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Schrödinger operators and Feynman-Kac semigroups with arbitrary nonnegative potentials

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Abstract. We give a brief survey on the analytic and probabilistic approach to Schrödinger operators $-\frac{1}{2}\Delta+V$ with arbitrary potentials $V \geq 0$ and on the canonical generalizations to more general "positive perturbations" of the free Hamiltonian. To be more precise, we investigate generalized Schrödinger operators $H^\mu = -\frac{1}{2}\Delta + \mu$ with measures μ charging no polar sets and compare them with generalized Feynman-Kac semigroups $(P_t^A)_{t>0}$ derived from additive functionals A. In fact, there is a canonical one-to-one correspondence between generalized Schrödinger semigroups $\left(e^{-tH^\mu}\right)_{t>0}$ and generalized Feynman-Kac semigroups $\left(P_t^A\right)_{t>0}$.

Introduction

In order to investigate Schrödinger operators $-\frac{1}{2}\Delta+V$ with singular potentials $V \geq 0$, there are two robust quantities to start with:
one is the symmetric form

$$(*) \qquad \mathcal{E}^V(f,g) := \frac{1}{2}\int \nabla f \nabla g \, dm + \int fg\, V\, dm,$$

the other is the Feynman-Kac semigroup

$$(**) \qquad P_t^V f(x) := \mathbb{E}^x\left[e^{-\int_0^t V(X_s)\,ds} \cdot f(X_t)\right].$$

We shall see that for all potentials $V \geq 0$ both approaches lead to the same result, — without any finiteness or integrability assumption on V. Both approaches also admit to treat more general perturbations in the same way as perturbations by functions $V \geq 0$.

In the analytic case, one can replace the measures $V \cdot m$ in the definition of the symmetric form $(*)$ by general measures μ on $\mathbb{R}^d$. The appropriate condition on the measures μ is that they do not charge polar sets. Starting with the symmetric forms $\mathcal{E}^\mu$ associated with such measures μ, we obtain generalized Schrödinger operators $H^\mu = -\frac{1}{2}\Delta + \mu$ and generalized Schrödinger semigroups $(e^{-tH^\mu})_{t>0}$. The reasons for considering not only usual Schrödinger operators $-\frac{1}{2}\Delta + V$ with potentials $V \geq 0$ but

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also generalized Schrödinger operators $-\frac{1}{2}\Delta + \mu$ with measures μ charging no polar sets are that

- the class of these generalized Schrödinger operators turns out to be closed with respect to strong resolvent convergence
- this class contains all the Dirichlet Laplacians on open subsets of $\mathbb{R}^d$
- in this class, the set of usual Schrödinger operators $-\frac{1}{2}\Delta + V$ with smooth potentials $V \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ is dense with respect to strong resolvent convergence.

In the probabilistic case, the additive functionals $t \mapsto \int_0^t V(X_s)\, ds$ in the definition of the Feynman-Kac semigroup (**) can be replaced by general additive functionals $A : t \mapsto A_t$ which leads to generalized Feynman-Kac semigroups $(P_t^A)_{t>0}$. One aim of this paper is to illustrate that there is a canonical one-to-one correspondence between additive functionals A (or, which comes to the same thing, multiplicative functionals) and measures μ charging no polar sets. This equivalence extends to (or comes from) a one-to-one correspondence between generalized Feynman-Kac semigroups $(P_t^A)_{t>0}$ derived from additive functionals A and generalized Schrödinger semigroups (e^{-tH^μ}) derived from measures μ.

From this correspondence one can deduce nice approximation results for these semigroups (via monotone convergence as well as via strong resolvent convergence). One obtains, for instance, also a characterization of the form domains $\mathcal{D}(\mathcal{E}^\mu)$, in particular, criteria for $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ being densely defined. Note that, in general, the form $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ is not densely defined on $L^2(\mathbb{R}^d, m)$. Let us mention some results in terms of the set $E^\mu \subset \mathbb{R}^d$ of permanent points for μ (see chap. 3) which plays a crucial rôle in various places:

- $\mathcal{D}(\mathcal{E}^\mu)$ is dense in $W^{1,2}(\mathbb{R}^d)$ if and only if $\text{cap}(\mathbb{R}^d \setminus E^\mu) = 0$
- $\mathcal{D}(\mathcal{E}^\mu)$ is dense in $L^2(\mathbb{R}^d, m)$ if and only if $m(\mathbb{R}^d \setminus E^\mu) = 0$.

It should be mentioned that many of the results presented in this paper actually hold true in a much more general context. Several of them are valid for perturbations of Dirichlet forms or symmetric Hunt processes on locally compact spaces, some of them even for perturbations of right processes on general state spaces. For recent generalizations in these directions, we refer to articles by S. Albeverio, Ph. Blanchard and Z. M. Ma [2], R. K. Getoor [22], K. Kuwae [26] and P. Stollmann and J. Voigt [34]. In the present paper, we restrict ourselves to the most easiest (and most important) case of perturbations of the classical Dirichlet form and the Brownian motion on $\mathbb{R}^d$. The emphasis is on describing the most general kind of "positive perturbations" of these objects.

In detail we proceed as follows: in the first chapter, we carry out the analytic approach. The probabilistic approach is described in chapter 2. The third chapter is devoted to the investigation of the set $E^\mu \subset \mathbb{R}^d$ of permanent points for a measure μ. (Let us mention that parts of chapters 1 and 3 dealing with the analytic point of view already appeared in [38].) At the end of the paper, we add a brief appendix on potential theoretic notions (like "regular", "fine", "quasi-", "polar", "capacity") which we use in the text without explicit definition. These notions are always understood with respect to the Laplace operator (or, equivalently, the Brownian motion) on $\mathbb{R}^d$.

Throughout this paper, m denotes the Lebesgue measure on $\mathbb{R}^d$. If not specified otherwise, *measurable* means measurable with respect to m. The *Borel* σ-field in $\mathbb{R}^d$ is denoted by $\mathcal{B}(\mathbb{R}^d)$. We recall that a measure μ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ is a *Radon measure* iff $\mu(K) < \infty$ for all compact sets $K \subset \mathbb{R}^d$.

1 The Analytic Approach

1.1 Schrödinger Operators $-\frac{1}{2}\Delta + V$ with Arbitrary Potentials $V \geq 0$.

The most reasonable analytic way to define a Schrödinger operator $H^V = -\frac{1}{2}\Delta + V$ for an arbitrary measurable potential $V \geq 0$ is to define H^V as the *form sum* of the free Hamiltonian $H^0 = -\frac{1}{2}\Delta$ and the operator of multiplication by V. Let us summarize some of the main steps of this approach. The precise results will be stated in the sequel in a more general context.

The quantity to start with is the nonnegative bilinear symmetric form

$$\mathcal{E}^V(f,g) := \frac{1}{2}\int \nabla f \nabla g \, dm + \int fg\, V\, dm$$

$$\mathcal{D}(\mathcal{E}^V) := W^{1,2}(\mathbb{R}^d) \cap L^2(\mathbb{R}^d, V \cdot m).$$

Note that for every measurable function $V \geq 0$ this defines a *closed* form $(\mathcal{E}^V, \mathcal{D}(\mathcal{E}^V))$ on $L^2(\mathbb{R}^d, m)$, — without any integrability or finiteness assumption on V!

In general, of course, this form will *not* be *densely defined* on $L^2(\mathbb{R}^d, m)$. However, there always exists a measurable set $E^V \subset \mathbb{R}^d$ (*set of permanent points for* V) such that

$$L^2_0(E^V, m) := \left\{ f \in L^2(\mathbb{R}^d, m) : \ f = 0 \ m\text{-a.e. on } \mathcal{C}E^V \right\}$$

is the closure of $\mathcal{D}(\mathcal{E}^V)$ in $L^2(\mathbb{R}^d, m)$. Let us mention that if V is locally integrable on $\mathbb{R}^d$ then $m(\mathbb{R}^d \setminus E^V) = 0$ and, hence, $(\mathcal{E}^V, \mathcal{D}(\mathcal{E}^V))$ is densely defined on $L^2(\mathbb{R}^d, m)$. For general potentials $V \geq 0$ we conclude that $(\mathcal{E}^V, \mathcal{D}(\mathcal{E}^V))$ is a densely defined, closed form on the Hilbert space $L^2_0(E^V, m)$.

Therefore, there always exists a unique nonnegative selfadjoint operator $(H^V, \mathcal{D}(H^V))$ on $L^2_0(E^V, m)$ which corresponds to $(\mathcal{E}^V, \mathcal{D}(\mathcal{E}^V))$ in the sense that $\mathcal{D}(H^V) \subset \mathcal{D}(\mathcal{E}^V)$ and for all $f \in \mathcal{D}(H^V)$

$$(*) \qquad \int H^V f \cdot g \, dm = \mathcal{E}^V(f,g) \qquad \text{for all } g \in \mathcal{D}(\mathcal{E}^V).$$

This operator H^V is the *form sum* of $H^0 = -\frac{1}{2}\Delta$ and V and is called *Schrödinger operator*. It will also be denoted by $-\frac{1}{2}\Delta + V$. Note that for $f \in \mathcal{D}(H^V) \cap \mathcal{C}^2(\mathbb{R}^d)$ one actually obtains from (*) by means of Green's formula

$$\int H^V f \cdot g\, dm \;=\; \int \left(-\frac{1}{2}\Delta f + Vf\right) \cdot g\, dm \qquad \text{for all } g \in \mathcal{D}(\mathcal{E}^V).$$

There also always exists a unique strongly continuous contraction semigroup $(e^{-tH^V})_{t>0}$ on $L_0^2(E^V, m)$ with generator $-H^V$. This semigroup trivially extends to a contraction semigroup on $L^2(\mathbb{R}^d, m)$, called *Schrödinger semigroup* and also denoted by $(e^{-tH^V})_{t>0}$.

It turns out that for *every* measurable potential $V \geq 0$ the Schrödinger semigroup $(e^{-tH^V})_{t>0}$ on $L^2(\mathbb{R}^d, m)$ (defined analytically by means of the form sum $H^V = -\frac{1}{2}\Delta + V$) coincides with the extension to $L^2(\mathbb{R}^d, m)$ of the probabilistically defined *Feynman-Kac semigroup* $(P_t^V)_{t>0}$.

The alternative way to define $-\frac{1}{2}\Delta + V$ analytically is to define it as the *operator sum.* If the potential V is locally integrable, this leads to the same selfadjoint operator as defined above. In the general case, however, this alternative approach is not satisfactory: one obtains neither the existence of a selfadjoint (extension of this) operator nor the existence of a reasonable semigroup associated to it.

1.2 Symmetric Forms with Measures Charging No Polar Sets.

If one wants to study "positive perturbations" of the free energy $\mathcal{E}^0$ which are more general than the above forms $\mathcal{E}^V$ with functions $V \geq 0$ one is lead in a natural way to consider forms of the type

$$\mathcal{E}^\mu(f,g) \;:=\; \frac{1}{2}\int \nabla f \nabla g\, dm + \int fg\, d\mu$$

where μ is some measure on $\mathbb{R}^d$. If μ charges no sets of Lebesgue measure 0 (that is, if $\mu = V \cdot m$ with a measurable function $V \geq 0$), this leads to the previous case. In some sense the minimal assumption on μ should be that it charges no polar sets.

(1.1) Definition. We say that a *measure* μ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ *charges no polar sets* if and only if

$$\mu(F) = 0 \qquad \text{for every polar set } F \in \mathcal{B}(\mathbb{R}^d).$$

The set of all measures on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ which do not charge polar sets will be denoted by $\mathcal{M}_0$.

For instance, the d-dimensional Lebesgue measure m and the δ-dimensional Hausdorff measures for $\delta > d-2$ are in $\mathcal{M}_0$ (cf. [3], [27]). For any measure $\mu \in \mathcal{M}_0$ and any Borel function $f \geq 0$ on $\mathbb{R}^d$ the measure $f \cdot \mu$ (having density f with respect to μ) is in $\mathcal{M}_0$, too. Special attention should be given to the fact that measures in $\mathcal{M}_0$ have neither to be regular nor to be σ-finite, in particular, they are not assumed to be Radon measures. A typical example is the measure

$$\overline{\infty}(.) := \infty \cdot \mathrm{cap}(.) :\; F \mapsto \begin{cases} 0, & \text{if } F \text{ is polar,} \\ \infty, & \text{else.} \end{cases}$$

By this, $\mathcal{M}_0$ can be characterized as the set of all measures on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ which are absolutely continuous with respect to $\overline{\infty}$.

We are now going to show that the condition $\mu \in \mathcal{M}_0$ *suffices* to define the form $\mathcal{E}^\mu$ on a reasonable domain $\mathcal{D}(\mathcal{E}^\mu)$. In order to see this, we make the following

(1.2) Remarks. a) Let $\overline{\mathcal{B}}(\mathbb{R}^d)$ denote the completion of $\mathcal{B}(\mathbb{R}^d)$ with respect to the measure $\overline{\infty}$, the so-called σ-field of *nearly Borel* sets. Then every measure $\mu \in \mathcal{M}_0$ can be extended in a trivial way to a measure on $(\mathbb{R}^d, \overline{\mathcal{B}}(\mathbb{R}^d))$. Note that every quasi-open or quasi-closed (in particular, every polar) set belongs to $\overline{\mathcal{B}}(\mathbb{R}^d)$. Similarly, all *quasi-continuous* functions on $\mathbb{R}^d$ are $\overline{\mathcal{B}}(\mathbb{R}^d)$-measurable, hence, they are μ-measurable for every $\mu \in \mathcal{M}_0$.

b) Every element f in the Sobolev space $W^{1,2}(\mathbb{R}^d)$ has a quasi-continuous version $\tilde{f}$. That is, there exists a quasi-continuous function $\tilde{f}$ which coincides m-a.e. with f. Such a function $\tilde{f}$ is q.e. uniquely determined (and can be defined arbitrarily on a polar set). Actually, one can choose $\tilde{f}$ to be the Lebesgue mean of f, i.e.

$$\tilde{f}(x) = \lim_{\epsilon \to 0} \frac{1}{m(B_\epsilon(x))} \int_{B_\epsilon(x)} f(y)\, dy$$

which converges for q.e. $x \in \mathbb{R}^d$ and coincides with $f(x)$ for m-a.e. $x \in \mathbb{R}^d$.

(1.3) Definition. For any measure $\mu \in \mathcal{M}_0$ we define the nonnegative symmetric form

$$\mathcal{E}^\mu(f,g) := \frac{1}{2}\int \nabla f \nabla g\, dm + \int fg\, d\mu$$

$$\mathcal{D}(\mathcal{E}^\mu) := \left\{ f \in W^{1,2}(\mathbb{R}^d) :\ \tilde{f} \in L^2(\mathbb{R}^d, \mu) \right\}.$$

According to the previous remarks, this form is always well-defined. The main observation ([33], Theorem 1.2) is

(1.4) Theorem. *For every $\mu \in \mathcal{M}_0$ the form $(Q^\mu, \mathcal{D}(\mathcal{E}^\mu))$ is a closed form on $L^2(\mathbb{R}^d, m)$.*

(1.5) Remarks. a) In general, the form $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ with $\mu \in \mathcal{M}_0$ is not *densely defined* on $L^2(\mathbb{R}^d)$, in particular, it is not *regular*. Concerning the question of being densely defined, we note that, of course, the form is always densely defined on the closure of $\mathcal{D}(\mathcal{E}^\mu)$ in $L^2(\mathbb{R}^d, m)$, cf. section 1.3. Actually we will prove that this closure coincides with the set

$$L_0^2(E^\mu, m) := \left\{ f \in L^2(\mathbb{R}^d, m) :\ f = 0\ m\text{-a.e. on } \mathcal{C}E^\mu \right\}$$

where E^μ denotes the set of permanent points for μ, cf. chapter 3.

b) For a measure $\mu \in \mathcal{M}_0$ the form $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ is regular if and only if μ is a *Radon measure* ([4], Theorem 2.2.2). In this case, the set

$$\mathcal{C}_0^\infty(\mathbb{R}^d) \text{ is a core for } (\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu)).$$

In other words, the symmetric form $(\mathcal{E}^\mu, C_0^\infty(\mathbb{R}^d))$ is closable on $L^2(\mathbb{R}^d, m)$ with closure $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$.

c) Now let conversely μ be an arbitrary Radon measure on $\mathbb{R}^d$. In order that

$$(\mathcal{E}^\mu, C_0^\infty(\mathbb{R}^d)) \text{ is closable on } L^2(\mathbb{R}^d, m)$$

it is *necessary* (and, as already stated, also sufficient) that μ *does not charge polar sets*, i.e. that $\mu \in \mathcal{M}_0$ ([3]; [12]; [27] Theorem 12.4/1).

We close this section with an important

(1.6) Example. If $\mu = 1_{\complement G} \cdot \overline{\infty}$ with a nearly Borel set $G \subset \mathbb{R}^d$, then

$$\mathcal{D}(\mathcal{E}^\mu) = W_0^{1,2}(G) := \left\{ f \in W^{1,2}(\mathbb{R}^d) : \tilde{f} = 0 \text{ q.e. on } \complement G \right\}.$$

For more details concerning the Sobolev spaces $W_0^{1,2}(G)$ for not necessarily open sets $G \subset \mathbb{R}^d$ we refer to [19]. We restrict ourselves to the following

(1.7) Remarks. a) If G is an open set, then this definition of the Sobolev space $W_0^{1,2}(G)$ coincides with the usual one, namely to be the closure of $C_0^\infty(G)$ in $W^{1,2}(\mathbb{R}^d)$.

b) For arbitrary nearly Borel sets $G \subset \mathbb{R}^d$, the Sobolev space $W_0^{1,2}(G)$ coincides with $W_0^{1,2}(\text{reg}(G))$. Note that in general $G \setminus \text{reg}(G)$ need not to be polar. For instance, if $G = \overline{B_r(x)}$ then $\text{reg}(G) = B_r(x)$.

c) For quasi-open sets $G \subset \mathbb{R}^d$, the measure $\mu = 1_{\complement G} \cdot \overline{\infty}$ can be used to produce complete absorption on the complement of G. In particular, it can be used to simulate homogeneous Dirichlet "boundary" conditions on $\complement G$.

d) One might be tempted to simulate Dirichlet "boundary" conditions on $\complement G$ also by a potential V which is $\equiv \infty$ on $\complement G$ and $\equiv 0$ in G, i.e. by $\mu = V \cdot m$ with $V = 1_{\complement G} \cdot \infty$. This, however, leads to

$$\mathcal{D}(\mathcal{E}^\mu) = \left\{ f \in W^{1,2}(\mathbb{R}^d) : f = 0 \text{ } m\text{-a.e. on } \complement G \right\}$$

which in general is a proper superset of $W_0^{1,2}(G)$, even if G is assumed to be open. It coincides with $W_0^{1,2}(G)$ (and produces the right boundary condition) if and only if the measures $1_{\complement G} \cdot \overline{\infty}$ and $1_{\complement G} \cdot \infty \cdot m$ are *equivalent* (cf. section 1.4 and [23]).

1.3 Schrödinger Operators and Schrödinger Semigroups

We recall that for arbitrary $\mu \in \mathcal{M}_0$ the closed form $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ is in general not densely defined on $L^2(\mathbb{R}^d)$. One goal will be to characterize the closure $\overline{\mathcal{D}}(\mathcal{E}^\mu)$ of $\mathcal{D}(\mathcal{E}^\mu)$ in $L^2(\mathbb{R}^d, m)$ and to state necessary and sufficient criteria for $\overline{\mathcal{D}}(\mathcal{E}^\mu) = L^2(\mathbb{R}^d, m)$, that is, for $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ being densely defined on $L^2(\mathbb{R}^d, m)$. For instance, in chapter 3 we will prove that this is the case if μ is a Radon measure on $\mathbb{R}^d$ (or more generally a Radon measure on an open set $G \subset \mathbb{R}^d$ with $m(\mathbb{R}^d \setminus G) = 0$). In general, $\overline{\mathcal{D}}(\mathcal{E}^\mu)$ will turn out to coincide with the set

$$L^2_0(E^\mu, m) := \left\{ f \in L^2(\mathbb{R}^d, m) : \ f = 0 \ m\text{-a.e. on } \complement E^\mu \right\}$$

where E^μ denotes the set of permanent points for μ.

At the moment we restrict ourselves with the fact that $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ is always a densely defined, closed form on the Hilbert space $\overline{\mathcal{D}}(\mathcal{E}^\mu)$ (equipped with the inner product of $L^2(\mathbb{R}^d, m)$).

Since there is a one-to-one correspondence between closed symmetric forms and self-adjoint operators (cf. [25] and [30]) we obtain

(1.8) Theorem. *For every $\mu \in \mathcal{M}_0$ there exists a unique nonnegative selfadjoint operator $(H^\mu, \mathcal{D}(H^\mu))$ on $\overline{\mathcal{D}}(\mathcal{E}^\mu)$ which corresponds to $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ in the sense that $\mathcal{D}(H^\mu) \subset \mathcal{D}(\mathcal{E}^\mu)$ and*

$$\int H^\mu f \cdot g \, d\mu = \mathcal{E}^\mu(f, g) \qquad \forall \, f \in \mathcal{D}(H^\mu), \ g \in \mathcal{D}(\mathcal{E}^\mu).$$

This operator H^μ is the *form sum* of the free Hamiltonian $H^0 = -\frac{1}{2}\Delta$ and of the operator of integration with respect to μ. It will also be denoted by $-\frac{1}{2}\Delta + \mu$ and is called the (generalized) *Schrödinger operator* associated with μ.

(1.9) Examples. a) If $\mu = V \cdot m$ with $V \in L^1_{loc}(\mathbb{R}^d, m)$ then H^μ is the usual Schrödinger operator $-\frac{1}{2}\Delta + V$. In other words, $(H^\mu, \mathcal{D}(H^\mu))$ is the Friedrichs extension of the operator $(-\frac{1}{2}\Delta + V, \ \mathcal{C}_0^\infty(\mathbb{R}^d))$.

b) If $\mu = 1_{\complement G} \cdot \infty$ with an open set $G \subset \mathbb{R}^d$ then H^μ is ($-\frac{1}{2}$ times) the Dirichlet Laplacian on G. In other words, $(H^\mu, \mathcal{D}(H^\mu))$ is the Friedrichs extension of the operator $(-\frac{1}{2}\Delta, \ \mathcal{C}_0^\infty(G))$.

There is also a one-to-one correspondence between self-adjoint operators and strongly continuous semigroups (resp. strongly continuous resolvents). That yields

(1.10) Corollary. *a) For every $\mu \in \mathcal{M}_0$ there exists a unique strongly continuous contraction semigroup $(e^{-tH^\mu})_{t>0}$ on $\overline{\mathcal{D}}(\mathcal{E}^\mu)$ with generator $-(H^\mu, \mathcal{D}(H^\mu))$. Defining e^{-tH^μ} to be 0 on the orthogonal complement of $\overline{\mathcal{D}}(\mathcal{E}^\mu)$ in $L^2(\mathbb{R}^d, m)$ this semigroup trivially extends to a contraction semigroup on $L^2(\mathbb{R}^d, m)$, called* Schrödinger semigroup *and also denoted by $(e^{-tH^V})_{t>0}$.*

b) Similarly, one obtains the existence of a unique strongly continuous resolvent $(H^\mu + \alpha)^{-1}$, $\alpha > 0$, on $\overline{\mathcal{D}}(\mathcal{E}^\mu)$ which in an analogous way will be extended to a resolvent on $L^2(\mathbb{R}^d, m)$.

(1.11) Remark. According to [7] (Prop. 2.1), for every $f \in L^2(\mathbb{R}^d, m)$ the function $(H^\mu + \alpha)^{-1} f$ is given as the unique minimal point of the functional

$$g \mapsto \mathcal{E}^\mu(g, g) - 2 \int f g \, dm$$

on $W^{1,2}(\mathbb{R}^d, m)$.

An essential observation now is that the symmetric form $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ as well as the Schrödinger operator $(H^\mu, \mathcal{D}(H^\mu))$ associated with a measure $\mu \in \mathcal{M}_0$ only depends on the equivalence class $(\mu, \sim)$ of μ under a certain equivalence relation $\sim$, — and not on the particular choice of the representant μ in $(\mu, \sim)$. Of course, the same is then also true for the associated semigroup and resolvent.

(1.12) Definition. Two measures μ, ν in $\mathcal{M}_0$ are called *equivalent* $(\mu \sim \nu)$ iff

$$(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu)) = (\mathcal{E}^\nu, \mathcal{D}(\mathcal{E}^\nu)).$$

The set of equivalence classes in $\mathcal{M}_0$ is denoted by $(\mathcal{M}_0, \sim)$.

This equivalence relation $\sim$ and the induced set of equivalence classes $(\mathcal{M}_0, \sim)$ will be investigated in more details in the next section. At the moment we turn our attention to a quite natural notion of convergence in $(\mathcal{M}_0, \sim)$ which makes this set to a nice topological space.

(1.13) Definition. Let $(\mu_n)_{n \in N}$ be a sequence in $\mathcal{M}_0$ and let $\mu \in \mathcal{M}_0$. We say that the sequence of Schrödinger operators $(H^{\mu_n})_{n \in N}$ *converges in the strong resolvent sense* to H^μ iff for some (hence all) $\alpha > 0$ the sequence of resolvent operators $([H^{\mu_n} + \alpha]^{-1})_{n \in N}$ on $L^2(\mathbb{R}^d, m)$ converges strongly to $[H^\mu + \alpha]^{-1}$.

In this case, we also say that the sequence of measures $(\mu_n)_{n \in N}$ is *γ-convergent* to μ (or, more precisely, that the sequence of equivalence classes $((\mu_n, \sim))_{n \in N}$ is γ-convergent to $(\mu, \sim)$).

For equivalent characterizations and many interesting properties of the γ-convergence we refer to [7] and literature cited there. We restrict ourselves to the follwing important result from [14].

(1.14) Theorem.

a) Under the γ-topology the set $(\mathcal{M}_0, \sim)$ is compact and metrizable.

b) The set $\{V \cdot m : V \in C_0^\infty(\mathbb{R}^d)\}$ as well as the set $\{1_K \cdot \overline{\infty} : K \subset \mathbb{R}^d$ compact$\}$ is γ-dense in $(\mathcal{M}_0, \sim)$

One of the reasons to consider generalized Schrödinger operators $-\frac{1}{2}\Delta + \mu$ with measures $\mu \in \mathcal{M}_0$ is that this class of operators is closed with respect to strong resolvent convergence.

(1.15) Corollary. *The strong resolvent limit of any sequence of Schrödinger operators $(-\frac{1}{2}\Delta + \mu_n)_{n \in N}$ with measures $\mu_n \in \mathcal{M}_0$, $n \in \mathbb{N}$, is (if it exists) again a Schrödinger operator $-\frac{1}{2}\Delta + \mu$ with a measure $\mu \in \mathcal{M}_0$.*

Actually, the above mentioned set of generalized Schrödinger operators is the smallest set which is closed with respect to strong resolvent convergence and which contains the usual Schrödinger operators $-\frac{1}{2}\Delta + V$ with smooth potentials $V \in C_0^\infty(\mathbb{R}^d)$.

(1.16) Corollary.

a) Every generalized Schrödinger operator $-\frac{1}{2}\Delta + \mu$ *with a measure* $\mu \in \mathcal{M}_0$ *is the limit in the strong resolvent sense of a sequence of usual Schrödinger operators* $(-\frac{1}{2}\Delta + V_n)_{n\in N}$ *with smooth potentials* $V_n \in C_0^\infty(\mathbb{R}^d)$, $n \in \mathbb{N}$.

b) On the other hand, every generalized Schrödinger operator $-\frac{1}{2}\Delta + \mu$ *with a measure* $\mu \in \mathcal{M}_0$ *is also the limit in the strong resolvent sense of a sequence of Dirichlet Laplacians (times* $-\frac{1}{2}$*) on open sets* $G_n \subset \mathbb{R}^d$ *(with* $\mathbb{R}^d \setminus G_n$ *being compact),* $n \in \mathbb{N}$.

1.4 Equivalence of Measures in $\mathcal{M}_0$

According to [7], the equivalence $\sim$ can also be expressed in terms of the fine topology.

(1.17) Theorem. *Two measures* μ, ν *in* $\mathcal{M}_0$ *are equivalent if and only if*

$$\mu(F) = \nu(F) \qquad \text{for all finely open sets } F \in \mathcal{B}(\mathbb{R}^d).$$

(1.18) Remarks. a) Obviously, always the following implications hold:

- if the measures μ, ν are identical (i.e. $\mu(F) = \nu(F)$ for all Borel sets $F \subset \mathbb{R}^d$), then they are equivalent
- if they are equivalent, then they satisfy $\mu(F) = \nu(F)$ for all open sets $F \subset \mathbb{R}^d$.

In general, however, none of the converse implications holds: An example of measures, which are equivalent but not identical, is given by the measures $\overline{\infty}$ and $\infty \cdot m$. On the other hand, let G be an open set which is dense but not finely dense in $\mathbb{R}^d$ (cf. [36], chap. 9). Then the measures $\overline{\infty}$ and $1_G \cdot \overline{\infty}$ obviously coincide for all open sets, but not for all finely open sets.

b) We emphasize that the equivalence of the measures μ and ν, in general, *does not* imply that for a given Borel set $G \subset \mathbb{R}^d$ the measures $1_{CG} \cdot \mu$ and $1_{CG} \cdot \nu$ are equivalent. For instance, consider the equivalent measures $\infty \cdot m$ and $\overline{\infty}$ and the open set $G = \mathbb{R}^d \setminus \partial B_1(0)$. Then $1_{CG} \cdot \infty \cdot m = 0$ (zero measure) but $1_{CG} \cdot \overline{\infty} \not\sim 0$.

(1.19) Proposition. *Let* $\mu, \nu \in \mathcal{M}_0$ *and let* $F, G \in \overline{\mathcal{B}}(\mathbb{R}^d)$.

a) If μ *is a Radon measure, then:* $\quad \mu \sim \nu \iff \mu = \nu$.

b) If $G \subset \mathbb{R}^d$ *is quasi-closed, then:* $\quad \mu \sim \nu \implies 1_{CG} \cdot \mu \sim 1_{CG} \cdot \nu$.

c) $\mathrm{reg}(G) = \mathrm{reg}(F) \iff 1_{CG} \cdot \overline{\infty} \sim 1_{CF} \cdot \overline{\infty}$.

d) $\mathrm{reg}(G) = \mathrm{reg}(\overline{G}) \implies 1_{CG} \cdot \overline{\infty} \sim 1_{CG} \cdot \infty \cdot m$.

Proof. a) - c) follows from [36]. In order to see d), note that, according to c), the assumption implies $1_{CG} \cdot \overline{\infty} \sim 1_{C\overline{G}} \cdot \overline{\infty}$. But the measures $\overline{\infty}$ and $\infty \cdot m$ are equivalent. Applying b) to these measures and to the set $\overline{G}$ yields $1_{C\overline{G}} \cdot \overline{\infty} \sim 1_{C\overline{G}} \cdot \infty \cdot m$, hence, $1_{CG} \cdot \overline{\infty} \sim 1_{C\overline{G}} \cdot \infty \cdot m$. This proves the claim since always $1_{C\overline{G}} \cdot \infty \cdot m \leq 1_{CG} \cdot \infty \cdot m \leq 1_{CG} \cdot \overline{\infty}$. □

(1.20) Remarks. a) Let us call a measure $\mu \in \mathcal{M}_0$ *maximal* (with respect to $\sim$) iff all measures $\nu \in \mathcal{M}_0$ equivalent to μ satisfy $\nu \leq \mu$. According to (1.19.a), every Radon measure is maximal. In Lemma (3.5) below, we shall see that for every measure μ there exists a *unique maximal* measure $\overline{\mu}$ equivalent to μ. For instance, the measure $\overline{\infty}$ is the unique maximal measure equivalent to the measure $\infty \cdot m$. Thus there is a one-to-one correspondence between the set $\overline{\mathcal{M}}_0 \subset \mathcal{M}_0$ of maximal measures and the set $(\mathcal{M}_0, \sim)$ of equivalence classes in $\mathcal{M}_0$.

b) Another way to obtain a unique, canonical representative in each equivalence class of $(\mathcal{M}_0, \sim)$ is to look at quasi-regular measures. Here a measure $\mu \in \mathcal{M}_0$ is called *quasi-regular* (from outside) iff $\mu(F) = \inf\{\mu(G) : G \supset F, G \text{ quasi-open}\}$ for all $F \in \mathcal{B}(\mathbb{R}^d)$. It is easy to see that for any measure $\mu \in \mathcal{M}_0$ there exists a *unique quasi-regular* measure $\mu^* \in \mathcal{M}_0$ equivalent to μ, namely

$$\mu^*(F) := \inf\{\mu(G) : G \supset F, G \text{ quasi-open}\} \qquad \text{for } F \in \mathcal{B}(\mathbb{R}^d).$$

Since any maximal measure is immediately proved to be quasi-regular, one actually obtains

$$\mu^* = \overline{\mu}.$$

We add parenthetically that any quasi-regular measure $\mu \in \mathcal{M}_0$ is not only quasi-regular from outside (as our definition states) but by itself also regular from inside, i.e. $\mu(F) = \sup\{\mu(K) : K \subset F, K \text{ compact}\}$ for every $F \in \mathcal{B}(\mathbb{R}^d)$ ([13], Theorem 4.4). This justifies our usage of "quasi-regular" instead of "quasi-regular from outside".

For $r > 0$ let us define the kernel

$$k(r) := \begin{cases} 1 & \text{if } d = 1 \\ \sup\{-\ln r, 0\} & \text{if } d = 2 \\ r^{2-d} & \text{if } d \geq 3. \end{cases}$$

(1.21) Definition. We say that a measure μ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ belongs to the *Dynkin class* ($\mu \in \mathcal{K}_\infty$) iff for some (hence all) $r > 0$

$$\sup_{x \in R^d} \int_{B_r(x)} k(\|x - y\|)\, \mu(dy) < \infty.$$

In the case $d \geq 2$, μ is said to belong to the *Kato class* ($\mu \in \mathcal{K}_0$) iff

$$\lim_{r \to 0} \sup_{x \in R^d} \int_{B_r(x)} k(\|x - y\|)\, \mu(dy) = 0.$$

In the case $d = 1$, the Kato class coincides by definition with the Dynkin class.

Of course, we always have $\mathcal{K}_0 \subset \mathcal{K}_\infty \subset \mathcal{M}_0$. According to the following result ([7], Prop. 2.5), every measure $\mu \in \mathcal{M}_0$ can be approximated monotonically by Kato measures, at least up to equivalence.

(1.22) Theorem. *For any measure $\mu \in \mathcal{M}_0$ there exist a Borel function $f \geq 0$ and a Kato measure $\nu \in \mathcal{K}_0$ satisfying*

$$\mu \sim f \cdot \nu.$$

Note that in this case, the measure $f \cdot \nu$ is the strong limit of the increasing sequence of Kato measures $f_n \cdot \nu$ (where f_n denotes the bounded function $x \mapsto \inf\{f(x), n\}$).

(1.23) Corollary. *For every measure $\mu \in \mathcal{M}_0$ there exists an increasing sequence of Kato measures μ_n, $n \in \mathbb{N}$, such that the form $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ is the increasing limit of the forms $(\mathcal{E}^{\mu_n}, \mathcal{D}(\mathcal{E}^{\mu_n}))$, $n \in \mathbb{N}$. Similarly, the semigroup (resolvent, resp.) associated with H^μ is the decreasing limit of the semigroups (resolvents, resp.) associated with H^{μ_n}, $n \in \mathbb{N}$.*

Dynkin measures (in particular, Kato measures) $\mu \in \mathcal{M}_0$ play an important rôle since the associated symmetric forms $\mathcal{E}^\mu$ have the same domain $\mathcal{D}(\mathcal{E}^\mu)$ as the unperturbed form $\mathcal{E}^0$ (cf. [5], [25], [40]). For an entirely different situation, recall Example (1.6).

(1.24) Proposition. *For every Dynkin measure $\mu \in \mathcal{K}_\infty$ we have*

$$\mathcal{D}(\mathcal{E}^\mu) = W^{1,2}(\mathbb{R}^d).$$

2 The Probabilistic Approach

2.1 Additive Functionals and Subordinated Semigroups

The probabilistic way to define a Schrödinger operator $H^V = -\frac{1}{2}\Delta + V$ for an arbitrary Borel function $V \geq 0$ is to define H^V as the negative generator of the Feynman-Kac semigroup

$$(*) \qquad P_t^V f(x) := \mathbb{E}^x \left[e^{-\int_0^t V(X_s)\,ds} \cdot f(X_t) \right],$$

more precisely, as the negative generator of the extension $(T_t^V)_{t>0}$ to $L^2(\mathbb{R}^d, m)$ of the Feynman-Kac semigroup $(P_t^V)_{t>0}$. It will actually turn out that this always leads to the same operator as defined before by means of the symmetric form $(\mathcal{E}^V, \mathcal{D}(\mathcal{E}^V))$, namely, to the form sum of H^0 and V.

In order to treat more general perturbations of the heat semigroup $(P_t)_{t>0}$ one has to replace the Feynman-Kac functional $\exp - \int_0^t V(X_s)\,ds$ in $(*)$ by a general multiplicative functional or, in other words, to replace the additive functional $\int_0^t V(X_s)\,ds$ by a general additive functional.

(2.1) Definition. A process $A : \mathbb{R}_+ \times \Omega \to [0, \infty]$ is called *additive functional* (of Brownian motion) iff it is adapted and right continuous and if for all $t, s \geq 0$:

(i) $A_{t+s} = A_t + A_s \circ \Theta_t$ a.s.

(ii) $t \mapsto A_{s-t} \circ \Theta_t$ is a.s. right continuous on $[0, s[$.

The set of additive functionals is denoted by $\mathbb{A}_+$. Two additive functionals A and A' are identified if they are indistinguishable, i.e. if there exists a set $\Omega_1 \subset \Omega$ of full measure (that is, $\mathbb{P}^x(\Omega_1) = 1$ for all $x \in \mathbb{R}^d$) such that $A_t(\omega) = A'_t(\omega)$ for all $t \geq 0$ and all $\omega \in \Omega_1$. In this case we write $A = A'$ a.s.

(2.2) Examples. a) For any Borel function $V \geq 0$ on $\mathbb{R}^d$, an additive functional $V \cdot I$ is defined by

$$V \cdot I : \ (t, \omega) \mapsto \lim_{\epsilon \to 0} \int_0^{t+\epsilon} V(X_s(\omega))\, ds.$$

(More generally, for any real-valued, continuous additive functional A, the product $V \cdot A$ of the function $V \geq 0$ and the additive functional A, defined by $(V \cdot A)(t, \omega) := \lim_{\epsilon \to 0} \int_0^{t+\epsilon} V(X_s(\omega))\, dA_s$, is again an additive functional, [35], Lemma 1.3.)

b) If $V = 1_F \cdot \infty$ with a Borel set $F \subset \mathbb{R}^d$ then the additive functional $V \cdot I$ is indistinguishable from the additive functional

$$\infty \cdot 1_{[[S_F, \infty[[} : \ (t, \omega) \mapsto \begin{cases} 0, & \text{if } t < S_F(\omega) \\ \infty, & \text{if } t \geq S_F(\omega), \end{cases}$$

where $S_F(\omega) := \inf\{s \geq 0 : \text{ Lebesgue measure of } \{t \in [0, s] : X_t(\omega) \in F\} > 0\}$ is the *penetration time* of F.

c) For any Borel set $F \subset \mathbb{R}^d$, a similar additive functional $\infty \cdot 1_{[[T_F, \infty[[}$ is defined by

$$\infty \cdot 1_{[[T_F, \infty[[}(t, \omega) := \begin{cases} 0, & \text{if } t < T_F(\omega) \\ \infty, & \text{if } t \geq T_F(\omega), \end{cases}$$

where $T_F(\omega) := \inf\{t > 0 : X_t(\omega) \in F\}$ is the *hitting time* of F.

(2.3) Remarks. a) Note that the condition (ii) in the definition of additive functionals may fail if we replace in the above Example c) the first hitting time T_F of F by the *debut* $T_F^0 = \inf\{t \geq 0 : X_t \in F\}$ of F. The process $\infty \cdot 1_{[[T_F^0, \infty[[}$ satisfies condition (ii) if and only if it is indistinguishable from the processes $\infty \cdot 1_{[[T_F, \infty[[}$ which is the case if and only if F is regular (cf. [29]).

Also note that, in general, the additive functionals $\infty \cdot 1_{[[T_F, \infty[[}$ and $\infty \cdot 1_{[[S_F, \infty[[}$ are not indistinguishable. We shall see that they are indistinguishable if and only if the measures $1_F \cdot \overline{\infty}$ and $1_F \cdot \infty \cdot m$ are equivalent.

b) Additive functionals $A \in \mathbb{A}_+$ need not to be continuous. However, along a.e. path ω the only discontinuity of $s \mapsto A_s(\omega)$ is at most one jump to infinity which may occur at the debut of the set $\mathbb{R}^d \setminus E^A$ (cf. chapter 3). But for every fixed $t > 0$ the map $s \mapsto A_s$ is a.s. continuous at $s = t$. Hence, for every fixed $t > 0$ we have in the situation of Example (2.2.a)

$$(V \cdot I)_t \ = \ \int_0^t V(X_s)\, ds \text{ a.s.}$$

c) Our definition of additive functionals differs in some respects from the usual one. The most striking difference is that we do not impose any finiteness condition on A. According to our definition, A is an additive functional if and only if

$$e^{-A}: (t,\omega) \mapsto e^{-A_t(\omega)} \qquad \text{(with the convention } e^{-\infty} = 0)$$

is an (adapted, right continuous, decreasing and exact) *multiplicative functional* (cf. [11], [29]).

(2.4) Definition. For an additive functional $A \in \mathbb{A}_+$ we define the generalized *Feynman-Kac semigroup* $(P_t^A)_{t>0}$ as a semigroup of kernels on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ by

$$P_t^A f(x) := \mathbb{E}^x \left[e^{-A_t} f(X_t) \right]$$

for Borel functions $f \geq 0$ on $\mathbb{R}^d$ (again with the convention $e^{-\infty} := 0$).

This is always a subordinated semigroup in the sense of the following

(2.5) Definition. A semigroup $(Q_t)_{t>0}$ of kernels on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ is called *subordinated* (to the heat semigroup $(P_t)_{t>0}$) if

$$Q_t f(x) \leq P_t f(x)$$

for all Borel functions $f \geq 0$ on $\mathbb{R}^d$, all $x \in \mathbb{R}^d$ and all $t > 0$.

If $(Q_t)_{t>0}$ is a subordinated semigroup, then for all bounded Borel functions $f \geq 0$ and all $\alpha > 0$ the functions

$$\int_0^\infty (P_t f - Q_t f) e^{-\alpha t}\, dt$$

are α-supermedian. If they are even α-excessive, then the semigroup $(Q_t)_{t>0}$ is called *exactly subordinated* (to the heat semigroup $(P_t)_{t>0}$). From [11] or [29] we quote

(2.6) Theorem. *There is a one-to-one correspondence between additive functionals and exactly subordinated semigroups. In particular, every generalized Feynman-Kac semigroup $(P_t^A)_{t>0}$ with $A \in \mathbb{A}_+$ is exactly subordinated and, conversely, every exactly subordinated semigroup $(Q_t)_{t>0}$ is actually a generalized Feynman-Kac semigroup, that is, there exists a (unique) additive functional A such that $Q_t = P_t^A$ for all $t > 0$.*

(2.7) Remark. a) We recall that our definition of additive functionals differs from the usual one because we do not require them to be finite. Usually, one requires that A_t is finite for all t or, at least, that A_0 is finite a.s. (A slightly more general definition is used by Fukushima [21] who allows A_0 to be infinite $\mathbb{P}^x$-a.s. for a polar set of starting points x.) For additive functionals in the usual sense the property (ii) in our definition (called exactness) is an immediate consequence of the additivity (i) and the right continuity. In our general case, however, one has to require this as an additional property.

b) There are several weaker versions of exactness (= property (ii)). These properties would also be sufficient for our purposes. However, we do not want to go into details here since there is no essential difference between all these notions, according to a suitable perfection procedure, cf. [29].

c) Let us mention the following useful effect of the exactness condition in the definition of additive functionals ([36], Lemma (2.5)): If two additive functionals A and A' in $\mathbb{A}_+$ coincide $\mathbb{P}^x$-a.s. for q.e. $x \in \mathbb{R}^d$, then they coincide a.s. (i.e. they are indistinguishable and can be identified).

2.2 Additive Functionals and Measures Charging No Polar Sets

Additive functionals on the path space correspond to measures on the state space. In order to make this assertion more precise, let us first of all turn to two special classes of "nice" additive functionals which will play a major rôle in the sequel. These classes of additive functionals will be compared with corresponding classes of measures on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$.

(2.8) Definition. An additive functional $A \in \mathbb{A}_+$ is called *Dynkin functional* ($A \in \mathbb{K}_\infty$) iff

$$\sup_{x \in R^d} \mathbb{E}^x [A_t] < \infty$$

for some (hence all) $t > 0$. It is called *Kato functional* ($A \in \mathbb{K}_0$) iff

$$\lim_{t \to 0} \sup_{x \in R^d} \mathbb{E}^x [A_t] = 0.$$

As our definitions suggest, there is a one-to-one correspondence between Dynkin functionals and Dynkin measures (which induces a similar correspondence between Kato functionals and Kato measures). This correspondence was established by E. B. Dynkin ([18], Theorem 8.4, cf. also [11], [21], [28]).

(2.9) Proposition. *There is a bijective map A from the set of Dynkin measures $\mathcal{K}_\infty$ onto the set of Dynkin functionals $\mathbb{K}_\infty$, specified by the relation*

$$A = A(\mu) \qquad \Longleftrightarrow \qquad \mathbb{E}^x [A_t] = \int_0^t \int p(s,x,y)\, \mu(dy)\, ds \quad \text{for } t > 0 \text{ and } x \in \mathbb{R}^d.$$

The inverse map assigns to each additive functional $A \in \mathbb{A}_+$ its Revuz measure $\mu(A)$.

(2.10) Remarks. a) It is evident that the subset of Kato measures $\mathcal{K}_0$ is mapped by A onto the set of Kato functionals $\mathbb{K}_0$.

b) The class $\mathbb{K}_\infty$ of additive functionals and the class $\mathcal{K}_\infty$ of measures have been studied in detail by E. B. Dynkin [18]. Therefore, we called them Dynkin classes. The

Kato class of additive functionals (resp. measures) is the canonical generalization of the so-called Kato class of functions (cf. [1]).

c) Obviously, an additive functional (measure, resp.) belongs to the Dynkin class if and only if for some (hence all) $\alpha > 0$ its α-potential is a bounded function on $\mathbb{R}^d$. Similarily, one can show that it belongs to the Kato class if and only if its α-potential is a bounded and uniformly continuous function on $\mathbb{R}^d$ ([35], Korollar 4.8).

We are now in a position to assign an additive functional $A(\mu)$ to each measure $\mu \in \mathcal{M}_0$.

(2.11) Construction of the map $A: (\mathcal{M}_0, \sim) \to \mathbb{A}_+$ (following [7]).

Step 1. If μ is a Kato measure there exists a unique additive functional $A(\mu)$ according to (2.9).

Step 2. If μ has density f with respect to a suitable Kato measure ν we define

$$A(\mu) = A(f \cdot \nu) := f \cdot A(\nu) : \quad t \mapsto \lim_{\epsilon \to 0} \int_0^{t+\epsilon} f(X_s)\, dA_s(\nu).$$

Step 3. For an arbitrary measure $\mu \in \mathcal{M}_0$ we choose $f \in \mathcal{B}_+$ and $\nu \in \mathcal{K}_0$ satisfying $f \cdot \nu \sim \mu$ and define

$$A(\mu) \ := \ A(f \cdot \nu).$$

The important fact is that this definition does *not depend* on the particular choice of f and ν. In other words,

$$A(\mu) = A(\nu) \text{ a.s.} \quad \Longleftrightarrow \quad \mu \sim \nu$$

([7]). Therefore, the map $A : \mathcal{M}_0 \to \mathbb{A}_+$ induces an injective map $A : (\mathcal{M}_0, \sim) \to \mathbb{A}_+$.

This map is actually bijective. In order to see this we should know a little bit more on the structure of general additive functionals $A \in \mathbb{A}_+$.

(2.12) Decomposition of additive functionals. Let us consider the function

$$\varphi^A : \ x \mapsto \mathbb{E}^x \int_0^\infty e^{-A_t} \cdot e^{-t}\, dt$$

on $\mathbb{R}^d$. It is easy to see that always $0 \le \varphi^A \le 1$ and that φ^A is finely continuous and upper semicontinuous on $\mathbb{R}^d$. This implies that the sets $G_n^A := \{\varphi^A > \frac{1}{n}\}$ are finely open and the sets $F_n^A := \{\varphi^A \ge \frac{1}{n}\}$ are closed (not only finely closed). Of course, $\overline{G_n^A} \subset F_n^A \subset G_{n+1}^A \subset (F_{n+1}^A)^{f-int}$ (:= fine interior of F_{n+1}^A) and

$$\bigcup_{n=1}^\infty G_n^A = \bigcup_{n=1}^\infty F_n^A = \{\varphi^A > 0\}.$$

In particular, $E^A := \{\varphi^A > 0\}$ turns out to be a finely open F_σ-set. This set E^A is called *the set of permanent points for A.*

By means of this function φ^A, we can decompose any additive functional $A \in \mathbb{A}_+$ in a unique way into a "singular" part and into a "smooth" part. In order to see this note that

$$A_t = \infty \quad \text{if (and only if)} \quad t \geq T_0(\mathbb{R}^d \setminus E^A)$$

and

$$A_t = (1_{F_n^A} \cdot A)_t \quad \text{if} \quad t < T(\mathbb{R}^d \setminus F_n^A)$$

where $1_{F_n^A} \cdot A$ is a Dynkin functional. (For the precise definition of the product $1_{F_n^A} \cdot A$ we refer to [36].)

This decomposition enables us to assign a measure $\mu = \mu(A)$ to each additive functional $A \in \mathbb{A}_+$. In fact, this measure will be maximal.

(2.13) Construction of the map $\mu: \mathbb{A}_+ \to \overline{\mathcal{M}}_0$ (following [36]).
For $A \in \mathbb{A}_+$ we define

$$\mu(A) := 1_{CE^A} \cdot \overline{\infty} + \uparrow \lim_{n \to \infty} \mu_n$$

where μ_n is the uniquely determined Dynkin measure associated with the Dynkin functional $1_{F_n^A} \cdot A$, $n \in \mathbb{N}$. The measure $\mu(A)$ defined in this way is always maximal. Therefore, we have indeed a map $\mu: \mathbb{A}_+ \to \overline{\mathcal{M}}_0$.

According to [36], this is the converse map to the map $A: \mathcal{M}_0 \mapsto \mathbb{A}_+$ defined above. Hence, we obtain

(2.14) Theorem. *The maps $\mu: \mathbb{A}_+ \to \overline{\mathcal{M}}_0$ from (2.13) and $A: (\mathcal{M}_0, \sim) \to \mathbb{A}_+$ from (2.11) establish a one-to-one correspondence between additive functionals and equivalence classes of measures in $\mathcal{M}_0$.*

Let us mention that the map $A: (\mathcal{M}_0, \sim) \to \mathbb{A}_+$ is linear and order preserving (which is a reason to consider additive functionals rather than multiplicative functionals). To be precise, for $\alpha_1, \alpha_2 \in \mathbb{R}_+$ and $\mu_1, \mu_2 \in \mathcal{M}_0$ the following holds a.s.:

$$\begin{aligned} A(\alpha_1 \cdot \mu_1 + \alpha_2 \cdot \mu_2) &= \alpha_1 \cdot A(\mu_1) + \alpha_2 \cdot A(\mu_2) \\ \mu_1 \leq \mu_2 &\Longrightarrow A(\mu_1) \leq A(\mu_2). \end{aligned}$$

Actually, one has also the following version of "nearly" σ-additivity: for every sequence $(\mu_n)_{n \in N}$ in $\mathcal{M}_0$ there exists a set $\Omega' \subset \Omega$ of full measure such that

$$A_t \left(\sum_{n=1}^{\infty} \mu_n \right)(\omega) = \lim_{\epsilon \to 0} \sum_{n=1}^{\infty} A_{t+\epsilon}(\mu_n)(\omega) \qquad \text{for all } t \geq 0 \text{ and all } \omega \in \Omega'$$

(i.e. $A(\Sigma \mu_n)$ coincides a.s. with the right continuous modification of $\sum A(\mu_n)$). For further versions of "nearly" σ-additivity, we refer to [36], chap. 4.

Finally, we are going to identify the additive functional $A(\mu)$ in some of the most important cases.

(2.15) Examples. a) For every Borel function $V \geq 0$ the additive functional associated with the measure $V \cdot m$ is $V \cdot I$, i.e.

$$A_t(V \cdot m) = \lim_{\epsilon \to 0} \int_0^{t+\epsilon} V(X_s)\, ds.$$

b) For every Borel set $F \subset I\!R^d$ the additive functional associated with the measure $1_F \cdot \infty \cdot m$ is $\infty \cdot 1_{[[S_F,\infty[[}$, i.e.

$$A_t(1_F \cdot \infty \cdot m) = \begin{cases} 0, & \text{for } t < S_F \\ \infty, & \text{for } t \geq S_F \end{cases}$$

where S_F is the penetration time of F.

c) In contrast to that, the additive functional associated with the measure $1_F \cdot \overline{\infty}$ is $\infty \cdot 1_{[[T_F,\infty[[}$, i.e.

$$A_t(1_F \cdot \overline{\infty}) = \begin{cases} 0, & \text{for } t < T_F \\ \infty, & \text{for } t \geq T_F \end{cases}$$

where T_F is the hitting time of F.

These examples together with the uniqueness result in Theorem (2.14) immediately yield

(2.16) Corollary. *For every Borel set $F \subset I\!R^d$*

$$S_F = T_F \ a.s. \iff 1_F \cdot \infty \cdot m \sim 1_F \cdot \overline{\infty}.$$

Combining that with (1.19) gives sufficient geometric conditions on F for S_F being indistinguishable from T_F (cf. also [23] and [39]).

(2.17) Corollary. *If F is quasi-open or if* $\text{reg}(\mathcal{C}F) = \text{reg}(\mathcal{C}\overline{F})$ *then*

$$S_F = T_F \ a.s.$$

2.3 Feynman-Kac Semigroups and Schrödinger Semigroups

Up to now we have considered the Feynman-Kac semigroup $(P_t^A)_{t>0}$ only as a semigroup of kernels on $(I\!R^d, \mathcal{B}(I\!R^d))$. It can of course also be regarded as a semigroup of operators on the Banach space of bounded Borel functions on $I\!R^d$ which obviously extends to a semigroup of operators on $L^\infty(I\!R^d, m)$ (since $P_t^A(x,dy) \leq P_t(x,dy) << m(dy)$ for every $x \in I\!R^d$ and every $t > 0$). This semigroup in turn extends in a canonical way to a semigroup of operators T_t^A, $t > 0$, on $L^2(I\!R^d, m)$. We call it extended Feynman-Kac semigroup.

(2.18) Theorem. *The additive functional $A \in \mathbb{A}_+$ corresponds to the measure $\mu \in \mathcal{M}_0$ in the sense of Theorem (2.14) if and only if the extended Feynman-Kac semigroup $(T_t^A)_{t>0}$ on $L^2(I\!R^d, m)$ coincides with the Schrödinger semigroup $(e^{-tH^\mu})_{t>0}$.*

In other words:

$$A = A(\mu) \iff T_t^A = e^{-tH^\mu} \quad \textit{for } t > 0.$$

Proof. This identity is well-known for measures and additive functionals in the Kato class (cf. [2 - 6], [8 - 10]). According to (1.23), however, the general case follows by a simple monotone convergence argument (cf. also [7]). □

(2.19) Corollary. *The negative generator of the extended Feynman-Kac semigroup $(T^A)_{t>0}$ on $L^2(\mathbb{R}^d, m)$ is the Schrödinger operator $(H^\mu, \mathcal{D}(H^\mu))$ (as defined in Theorem (1.8)) where μ corresponds to A according to Theorem (2.14).*

This correspondence between the analytic approach (via symmetric forms, self-adjoint operators and semigroups on $L^2(\mathbb{R}^d, m)$) and the probabilistic approach (via semigroups of kernels on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$) has a particularly natural form in the case where $\mu = V \cdot m$ (and hence $A = V \cdot I$).

(2.20) Corollary. *For every Borel function $V \geq 0$ the Schrödinger semigroup $(e^{-tH^V})_{t>0}$ on $L^2(\mathbb{R}^d, m)$ associated with the Schrödinger operator $H^V = -\frac{1}{2}\Delta + V$ (form sum) is given by the* Feynman-Kac formula, *i.e. for $f \in L^2(\mathbb{R}^d, m)$ the function $e^{-tH^V} f \in L^2(\mathbb{R}^d, m)$ is given for m-a.e. $x \in \mathbb{R}^d$ by*

$$e^{-tH^V} f(x) = \mathbb{E}^x \left[e^{-\int_0^t V(X_s)\,ds} \cdot f(X_t) \right].$$

Of course, also the resolvent and the symmetric form derived from an additive functional $A \in \mathbb{A}_+$ by means of the extended Feynman-Kac semigroup $(T_t^A)_{t>0}$ coincide with the corresponding quantities derived from the associated measure $\mu = \mu(A) \in \mathcal{M}_0$ by means of the symmetric form $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$.

Note, however, that if $A = A(\mu)$ then the semigroup $(P_t^A)_{t>0}$ of operators on $\mathcal{B}_b$, the set of bounded Borel functions on $\mathbb{R}^d$, bears more information than its extension $(e^{-tH^\mu})_{t>0}$ to $L^2(\mathbb{R}^d, m)$. In particular, one can not obtain $(P_t^A)_{t>0}$ directly from $(e^{-tH^\mu})_{t>0}$. Besides $(e^{-tH^\mu})_{t>0}$ it is reasonable also to investigate the semigroup $(P_t^\mu)_{t>0} := (P_t^{A(\mu)})_{t>0}$. For the set $E^{A(\mu)}$ we also use the notation E^μ.

One of the main advantages of the semigroup $(P_t^\mu)_{t>0}$ is that it has a pointwise well-defined density. Note that, in general, one can not expect that there exists a continuous (and therefore pointwise uniquely defined) density $p^\mu(t, x, y)$ for $P_t^\mu(x, dy)$. A sufficient condition to ensure continuity is that $\mu \in \mathcal{K}_0$ ([10]).

(2.21) Theorem ([37]). *For any measure $\mu \in \mathcal{M}_0$ there exists a unique integral kernel $p^\mu : \mathbb{R}_+^* \times E \times E \to \overline{\mathbb{R}}_+$ with the following properties:*

1. *for all $t > 0$ and $x \in E$ the function $p^\mu(t, x, .)$ is a density for $P_t^\mu(x, .)$, i.e. for any Borel function $f \geq 0$ on $\mathbb{R}^d$*

$$\int p^\mu(t, x, y) \cdot f(y)\,dy = P_t^\mu f(x)$$

which implies that for any $f \in L^2(\mathbb{R}^d, m)$ and for m-a.e. $x \in \mathbb{R}^d$

$$\int p^\mu(t,x,y) \cdot f(y)\,dy \;=\; e^{-tH^\mu} f(x);$$

2. *for all $x \in E$ the function $p^\mu(.,x,.) : (t,y) \mapsto p^\mu(t,x,y)$ is space-time finely continuous on $\mathbb{R}_+^* \times E^\mu$ (and is identically 0 on $\mathbb{R}_+^* \times (\mathbb{R}^d \setminus E^\mu)$), i.e. for all $t > 0$ and all $x, y \in \mathbb{R}^d$ the function*

   $$s \mapsto p^\mu(t-s, x, X_s) \quad \text{is right continuous on } \{s \in [0,t] : X_s \in E^\mu\} \quad \mathbb{P}^y - a.s.$$

   (and identically 0 on $\{s \in [0,t] : X_s \notin E^\mu\}$);

3. *$p^\mu(t,x,y)$ is symmetric in x and y for all $t > 0$ and $x, y \in \mathbb{R}^d$, i.e.*

   $$p^\mu(t,x,y) \;=\; p^\mu(t,y,x);$$

4. *p^μ satisfies the Chapman-Kolmogorov equation, i.e. for all $s, t > 0$ and $x, y \in \mathbb{R}^d$:*

   $$p^\mu(s+t,x,y) \;=\; \int p^\mu(s,x,z) \cdot p^\mu(t,z,y)\,dz;$$

5. *if $(\mu_n)_{n\in N}$ is any increasing sequences of Kato measures satisfying*

   $$\uparrow \lim_{n\to\infty} \mu_n \sim \mu$$

   then p^μ is the pointwise decreasing limit of $(p^{\mu_n})_{n\in N}$, i.e. for all $t > 0$ and $x, y \in \mathbb{R}^d$

   $$p^\mu(t,x,y) \;= \downarrow \lim_{n\in N} p^{\mu_n}(t,x,y).$$

The uniqueness of this function p^μ follows already from the first two properties, i.e. from being a space-time finely continuous density for P^μ.

In the case of usual Schrödinger operators $-\frac{1}{2}\Delta + V$ the density p^V admits a nice representation (cf. also [17]).

(2.22) Proposition ([37]). *Let $V \geq 0$ be any Borel function. Then the pointwise uniquely defined density p^V associated (by the preceding construction) to the measure $V \cdot m$ is given by*

$$p^V(t,x,y) \;=\; p(t,x,y) \cdot \mathbb{E}_{t,y}^{0,x}\left[e^{-\int_0^t V(X_s)\,ds}\right],$$

where $\mathbb{E}_{t,y}^{0,x}$ denotes expectation with respect to the Brownian bridge starting in x at time 0 and arriving in y at time $t > 0$.

3 The Set of Permanent Points

Throughout this chapter we assume that μ is a given (fixed) measure in $\mathcal{M}_0$ and that $A = A(\mu)$ is the associated additive functional.

3.1 Local Characterizations of E^μ

(3.1) Definition. We define the set E^μ of *permanent points* for μ to be the set of all points $x \in \mathbb{R}^d$ for which
$$A_0 = 0 \ \mathbb{P}^x\text{-a.s.}$$

Note that for every $x \in \mathbb{R}^d$ we have either $A_0 = 0$ $\mathbb{P}^x$-a.s. or $A_0 = \infty$ $\mathbb{P}^x$-a.s., according to the additivity of A and to Blumenthal's 0-1 law. Therefore, this definition is consistent with the definition $E^\mu = \{x \in \mathbb{R}^d : \varphi^A(x) = 0\}$ used in chapter 2. Due to (2.12), E^μ is always a finely open Borel set (even a F_σ-set).

(3.2) Examples. a) If $\mu = 1_{CG} \cdot \overline{\infty}$ with a Borel set $G \subset \mathbb{R}^d$ then
$$E^\mu = \mathrm{reg}(G).$$

b) If $\mu = V \cdot m$ with a Borel function $V \geq 0$ on $\mathbb{R}^d$ then $x \in E^\mu$ if and only if
$$\mathbb{P}^x \left\{ \int_0^\epsilon V(X_s)\, ds < \infty \quad \text{for some } \epsilon > 0 \right\} > 0.$$

(3.3) Theorem ([7]). *E^μ is the set of all points $x \in \mathbb{R}^d$ which have a finely open neighbourhood $G \subset \mathbb{R}^d$ satisfying*

$$(*) \qquad \int_G k(\|x - y\|)\, \mu(dy) < \infty.$$

(3.4) Remarks. a) The conditon $(*)$ can equivalently be replaced by the condition
$$\int_G k(\|z - y\|)\, \mu(dy) < \infty \qquad \text{for all } z \in \mathbb{R}^d$$
or even by
$$\sup_{z \in R^d} \int_G k(\|z - y\|)\, \mu(dy) < \infty,$$
which is to say $1_G \cdot \mu \in \mathcal{K}_\infty$ ([36], Theorem (5.1)).

b) The set E^μ is closely related with the set F^μ, the *set of finiteness* for μ, which is by definiton the union F^μ of all finely open sets G with finite μ-measure. In other words, F^μ is the set of all points $x \in \mathbb{R}^d$ which have a finely open neighbourhood

$G \subset \mathbb{R}^d$ satisfying $\mu(G) < \infty$. Obviously, the set F^μ is finely open. It is also easy to see that $E^\mu \subset F^\mu$ and that $F^\mu \setminus E^\mu$ is always polar. Hence, the sets

$$E^\mu \text{ and } F^\mu \text{ differ at most by a polar set.}$$

In many of the following analytic statements one is therefore allowed to replace E^μ by F^μ. However, the probabilistic approach shows that the right quantity to look at is indeed E^μ.

The set of permanent points (as well as the set of finiteness) can also be used to characterize the equivalence relation $\sim$.

(3.5) Lemma ([36]). *Two measures μ and ν in $\mathcal{M}_0$ are equivalent if and only if*

$$E^\mu = E^\nu \qquad \text{and} \qquad 1_{E^\mu} \cdot \mu = 1_{E^\nu} \cdot \nu.$$

In particular, every measure $\mu \in \mathcal{M}_0$ is equivalent to the measure

$$\overline{\mu} := 1_{\mathcal{C}E^\mu} \cdot \overline{\infty} + 1_{E^\mu} \cdot \mu$$

which is the unique maximal (resp. quasi-regular) measure equivalent to μ.

Be careful: the equivalence of μ and $\overline{\mu}$ *does not imply* that $1_{\mathcal{C}E^\mu} \cdot \mu$ is equivalent to $1_{\mathcal{C}E^\mu} \cdot \overline{\infty}$.

3.2 Decomposition Theorem

The measures $\mu \in \mathcal{M}_0$ which we have considered up to now have

- either been extremely singular, like $\mu_1 = 1_{\mathcal{C}G} \cdot \overline{\infty}$ with $G \in \mathcal{B}(\mathbb{R}^d)$ (which implies $\mathcal{D}(\mathcal{E}^{\mu_1}) = W_0^{1,2}(G)$)
- or rather smooth, like $\mu_2 \in \mathcal{K}_\infty$ (which implies $\mathcal{D}(\mathcal{E}^{\mu_2}) = W^{1,2}(\mathbb{R}^d)$).

Of course, we can compose such measures in order to obtain a measure $\mu = \mu_1 + \mu_2$ which is "singular" on $\mathcal{C}G$ and "smooth" on G.

However, much more interesting is the converse question, namely whether one can decompose an *arbitrary measure* $\mu \in \mathcal{M}_0$ into a "singular" part μ_1 and a "smooth" part μ_2. The main result in this section is that such a decomposition is always possible.

(3.6) Theorem ([36]).

a) On $\mathcal{C}E^\mu$ the measure μ is "singular" in the sense that

$$\mu \sim 1_{\mathcal{C}E^\mu} \cdot \overline{\infty} + 1_{E^\mu} \cdot \mu.$$

b) On E^μ the measure μ is "smooth" in the sense that for $n \in \mathbb{N}$

$$1_{F_n} \cdot \mu \in \mathcal{K}_\infty$$

where $F_n := \{\varphi^\mu \geq \frac{1}{n}\}$. *Note that* $(F_n)_{n \in N}$ *is an increasing sequence of closed sets with the property that the fine interiors* F_n^{f-int} *of the sets* F_n *increase to* E^μ.

This "smoothness" property of μ on E^μ is indeed closely related to the notion of smoothness in the sense of [21], cf. also [36], chap. 8.

(3.7) Definition. A measure μ is called *smooth* on a finely open set $G \subset \mathbb{R}^d$ iff $\mu \in \mathcal{M}_0$ and there exists an increasing sequence of compact sets $F_n \subset G$ satisfying

- $\text{cap}(K \setminus F_n) \xrightarrow{n \to \infty} 0$ for any compact set $K \subset G$ and
- $1_{F_n} \cdot \mu$ is a Radon measure for any $n \in \mathbb{N}$.

(3.8) Remarks. a) Due to Ph. Blanchard and Z. M. Ma ([9], Theorem 2.1), the condition "$1_{F_n} \cdot \mu$ is a Radon measure" may be replaced by the condition "$1_{F_n} \cdot \mu$ is a Kato measure (i.e. $1_{F_n} \cdot \mu \in \mathcal{K}_0$)".

b) According to [36], the following conditions are equivalent:

- $\text{cap}(K \setminus F_n) \xrightarrow{n \to \infty} 0$ for compact sets $K \subset G$;
- $G \setminus \bigcup_n F_n^{f-int}$ is polar;
- $\tau(F_n) \xrightarrow{n \to \infty} \tau(G)$ $\mathbb{P}^x$-a.s. for q.e. $x \in \mathbb{R}^d$.

(3.9) Corollary. *A measure* μ *is smooth on a finely open set* $G \subset \mathbb{R}^d$ *iff*

$$G \setminus E^\mu \text{ is polar.}$$

In particular, μ *is smooth on* $\mathbb{R}^d$ *if and only if*

$$\mathbb{R}^d \setminus E^\mu \text{ is polar.}$$

The condition that $\mathbb{R}^d \setminus E^\mu$ is polar plays also a crucial rôle in the proof of a multidimensional analogue to the 0-1-law of Engelbert-Schmidt ([24]). We refer to [24] for various analytic and probabilistic conditions equivalent to that condition. For further conditions equivalent to it, we refer to (3.12.a) and (3.17) below.

3.3 Characterization of $\mathcal{D}(\mathcal{E}^\mu)$

We are now in a position to characterize the form domain $\mathcal{D}(\mathcal{E}^\mu)$ for an arbitrary measure $\mu \in \mathcal{M}_0$. For that purpose, let $G_n^\mu := \{\varphi^\mu > \frac{1}{n}\}$, $n \in \mathbb{N}$, and recall that $E^\mu := \{\varphi^\mu > 0\}$.

(3.10) Theorem. *For every* $n \in \mathbb{N}$ *the following holds*

$$W_0^{1,2}(G_n^\mu) \subset \mathcal{D}(\mathcal{E}^\mu) \subset W_0^{1,2}(E^\mu).$$

Proof. The first inclusion follows immediately from Theorem (3.6.b) and Proposition (1.24), the second one from Theorem (3.6.a) and Example (1.6). □

(3.11) Corollary.
a) The closure of $\mathcal{D}(\mathcal{E}^\mu)$ in $W^{1,2}(\mathbb{R}^d)$ is

$$W_0^{1,2}(E^\mu) = \{f \in W^{1,2}(\mathbb{R}^d) : \ \tilde{f} = 0 \ \text{q.e. on } CE^\mu\}.$$

b) The closure of $\mathcal{D}(\mathcal{E}^\mu)$ in $L^2(\mathbb{R}^d, m)$ is

$$L_0^2(E^\mu, m) = \{f \in L^2(\mathbb{R}^d, m) : \ f = 0 \ m\text{-a.e. on } CE^\mu\}.$$

(3.12) Corollary.
a) $\mathcal{D}(\mathcal{E}^\mu)$ is dense in $W^{1,2}(\mathbb{R}^d)$ $\iff$ $cap(\mathbb{R}^d \setminus E^\mu) = 0$.
b) $\mathcal{D}(\mathcal{E}^\mu)$ is dense in $L^2(\mathbb{R}^d, m)$ $\iff$ $m(\mathbb{R}^d \setminus E^\mu) = 0$.

In other words, the last assertion says that $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ is a densely defined form on $L^2(\mathbb{R}^d, m)$ if and only if the set $\mathbb{R}^d \setminus E^\mu$ has Lebesgue measure 0. A quite different criterion for $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ being densely defined was given by P. Stollmann [33].

3.4 Limits of Schrödinger Operators

The characterization of the form domain $\mathcal{D}(\mathcal{E}^\mu)$ allows the investigation of the limits of the sequences $(\frac{1}{n} \cdot \mu)_{n \in N}$ as well as $(n \cdot \mu)_{n \in N}$ with respect to γ-convergence. In the case of the sequence $(n \cdot \mu)_{n \in N}$ we obviously have $n \cdot \mu \longrightarrow \infty \cdot \mu$ (for $n \to \infty$) in the sense of γ-convergence (and in the sense of monotone convergence). It is also easy to see that $\infty \cdot \mu \sim 1_{CE^{\infty \cdot \mu}} \cdot \overline{\infty} = 1_{CF^{\infty \cdot \mu}} \cdot \overline{\infty}$.

(3.13) Proposition. *In the sense of γ-convergence, for $n \to \infty$*

$$n \cdot \mu \longrightarrow 1_{CF^{\infty \cdot \mu}} \cdot \overline{\infty}$$

where $F^{\infty \cdot \mu}$ (the set of finiteness for the measure $\infty \cdot \mu$) coincides with the fine support f-supp(μ) of μ.

(3.14) Corollary. *The Schrödinger operators $H^{n \cdot \mu} = -\frac{1}{2}\Delta + n \cdot \mu$ converge for $n \to \infty$ in the strong resolvent sense to ($-\frac{1}{2}$ times) the Dirichlet Laplacian on the finely open set f-supp(μ).*

If $\mu = 1_{CG} \cdot m$ with a Borel set $G \subset \mathbb{R}^d$ which is either quasi-closed or satisfies reg(G) = reg($\overline{G}$) (cf. (1.19) or (2.17)), then the operators $H^{n \cdot \mu}$ converge for $n \to \infty$ in the strong resolvent sense to ($-\frac{1}{2}$ times) the Dirichlet Laplacian on G (or, which comes to the same, on reg(G)). For similar results we refer to [17].

On the other hand, we have in the case of the sequence $(\frac{1}{n} \cdot \mu)_{n \in N}$

(3.15) Proposition. *In the sense of γ-convergence, for $n \to \infty$*

$$\frac{1}{n} \cdot \mu \longrightarrow 1_{CE^{\mu}} \cdot \overline{\infty}.$$

(3.16) Corollary. *The Schrödinger operators $H^{\frac{1}{n}\cdot\mu} = -\frac{1}{2}\Delta + \frac{1}{n} \cdot \mu$ converge for $n \to \infty$ in the strong resolvent sense to ($-\frac{1}{2}$ times) the Dirichlet Laplacian on the finely open set E^{μ}.*

(3.17) Corollary. *The following statements are equivalent:*

- *$\mathbb{R}^d \setminus E^{\mu}$ is polar*
- *$\frac{1}{n} \cdot \mu \longrightarrow 0$ (for $n \to \infty$) in the sense of γ-convergence*
- *$H^{\frac{1}{n}\cdot\mu} \longrightarrow H^0 = -\frac{1}{2}\Delta$ (for $n \to \infty$) in the strong resolvent sense.*

For related results, cf. [33] and [40].

Appendix: Potential Theoretic Notions

All potential theoretic notions like *fine, polar,* etc. are those of classical potential theory. The underlying Markov process $\mathbb{X}$ will always be the Brownian motion in $\mathbb{R}^d$ (with $\Omega = C(\mathbb{R}_+, \mathbb{R}^d)$).

By *cap* we denote the capacity (Newtonian resp. logarithmical). A statement is said to hold *q.e. (=quasi everywhere)* (on $\mathbb{R}^d$) if it holds except on a polar subset of $\mathbb{R}^d$. The phrase *a.s.* is used to say that a statement holds $\mathbb{P}^x$-a.s. (on Ω) for every $x \in \mathbb{R}^d$.

It is well-known (and can also be used as a definition) that a Borel function u on $\mathbb{R}^d$ is *finely continuous* if and only if a.s. the map

$$t \mapsto u(X_t) \quad \text{is continuous on } [0, \infty[$$

and that a Borel set $F \subset \mathbb{R}^d$ is *finely open* if and only if a.s. the set

$$\{t \in \mathbb{R}_+ : X_t \in F\} \quad \text{is open in } \mathbb{R}_+.$$

The fine topology is the coarsest topology rendering all superharmonic (resp. all α-excessive) functions continuous.

There is a close connection between fine continuity and quasi-continuity. We recall that by definition a numerical function u on $\mathbb{R}^d$ is *quasi-continuous* iff for any $\epsilon > 0$ there exists an open set $D = D_\epsilon$ such that $\text{cap}(D) \leq \epsilon$ and

$$u|_{CD} : CD \to \overline{\mathbb{R}} \quad \text{is continuous.}$$

Indeed, a numerical function u on $\mathbb{R}^d$ is quasi-continuous if and only if it is finely continuous q.e. It should be clear that similar results also hold for quasi-open (resp. quasi-closed) sets and finely open (resp. finely closed) sets. For instance, a set $F \subset \mathbb{R}^d$ is quasi-open if and only if it is the union of a finely open set and a polar set. In particular, every finely open set G is quasi-open, that is, for any $\epsilon > 0$ there exists an open set D_ϵ such that $\text{cap}(D_\epsilon) \leq \epsilon$ and such that the set $G_\epsilon := G \cup D_\epsilon$ is open.

We define the *regularization* $\text{reg}(G)$ of a (nearly) Borel set $G \subset \mathbb{R}^d$ to be the finely open Borel set

$$\left\{x \in \mathbb{R}^d : \mathbb{P}^x\left\{T(\mathbb{R}^d \setminus G) > 0\right\} = 1\right\}.$$

Do not confuse $\text{reg}(G)$ with the set of regular points for the stopping time $T(\mathbb{R}^d \setminus G)$ (which in the literature is often denoted by $(\mathbb{R}^d \setminus G)^r$ and which coincides with $\mathbb{R}^d \setminus \text{reg}(G)$).

$\text{reg}(G)$ is the largest finely open set such that $\text{reg}(G) \setminus G$ is polar. The initial set G is finely open if and only if $G \subset \text{reg}(G)$ and it is quasi-open if and only if $G \setminus \text{reg}(G)$ is polar. G is called regular iff $G = \text{reg}(G)$.

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