mathrm{d}s - \int_1^2 e^{-2K\tau}\mathrm{d}s\right] \;\le\; N'\cdot\Big[2U_{N'}(\rho_{1,\tau}) - U_{N'}(\rho_0) - U_{N'}(\rho_2)\Big]\mathrm{d}s \;.$$

Lower semi-continuity of the entropy implies that in the limit $t \to 0$ the RHS will be bounded from above by

$$N'\cdot\left[2U_{N'}(\rho_1) - U_{N'}(\rho_0) - U_{N'}(\rho_2)\right] \;.$$

Finally, by the very definition of τ,

$$\begin{aligned}
\lim_{t\to 0}\frac{1}{t}\left[2 - \int_0^1 e^{-2K\tau}\mathrm{d}s - \int_1^2 e^{-2K\tau}\mathrm{d}s\right] &= -2K\int_0^2 \partial_t\tau_{s,t}\,\mathrm{d}s \\
&= -2K\left[\int_0^1 sU_{N'}(\rho_s)\mathrm{d}s + \int_1^2 (2-s)U_{N'}(\rho_s)\mathrm{d}s\right] \\
&= -4K\int_0^2 g(s,1)\,U_{N'}(\rho_s)\,\mathrm{d}s.
\end{aligned}$$

Thus we end up with

$$-\frac{K}{2}W_2^2(\rho_0,\rho_2)\cdot\int_0^2 g(s,1)\,U_{N'}(\rho_s)\,\mathrm{d}s \;\le\; N'\cdot\Big[2U_{N'}(\rho_1) - U_{N'}(\rho_0) - U_{N'}(\rho_2)\Big] \;.$$

Since $|\dot\rho|^2 = W_2^2(\rho_0,\rho_2)/4$, this proves the claim. □

Remark 4.18. A simple rescaling argument yields that for each geodesic $(\rho_s)_{s\in[0,1]}$ in $\mathscr{P}_2(X,d,m)$ with $\rho_0,\rho_1 \in D(\mathrm{Ent})$ and $r \in [0,1]$:

$$U_N(\rho_r) \;\ge\; (1-r)\cdot U_N(\rho_0) + r\cdot U_N(\rho_1) + \frac{K}{N}|\dot\rho|^2\cdot\int_0^1 g\,(s,r)\,U_N(\rho_s)\,\mathrm{d}s \qquad (4.40)$$

where $g(s,r) = \min\{s(1-r), r(1-s)\}$ now denotes the Green function on the interval $[0,1]$.

Theorem 4.19. *Let (X,d,m) be a infinitesimally Hilbertian mms satisfying the exponential integrability condition* (3.6) *and* $\mathrm{BL}(K,N)$. *Then the strong* $\mathrm{CD}^e(K,N)$ *condition holds. In particular, (X,d,m) is a* $\mathrm{RCD}^*(K,N)$ *space and the heat flow satisfies* $\mathrm{EVI}_{K,N}$.

Proof. By virtue of Lemma 2.8, this is merely a consequence of Proposition 4.17 and (4.40). □

Remark 4.20. In the special case $K = 0$ it turns out to be possible to derive the $\mathrm{EVI}_{0,N}$ property directly from the action estimate in Proposition 4.16. Let us give an alternative argument in this case.

We want to show that for any $\rho, \sigma \in \mathscr{P}_2(X, d)$ we have for all $t > 0$:

$$\frac{\mathrm{d}^+}{\mathrm{d}t}\frac{1}{2}W_2^2(H_t\rho, \sigma) \;\leq\; N \cdot \left[1 - \frac{U_N(\sigma)}{U_N(H_t\rho)}\right]. \tag{4.41}$$

Obviously, it is sufficient to prove that (4.41) is satisfied for any $N' > N$ and then let $N' \to N$. Moreover, by the semigroup property and Proposition 2.18 it is sufficient to assume that $\rho, \sigma \in D(\mathrm{Ent})$ and show that (4.41) holds at $t = 0$. So let us fix $N' > N$ and a geodesic $(\rho_s)_{s\in[0,1]}$ in $\mathscr{P}_2(X, d, m)$ connecting $\rho_0 = \sigma$ to $\rho_1 = \rho$. Since we already know that (X, d, m) is a strong $\mathrm{CD}(0, \infty)$ space we have that $s \mapsto \mathrm{Ent}(\rho_s)$ is convex and thus continuous. By approximating the geodesic (ρ_s) by regular curves one can show as in the proof of Proposition 4.17 that

$$\frac{1}{t}\left[\frac{1}{2}W_2^2(\rho_0, \rho_{1,\tau}) - \frac{1}{2}W_2^2(\rho_0, \rho_1)\right] \;\leq\; N' \cdot \left[U_{N'}(\rho_{1,\tau}) - U_{N'}(\rho_0)\right] .$$

Thus passing to the limit $t \to 0$ yields

$$\frac{\mathrm{d}^+}{\mathrm{d}t}\frac{1}{2}W_2^2(\rho_0, H_t\rho_1)\Big|_{t=0} \cdot \frac{d}{dt}\tau_{1,t}\Big|_{t=0} \;=\; \frac{\mathrm{d}^+}{\mathrm{d}t}\frac{1}{2}W_2^2(\rho_0, H_{\tau_{1,t}}\rho_1)\Big|_{t=0} \;\leq\; N' \cdot \left[U_{N'}(\rho_1) - U_{N'}(\rho_0)\right] .$$

Since $\frac{d}{dt}\tau_{1,t}\Big|_{t=0} = U_{N'}(\rho_1)$, this finally yields the $\mathrm{EVI}_{0,N'}$ inequality:

$$\frac{\mathrm{d}^+}{\mathrm{d}t}\frac{1}{2}W_2^2(\rho_0, H_t\rho_1)\Big|_{t=0} \;\leq\; N' \cdot \left[1 - \frac{U_{N'}(\rho_0)}{U_{N'}(\rho_1)}\right] .$$

To finish this section let us consider the classical case of weighted Riemannian manifolds. More precisely, let (M, d) be a n-dimensional smooth, complete Riemannian manifold and let $V : M \to \mathbb{R}$ be a smooth function bounded below. Consider the metric measure space $(M, d, \mathrm{e}^{-V}\mathrm{vol})$. The associated weighted Laplacian is given by

$$Lu \;=\; \Delta u - \nabla V \cdot \nabla u .$$

It is well known (see e.g. [40, Thm. 14.8]) that the operator L satisfies the Bakry–Émery condition $\mathrm{BE}(K, N)$ if and only if the generalized Ricci tensor

$$\mathrm{Ric}_{N,V} \;:=\; \mathrm{Ric} + \mathrm{Hess}\, V - \frac{1}{N - n}\nabla V \otimes \nabla V$$

is bounded below by K. As an immediate consequence of our equivalence result we thus obtain the following

Proposition 4.21. *The mms* $(M, d, \mathrm{e}^{-V}vol)$ *satisfies the* $\mathrm{CD}^e(K, N)$*-condition if and only if*

$$\mathrm{Ric} + \mathrm{Hess}\, V \;\geq\; K + \frac{1}{N - n}\nabla V \otimes \nabla V .$$

4.3. **The sharp Lichnerowicz inequality (spectral gap).** Here we provide a first application of the Bochner formula on infinitesimally Hilbertian metric measure spaces. Namely we establish the sharp spectral gap estimate on $\mathrm{RCD}^*(K, N)$ spaces in the case of positive curvature $K > 0$.

We consider an infinitesimally Hilbertian metric measure space (X, d, m). Recall that we denote by Δ the canonical Laplacian on (X, d, m), i.e. the generator of the heat semigroup in L^2 which is given as the L^2-gradient flow of the Cheeger energy Ch, see Section 3.2.

Theorem 4.22 (Spectral gap estimate)**.** *Let* (X, d, m) *be a mms satisfying the Riemannian curvature dimension condition* $\mathrm{RCD}^*(K, N)$ *with* $K > 0$ *and* $N > 1$. *Then the spectrum of* $(-\Delta)$ *is discrete and the first non-zero eigenvalue* $\lambda_1(X, d, m)$ *satisfies the following bound:*

$$\lambda_1(X, d, m) \;\geq\; \frac{N}{N - 1}K . \tag{4.42}$$

Proof. First recall that the $\mathrm{RCD}^*(K,N)$ condition with $K>0$ implies that (X,d,m) is doubling by Proposition 3.6 and compact by Corollary 3.7. In combination with the result in [36] this yields that (X,d,m) supports a global Poincaré inequality. Moreover, the $\mathrm{CD}^*(K,N)$ condition implies a global Sobolev inequality, by adapting [40, Thm. 30.23]. These ingredients yield the following Rellich–Kondrachov compactness property(c.f. [22, Thm. 8.1]): for any sequence of functions $(f_n)_n \subset W^{1,2}(X,d,m)$ with

$$\sup_n \left(\|f_n\|_{L^2(X,m)} + \mathrm{Ch}(f_n)\right) \ < \ \infty$$

we have that up to extraction of a subsequence $f_n \to f$ in $L^2(X,m)$ for some $f \in L^2(X,m)$. This compactness theorem is sufficient to prove that the spectrum of $(-\Delta)$ is discrete, e.g. by following verbatim the proof in [12] of the corresponding result for Riemannian manifolds.

For the eigenvalue estimate we follow the argument in [17]. Let $\lambda>0$ be a non-zero eigenvalue of $(-\Delta)$ and let $\psi \in D(\Delta)$ be a corresponding eigenfunction. We apply the Bochner inequality of Theorem 4.8 to $f=\psi$ and the test function $g\equiv 1$. Note that this pair is admissible since X is compact. Thus we obtain using the integration by parts formula (3.21):

$$\begin{aligned} 0 \ &\geq \ \int \langle \nabla(\Delta\psi), \nabla\psi\rangle \mathrm{d}m + K\int |\nabla\psi|_w^2 \mathrm{d}m + \frac{1}{N}\int (\Delta\psi)^2 \mathrm{d}m \\ &= \ (K-\lambda)\int |\nabla\psi|_w^2 \mathrm{d}m - \frac{\lambda}{N}\int \psi\Delta\psi \mathrm{d}m \\ &= \ \left(K-\lambda+\frac{\lambda}{N}\right)\int |\nabla\psi|_w^2 \mathrm{d}m \ . \end{aligned}$$

Since $\mathrm{Ch}(\psi)>0$ it follows that $\lambda \geq KN/(N-1)$ which yields the claim. □

Note that this estimate of the spectral gap is sharp. This can be seen by considering the model space

$$X = \left(-\frac{\pi}{2}\sqrt{\frac{N-1}{K}}, \frac{\pi}{2}\sqrt{\frac{N-1}{K}}\right)\,, \quad d(x,y) = |x-y|\,, \quad m(\mathrm{d}x) = \cos\left(x\sqrt{\frac{K}{N-1}}\right)^{N-1} \mathrm{d}x \ .$$

The corresponding operator is given by

$$Lf(r) = f''(r) - \sqrt{K(N-1)}\tan\left(r\sqrt{K/(N-1)}\right) f'(r)$$

with Neumann boundary conditions. By Proposition 4.21 the metric measure space (X,d,m) satisfies $\mathrm{RCD}^*(K,N)$. It is well known that the first non-zero eigenvalue of the Neumann problem associated to L is given by $KN/(N-1)$.

5. Dirichlet form point of view

Up to now we have formulated our results in the setting of metric measure spaces. Here the Cheeger energy, if assumed to be a quadratic form, gives rise to a canonical Dirichlet form. In this final section we take a different point of view and reformulate our results starting from a Dirichlet form. The relation between the two points of view and the compatibility of metric measure structures and Energy structures has been discussed extensively in [5] as well as in [25].

Let X be a Polish space and let m be a locally finite Borel measure on X. Let $\mathcal{E}$ be a strongly local Dirichlet form on $L^2(X,m)$ with domain $D(\mathcal{E})$. Denote the associated Markov semigroup in $L^2(X,m)$ by $(P_t)_{t>0}$ and its generator by Δ. Given a function $f \in D(\mathcal{E})$ we denote by $\Gamma(f)$ the associated energy measure defined by the relation

$$\int \varphi \mathrm{d}\Gamma(f) \ = \ \mathcal{E}(f, f\varphi) - \frac{1}{2}\mathcal{E}(f^2,\varphi) \quad \forall \varphi \in D(\mathcal{E})\cap L^\infty(X,m)\ .$$

If $\Gamma(f)$ is absolutely continuous w.r.t. m we will also denote its density with $\Gamma(f)$. The natural notion of a (pseudo-)distance on X associated to $\mathcal{E}$ is the intrinsic $d_{\mathcal{E}}$ defined by

$$d_{\mathcal{E}}(x,y) \ := \ \sup\left\{|f(x)-f(y)| \ : \ f \in D(\mathcal{E})\cap C(X), \Gamma(f)\leq m\right\} \ .$$

For the sequel, assume that $d_{\mathcal{E}}$ is a finite, complete distance on X inducing the given topology and assume that $(X, d, m, \mathcal{E})$ is upper regular energy measure space in the sense of [5, Def.3.6, Def. 3.13].

Corollary 5.1. *Under the previous assumptions, the following are equivalent:*

(i) *Assumption 4.2 and* $\mathrm{BL}(K, N)$ *holds, i.e. for any* $f \in D(\mathcal{E})$ *with* $\Gamma(f) \leq m$ *and* $t > 0$, f *is 1-Lipschitz and*

$$|\Gamma P_t f|^2 + \frac{1 - \mathrm{e}^{-2Kt}}{NK} |\Delta P_t f|^2 \;\leq\; e^{-2Kt} P_t \Gamma(f) \,.$$

(ii) $(X, d_{\mathcal{E}}, m)$ *is an* $\mathrm{RCD}^*(K, N)$ *space.*

Proof. Under the assumptions on $d_{\mathcal{E}}$ and $\mathcal{E}$, it is shown in [5, Thm. 3.14] that $\mathcal{E}$ coincides with the Cheeger energy on $(X, d_{\mathcal{E}}, m)$. Thus $(X, d_{\mathcal{E}}, m)$ is infinitesimally Hilbertian and for any $f \in D(\mathcal{E})$ we have $\Gamma(f) \ll m$ with density $|\nabla f|_w^2$. The equivalence of (i) and (ii) then follows from Theorems 4.19, 4.3. □

Remark 5.2. According to [5, Cor. 2.3] conditions (i) and (ii) of the previous result are in turn equivalent to the Bakry–Émery inequality $\Gamma_2(f) \geq K\Gamma(f) + \frac{1}{N}(\Delta f)^2$ in the form of $\mathrm{BE}(K, N)$, see Definition 4.7.

Note added in proof. *Since the first version of this article was published on arxiv, several remarkable follow-up papers appeared. Garofalo and Mondino have* [18] *have established the Li–Yau estimates on metric measure spaces satisfying* $\mathrm{RCD}^*(K, N)$*. Contraction properties of the heat flow reflecting dimensional effects have been exhibited by Bolley, Gentil and Guillin* [13]*, their approach however being very different from ours, based on a new transportation distance instead of the* L^2*-Wasserstein distance. The concept of* (K, N)*-convexity has been adopted by Naber* [30] *in the study of upper and lower Ricci bounds on metric measure spaces and the relation with spectral gaps on the associated path space*

The authors also would like to mention the closely related, independent work in progress of Ambrosio, Mondino and Savaré [7]*, where partly similar results as in the present article are obtained via a study of the porous medium equation in metric measure spaces.*

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