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Grundzüge der stochastischen Analysis

1. Exercise sheet from 10/17/2008

Exercise 1 – Path properties of a brownian motion (10 points)

Revision: A function $f : [a,b] \to \mathbb{R}$ is of finite variation iff

$$\sup_{a=t_0<\ldots<t_k=b} \sum_{i=1}^{k} |f(t_i) - f(t_{i-1})| < \infty.$$

Let $\{B_t : t \geq 0\}$ be a standard brownian motion. Consider for $a, b \in \mathbb{R}, a < b$ and $n \in \mathbb{N}$ the random variable

$$X_n = \sum_{j=1}^{2^n} \left(B_{a+k(b-a)2^{-n}} - B_{a+(k-1)(b-a)2^{-n}} \right)^2.$$

i) Calculate $\mathbb{E}[X_n]$ and $\mathrm{var}(X_n)$.

ii) Prove: $\lim_{n\to\infty} X_n = b - a$ almost surely.

iii) Conclude that brownian paths are of finite variation almost surely on no interval $[a, b]$.

Hint: Show fast stochastic convergence in ii).

Exercise 2 - Stopping times I (10 points)

Let T and S be stopping times. Prove:

i) $\forall A \in \mathcal{F}_S : A \cap \{S \leq T\} \in \mathcal{F}_T$.

ii) $S \leq T$ almost surely $\Rightarrow \mathcal{F}_S \subseteq \mathcal{F}_T$.

iii) $\mathcal{F}_{T\wedge S} = \mathcal{F}_T \cap \mathcal{F}_S$.

iv) $\{\{T < S\}, \{T \leq S\}, \{T = S\}\} \subseteq \mathcal{F}_T \cap \mathcal{F}_S$.

Exercise 3 - Stopping times II (10 points)

Let T and S be stopping times and let Z be an integrable random variable. Prove:

i) $\mathbb{E}[Z|\mathcal{F}_T] = \mathbb{E}[Z|\mathcal{F}_{T\wedge S}]$ almost surely on the set $\{T \leq S\}$.

ii) $\mathbb{E}[\mathbb{E}[Z|\mathcal{F}_T]|\mathcal{F}_S] = \mathbb{E}[Z|\mathcal{F}_{T\wedge S}]$ almost surely.

Exercise 4 - Galmarino's test (10 points)

Let $\Omega := \mathcal{C}(\mathbb{R}_+, \mathbb{R}^d)$. Let $\{X_t : t \geq 0\}$ denote the coordinate process (that is $X_t(\omega) := \omega(t)$) and let T be a random variable. Then the following are equivalent:

i) T is a stopping time with respect to the filtration $\mathcal{F}_t^0 := \sigma(X_s : s \leq t)$.

ii) $\forall t \geq 0 \forall \omega \in \Omega \forall \omega' \in \Omega((T(\omega) \leq t \wedge \forall s \leq t X_s(\omega) = X_s(\omega')) \Rightarrow T(\omega) = T(\omega'))$.

Hint: First show that ii) is equivalent to

$$ii') \quad \forall t \geq 0 \forall \omega \in \Omega \forall \omega' \in \Omega((T(\omega) \leq t \wedge \forall s \leq t X_s(\omega) = X_s(\omega')) \Rightarrow T(\omega') \leq t).$$

In order to prove the implication $ii) \Rightarrow i)$ define the function $a_t : \Omega \to \Omega$, $\omega \mapsto \omega(. \wedge t)$ and show that $\{T \leq t\} = a_t^{-1}[\{T \leq t\}]$.