Universität Bonn
Wintersemester 2023/2024

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Graduate seminar on Interacting Random Systems: Selected topics in random matrices and random operators

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Content

During this seminar, the main object of study is the operator

$$H_\lambda := -\Delta + \lambda V \tag{1}$$

on $\ell^2(\mathbb{Z}^d)$. Roughly speaking, we will be interested in various, fine spectral properties of H_λ.

Here, Δ is the discrete analogue, on $\mathbb{Z}^d$, of the Laplacian, on $\mathbb{R}^d$, i.e., $-\Delta\colon \ell^2(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d)$ is defined through

$$(-\Delta u)(x) := -\sum_{y\in\mathbb{Z}^d,\,|y|=1} (u(x+y) - u(x))$$

for all $x \in \mathbb{Z}^d$ and $u = (u(x))_{x\in\mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d)$, where $|y| = |y_1| + \ldots + |y_d|$ for $y = (y_i)_{i=1}^d \in \mathbb{Z}^d$. Moreover, $\lambda \in [0,\infty)$ and assume that $(v_x)_{x\in\mathbb{Z}^d}$ are independent, identically distributed random variables on $\mathbb{R}$ such that v_x is almost surely bounded, i.e. $|v_x| \leqslant K$ almost surely for some constant $K > 0$, for all $x \in \mathbb{Z}^d$. Define $V\colon \ell^2(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d)$ to be the multiplication operator with $(v_x)_{x\in\mathbb{Z}^d}$, i.e.

$$(Vu)(x) := v_x u(x)$$

for all $x \in \mathbb{Z}^d$ and $u = (u(x))_{x\in\mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d)$. The assumptions made above imply that, for any $\lambda \in [0,\infty)$, H_λ from (1) is a bounded linear operator on $\ell^2(\mathbb{Z}^d)$ and self-adjoint.

In a seminal work, the physicist (and later Nobel laureate) Philip W. Anderson proposed 1958 H_λ as quantum mechanical model that describes the behaviour of an electron in a disordered crystal and, therefore, explains its characteristics as conductor or insulator. The operator H_λ is called *random Schrödinger operator*, *tight-binding model* or *Anderson model*.

Quantum mechanics tell us that the time evolution of such system is described by the solution to the Schrödinger equation

$$\mathrm{i}\partial_t \psi_t = H_\lambda \psi_t,$$

which is given by $\psi_t = \mathrm{e}^{-\mathrm{i}tH_\lambda}\psi_0$. Hence, understanding this time evolution requires the analysis of the spectral properties of H_λ.

An important question is whether, depending on the strength λ of the disorder and the dimension d, the spectrum of H_λ consists only of eigenvalues with *localized eigenvectors* or whether also *extended states* can emerge. This *extended states conjecture* is probably one of the most important open problems in mathematical physics. See Figure 1.2 in [AW15] for the expected phase diagram of H_λ.

Note the very different spectral behaviours of V and $-\Delta$. For each $x \in \mathbb{Z}^d$, the standard basis vector $e_x := (\delta_{xy})_{y\in\mathbb{Z}^d}$ is an eigenvector of V with eigenvalue v_x. So the eigenvectors of V are very “localized”. In contrast, for any $k = (k_i)_{i=1}^d \in [0,2\pi)^d$, the plane wave ϕ_k, where $\phi_k(x) := \mathrm{e}^{\mathrm{i}k\cdot x}$ for each $x \in \mathbb{Z}^d$ with $k\cdot x := k_1x_1 + \ldots + k_dx_d$, is a generalized eigenvector of $-\Delta$. Clearly, ϕ_k is not localized and rather “extended”.

Some natural questions that we want to study in this seminar are:

- What is the shape of the spectrum of H_λ for $\lambda > 0$?
- Are there "localized" eigenvectors of "extended"/generalized eigenvectors?

References

[AW15] Michael Aizenman and Simone Warzel, *Random operators*, Graduate Studies in Mathematics, vol. 168, American Mathematical Society, Providence, RI, 2015, Disorder effects on quantum spectra and dynamics. MR 3364516