Spin systems

S countable set "sites"
$\{0,1\}^S$ state space. (note: this is compact)
$x, y \in S$, $\eta, \xi \in \{0,1\}^S$ generic elements
$c(x,\eta)$ bounded function $S \times \{0,1\}^S \to \mathbb{R}$

For $x \in S$, $\eta \in \{0,1\}^S$, write $\eta_x \in \{0,1\}^S$ for

$$\eta_x(y) = \begin{cases} \eta(y) & \text{if } y \neq x \\ 1-\eta(y) & \text{if } y = x \end{cases}$$

Def: given c, define L by

$$Lf(\eta) = \sum_x c(x,\eta)\,[f(\eta_x) - f(\eta)]$$

on $$D = \{ f \in C\{0,1\}^S : \sum_x \sup_\eta |f(\eta_x) - f(\eta)| < \infty \}$$

Remarks

- If S were finite, then L would be the generator of the Markov chain with
$$q(\eta, \xi) = \begin{cases} c(x,\eta) & \text{if } \xi = \eta_x \\ 0 & \text{if } \xi \neq \eta,\ \xi \neq \eta_x \\ -\sum_x c(x,\eta) & \text{if } \xi = \eta \end{cases}$$

- Define $|||f||| = \sum_x \sup_\eta |f(\eta_x) - f(\eta)|$. For $f \in C\{0,1\}^S$, $|||f||| < \infty$ iff $\exists$ summable $\alpha : S \to \mathbb{R}$ such that f is Lipschitz continuous with respect to the metric $\rho(\eta, \xi) = \sum_x \alpha(x)\,|\eta(x) - \xi(x)|$.

Proof: If $|||f||| < \infty$, take $\alpha(x) = \sup_\eta |f(\eta_x) - f(\eta)|$. This is non-negative and summable. Let $x_1, x_2, \ldots$ be an enumeration of S. For $\eta, \xi \in \{0,1\}^S$, Define η_i by

$$\eta_i(x_j) = \begin{cases} \eta(x_j) & \text{if } j \geq i \\ \xi(x_j) & \text{if } j < i. \end{cases}$$

Then $\eta_0 = \eta$ and $\eta_i \to \xi$ as $i \to \infty$. Since f is continuous,

$$f(\xi) = f(\eta) + \sum_{i=0}^{\infty} f(\eta_{i+1}) - f(\eta_i)$$

$$\Rightarrow |f(\xi) - f(\eta)| \leq \sum_{i=0}^{\infty} |f(\eta_{i+1}) - f(\eta_i)|$$

$$\leq \sum_{\substack{i \geq 0 \\ \eta(x_i) \neq \xi(x_i)}} \sup_\eta |f(\eta_{x_i}) - f(\eta)|$$

$$= \sum_{i \geq 0} \alpha(x_i)\,|\eta(x_i) - \xi(x_i)|$$

$$= \rho(\eta, \xi) \quad \Rightarrow f \text{ Lipschitz}$$

Conversely, suppose $\exists\, \alpha$ such that f is ρ-Lipschitz. Then $\forall \eta$,
$$|f(\eta_x) - f(\eta)| \leq \rho(\eta_x, \eta) = \alpha(x)$$
$$\Rightarrow \sup_\eta |f(\eta_x) - f(\eta)| \leq \alpha(x) \text{ is summable}$$ $\square$

- $|||f||| < \infty$ does not imply $f \in C\{0,1\}^S$. Example: $f(\eta) = \begin{cases} 1 & : \eta \text{ has infinitely many ones} \\ 0 & \text{otherwise.} \end{cases}$

  Then $f(\eta_x) = f(\eta) \ \forall \eta \Rightarrow |||f||| = 0$, but f is not continuous because $\eta_i = (\underbrace{0,0,\dots,0}_{i},1,1,\dots)$ satisfies

  - $f(\eta_i) = 1 \ \forall i$
  - $\eta_i \to (0,0,\dots)$
  - $f(\lim_{i\to\infty} \eta_i) = 0$.

- For $f \in D$, the series $\sum_x c(x,\eta)[f(\eta_x) - f(\eta)]$ converges uniformly, and so Lf is continuous.

- If $c(x,\cdot) \equiv 0$ except for finitely many x, then we can define L on all of $C\{0,1\}^S$.

Construction of spin systems

Thm: Let $\gamma(x,u) = \sup_\eta |c(x,\eta_u) - c(x,\eta)|$

$$M = \sup_x \sum_{u \neq x} \gamma(x,u).$$

If $M < \infty$ then L is a generator.

First part of proof:

a) D is an algebra: clearly, it is closed under sums, and

$$\begin{aligned} |||fg||| &= \sum_x \sup_\eta |f(\eta_x)g(\eta_x) - f(\eta)g(\eta)| \\ &\leq \sum_x \sup_\eta \big(|f(\eta_x)g(\eta_x) - f(\eta_x)g(\eta)| + |f(\eta_x)g(\eta) - f(\eta)g(\eta)|\big) \\ &\leq \sum_x \big(\|f\| \sup_\eta |g(\eta_x) - g(\eta)| + \|g\| \sup_\eta |f(\eta_x) - f(\eta)|\big) \\ &\leq \|f\| \, |||g||| + \|g\| \, |||f||| < \infty \end{aligned}$$

D separates points: if $\eta \neq \xi$ then $\exists x \in S: \eta(x) \neq \xi(x)$. Then $f(\eta) := \eta(x)$ satisfies $f \in D$ and $f(\eta) \neq \xi(x)$.

D contains constant functions
$\{0,1\}^S$ compact
$\Rightarrow D$ dense in $C\{0,1\}^S$ (Stone-Weierstrass)

b) Suppose $f \in D$, $\lambda \geq 0$, $(I - \lambda L)f = g$. Compactness $\Rightarrow \exists \eta$ such that $f(\eta) = \inf_\xi f(\xi)$. Then $Lf(\eta) \geq 0$ by the definition of L.

$$\Rightarrow \min_\xi f(\xi) = f(\eta) \geq g(\eta) \geq \min_\xi g(\xi).$$

d) $1 \in D$, $L1 = 0$.

It remains to show (c).

Definitions:

- $l_1(S)$ is the Banach space of functions $\alpha: S \to \mathbb{R}$ satisfying

$$\|\alpha\|_{l_1} := \sum_x |\alpha(x)| < \infty$$

- for $f \in C\{0,1\}^S$, define $\Delta_f : S \to \mathbb{R}$ by

$$\Delta_f(x) = \sup_\eta | f(\eta_x) - f(\eta)|$$

Note: $\|\Delta_f\|_{l_1} = |||f|||$

- $\varepsilon = \inf_{u,\eta} [c(u,\eta) + c(u,\eta_u)]$

- if $M < \infty$, define $\Gamma : l_1(S) \to l_1(S)$ by

$$\Gamma\alpha(u) = \sum_{x \neq u} \alpha(x)\,\gamma(x,u).$$

Then $\|\Gamma\| = \sup_{\alpha \neq 0} \frac{\|\Gamma\alpha\|_{l_1}}{\|\alpha\|_{l_1}} = M.$

Prop: Suppose that either
a) $f \in D$, or
b) $f \in C\{0,1\}^S$ and $c(x,\cdot) = 0$ for all but finitely many x

If $(I - \lambda L)f = g \in D$, $\lambda > 0$, $\lambda M < 1 + \lambda\varepsilon$ then

$$\Delta_f \leq [(1+\lambda\varepsilon)I - \lambda\Gamma]^{-1} \Delta_g \quad \text{pointwise,}$$

where $((1+\lambda\varepsilon)I - \lambda\Gamma)^{-1}\alpha = \frac{1}{1+\lambda\varepsilon} \sum_{k=0}^{\infty} \left(\frac{\lambda}{1+\lambda\varepsilon}\right)^k \Gamma^k \alpha$

Proof: Since $(\eta_u)_u = \eta$,

$$\begin{aligned} g(\eta_u) - g(\eta) &= f(\eta_u) - f(\eta) - \lambda Lf(\eta_u) + \lambda Lf(\eta) \\ &= f(\eta_u) - f(\eta) - \lambda \sum_x \begin{pmatrix} c(x,\eta_u)[f((\eta_u)_x) - f(\eta_u)] \\ - c(x,\eta)[f(\eta_x) - f(\eta)] \end{pmatrix} \\ &= f(\eta_u) - f(\eta) - \lambda \sum_{x \neq u} \begin{pmatrix} c(x,\eta_u)[f((\eta_u)_x) - f(\eta_u)] \\ - c(x,\eta)[f(\eta_x) - f(\eta)] \end{pmatrix} \\ &\quad + \lambda[c(u,\eta) + c(u,\eta_u)][f(\eta_u) - f(\eta)] \end{aligned}$$

$$\Rightarrow [f(\eta_u) - f(\eta)][1 + \lambda c(u,\eta) + \lambda c(u,\eta_u)]$$
$$= g(\eta_u) - g(\eta) + \lambda \sum_{x \neq u} \begin{pmatrix} c(x,\eta_u)[f((\eta_u)_x) - f(\eta_u)] \\ + c(x,\eta)[f(\eta_x) - f(\eta)] \end{pmatrix}$$

For fixed u, $f(\eta_u) - f(\eta)$ is continuous in η.
$\Rightarrow \exists\, \xi$ such that $f(\eta_u) - f(\eta) \leq f(\xi_u) - f(\xi) \quad \forall \eta$

Also,
$$\begin{aligned} f((\xi_u)_x) - f(\xi_u) = f((\xi_x)_u) - f(\xi_u) &= f((\xi_x)_u) - f(\xi_x) + f(\xi_x) - f(\xi_u) \\ &\leq f(\xi_u) - f(\xi) + f(\xi_x) - f(\xi_u) \\ &= f(\xi_x) - f(\xi) \end{aligned}$$

Now,

$$\Delta_f(u)(1+\lambda\varepsilon) \le \Delta_g(u)\,[1+\lambda c(u,\eta) + \lambda c(u,\eta_u)]$$

$$= [f(\xi_u) - f(\xi)]\,[1 + \lambda c(u,\eta) + \lambda c(u,\eta_u)]$$

$$\le \Delta_g(u) + \lambda \sum_{x \neq u} [f(\xi_x) - f(\xi)]\,[c(x,\eta_u) - c(x,\eta)]$$

$$\le \Delta_g(u) + \lambda \sum_{x \neq u} \gamma(x,u)\,\Delta_f(x).$$

Under assumption (b), only finitely many $\gamma(x,u)$ are non-zero $\Rightarrow \Delta_f \in \ell_1(S)$

Under assumption (a), $\Delta_f \in \ell_1(S)$ by assumption.

In either case, $(1+\lambda\varepsilon)\,\Delta_f \le \Delta_g + \lambda\Gamma\Delta_f.$

$$\Delta_f \le \frac{\Delta_g}{1+\lambda\varepsilon} + \frac{\lambda\Gamma\Delta_f}{1+\lambda\varepsilon}$$

Γ non-negative $\Rightarrow$ by induction,

$$\Delta_f \le \frac{1}{1+\lambda\varepsilon}\sum_{k=0}^{n}\left(\frac{\lambda}{1+\lambda\varepsilon}\right)^k \Gamma^k \Delta_g + \left(\frac{\lambda}{1+\lambda\varepsilon}\right)^{n+1}\Gamma^{n+1}\Delta_f.$$

taking $n \to \infty$ completes the proof.

Thm: If $M < \infty$ then $\mathcal{R}(I-\lambda L)$ is dense in $C\{0,1\}^S$, and hence $\bar{L}$ is a generator. The corresponding semigroup $T(t)$ satisfies

$$|||T(t)f||| \le e^{(M-\varepsilon)t} |||f|||$$

In particular, $f \in D \Rightarrow T(t)f \in D$.

Pf: Take S_n finite, increasing to S. Set

$$L_n f(\eta) = \sum_{x \in S_n} c(x,\eta) \left[f(\eta_x) - f(\eta) \right].$$

Then L_n is a generator $\forall n$ (because it comes from a finite-state Markov chain). In particular, $\mathcal{R}(I - \lambda L_n) = C\{0,1\}^S \quad \forall \lambda > 0$.

Take $g \in D$, define f_n by $(I - \lambda L_n) f_n = g$, where $\lambda > 0$ satisfies $\lambda M < 1 + \lambda\varepsilon$. By the previous proposition, $f_n \in D$ and

(*)
$$\Delta_{f_n} \le \left((1+\lambda\varepsilon)I - \lambda\Gamma\right)^{-1} \Delta_g.$$

Let $g_n = (I - \lambda L) f_n$, let $K = \sup_{x,\eta} c(x,\eta)$.

$$\|(L - L_n) f_n\| = \sup_\eta \left| \sum_{x \notin S_n} c(x,\eta) \left[f(\eta_x) - f(\eta) \right] \right|$$

$$\le K \sum_{x \notin S_n} \Delta_{f_n}(x).$$

$$\|g_n - g\| = \lambda \|(L - L_n) f_n\| \le \lambda K \sum_{x \notin S_n} \Delta_{f_n}(x)$$

$$\le \lambda K \sum_{x \notin S_n} \left((1+\lambda\varepsilon)I - \lambda\Gamma\right)^{-1} \Delta_g.$$

$\longrightarrow 0$ as $n \to \infty$ because $\Delta_g \in \ell_1(S)$.

$$g_n \in \mathcal{R}(I - \lambda L) \Rightarrow g \in \overline{\mathcal{R}(I - \lambda L)}$$
$$\Rightarrow \overline{\mathcal{R}(I - \lambda L)} \text{ dense in } D$$
$$\Rightarrow \overline{\mathcal{R}(I - \lambda L)} \text{ dense in } C\{0,1\}^S.$$

To prove the bound, let f solve $(I - \lambda\bar{L}) f = g$. Since $(I - \lambda\bar{L}) f_n \to g$ and $I - \lambda\bar{L}$ is an expansion, $f_n \to f$.

$\Rightarrow \Delta_{f_n} \to \Delta_f$ in $\ell_\infty(S)$

but $\|\Delta_{f_n}\|_{\ell_1(S)}$ uniformly bounded $\Rightarrow \Delta_f \in \ell_1(S)$, $\Delta_{f_n} \to \Delta_f$ in $\ell_1(S)$

$$\Rightarrow \Delta_f \le \left((1+\lambda\varepsilon)I - \lambda\Gamma\right)^{-1} \Delta_g.$$

$$\Rightarrow \Delta_{(I-\lambda\bar{L})^{-1} g} \le \left((1+\lambda\varepsilon)I - \lambda\Gamma\right)^{-1} \Delta_g$$

$$\Rightarrow \Delta_{(I - \frac{t}{n}\bar{L})^{-n} g} \le \left((1+\tfrac{t}{n}\varepsilon)I - \tfrac{t}{n}\Gamma\right)^{-n} \Delta_g$$

$$\downarrow \qquad\qquad \downarrow$$

$$\Delta_{T(t)g} \le e^{-\varepsilon t} e^{t\Gamma} \Delta_g$$

$$\Rightarrow |||T(t)g||| \le e^{-\varepsilon t} e^{Mt} |||g||| \qquad \square$$

Ergodicity

Def: A spin system is *ergodic* if there is a unique stationary distribution μ, and $T(t)f(\eta) \xrightarrow[t\to\infty]{} \int f\,d\mu \quad \forall f \in C\{0,1\}^S$.

(Equivalently, $\exists \mu$: $\nu T(t) \to \mu$ in distribution $\forall \nu$)

Thm: If $M < \varepsilon$ then L is ergodic.

Pf: First, recall that $|f(\eta) - f(\xi)| \leq \sum_{x:\eta(x)\neq\xi(x)} \Delta_f(x)$

$$\Rightarrow \sup_{\eta,\xi} |f(\eta) - f(\xi)| \leq \sum_x \Delta_f(x) = |||f|||$$

$$\Rightarrow \sup_{\eta,\xi} |T(t)f(\eta) - T(t)f(\xi)| \leq |||T(t)f|||$$

$$\leq e^{(M-\varepsilon)t}|||f|||$$

Let μ be any stationary dist'n, ν any prob. measure on $\{0,1\}^S$.
Take $f \in D$.

$$\left|\int f\,d\mu - \int f\,d\nu T(t)\right|$$
$$= \left|\int T(t)f\,d\mu - \int T(t)f\,d\nu\right|$$
$$= \left|\iint T(t)f(\eta) - T(t)f(\xi) \quad d\mu(\eta)\,d\nu(\xi)\right|$$
$$\leq e^{(M-\varepsilon)t}|||f|||$$
$$\to 0 \quad \text{as } t\to\infty$$

Since D is dense, $\nu T(t) \to \mu$ in distribution $\square$

Remark: this is very useful, because it is widely applicable. However, it is not usually sharp.

Examples:

- Contact process

  Let S be a graph of bounded degree
  $x \sim y$ denotes neighbors
  $\deg(x) = \#\{y : y \sim x\}$
  $D = \max_x \deg(x) < \infty$

  Define $c(x,\eta) = \begin{cases} 1 & \text{if } \eta(x)=1 \\ \lambda\#\{y\sim x: \eta(y)=1\} & \text{if } \eta(x)=0 \end{cases}$

  $\varepsilon = 1$
  $M = \lambda D$
  $\Rightarrow$ the process is defined $\forall \lambda$
  it is ergodic if $\lambda < \frac{1}{D}$.

  Note: δ_0 is always a stationary distribution.

- Ising model

$S = \mathbb{Z}^d, \quad \beta > 0$

$$c(x,\eta) = \exp\Big(-\beta \sum_{y\sim x} (-1)^{\eta(x)+\eta(y)}\Big)$$

$$= \exp\Big(-\beta \sum_{y\sim x} (2\eta(x)-1)(2\eta(y)-1)\Big)$$

$$c(x,\eta) + c(x,\eta_x) = \exp\Big(-\beta \sum_{y\sim x} (-1)^{\eta(y)}\Big) + \exp\Big(\beta \sum_{y\sim x} (-1)^{\eta(y)}\Big)$$

$$\geq 2 \qquad \Big(\text{AM-GM inequality } \tfrac{a+b}{2} \geq \sqrt{ab}\Big)$$

$\Rightarrow \varepsilon \geq 2.$

$$c(x,\eta) - c(x,\eta_y) = \exp(-\beta k) - \exp(-\beta(k\pm 2)) \quad \text{if } k = \sum_{y\sim x} (-1)^{\eta(x)+\eta(y)}$$

$$\leq \exp(2\beta d)(1 - \exp(-2\beta)) \qquad (\text{for } k = -2d)$$

$\Rightarrow M \leq 2d e^{2d\beta}(1-e^{-2\beta})$

$\Rightarrow$ process is defined $\forall \beta$,
ergodic for small β (depending on d)

Thm: $d=1 \Rightarrow$ always ergodic
$d=2 \Rightarrow$ ergodic iff $\beta \leq \frac{1}{2}\log(1+\sqrt{2})$
$d\geq 3 \Rightarrow$ ergodic for small β, non-ergodic for large β.

Open problem: what is the correct range of β?

Coupling, monotonicity, attractiveness

A coupling is a construction of two random variables on the same space.

Write η_t, ξ_t for spin systems.
Say $\eta \leq \xi$ if $\eta(x) \leq \xi(x)$ $\forall x \in S$.

Thm: Let η_t, ξ_t be spin systems with rates c_1 and c_2. Suppose that $\forall \eta \leq \xi$,

$$c_1(x,\eta) \leq c_2(x,\xi) \quad \text{if} \quad \eta(x) = \xi(x) = 0$$
$$c_1(x,\eta) \geq c_2(x,\xi) \quad \text{if} \quad \eta(x) = \xi(x) = 1$$

Then $\forall \eta \leq \xi$ $\exists$ coupling $P^{(\eta,\xi)}$ such that

$$P^{(\eta,\xi)}(\eta_t \leq \xi_t \ \ \forall t \geq 0) = 1.$$

Proof (sketch): We define the coupling as a Feller process on $\{(0,0),(0,1),(1,1)\}^S$.
At each site x, make transitions

$$(0,0) \to \begin{cases} (1,1) & \text{with rate } c_1(x,\eta) \\ (0,1) & \text{with rate } c_2(x,\xi) - c_1(x,\eta) \end{cases}$$

$$(0,1) \to \begin{cases} (0,0) & \text{with rate } c_2(x,\xi) \\ (1,1) & \text{with rate } c_1(x,\eta) \end{cases}$$

$$(1,1) \to \begin{cases} (0,0) & \text{with rate } c_2(x,\xi) \\ (0,1) & \text{with rate } c_1(x,\eta) - c_2(x,\xi) \end{cases}$$

More precisely, let $c(x, \sigma, \eta, \xi)$ be the rate of switching site x to value σ at configuration (η, ξ). Define the generator

$$(Lf)(\eta, \xi) = \sum_{\substack{x \in S \\ \sigma}} c(x, \sigma, \eta, \xi) \left[f(\underbrace{(\eta, \xi)_{x,\sigma}}_{\text{change } (\eta,\xi) \text{ to } \sigma \text{ at site } x}) - f(\eta, \xi) \right]$$

Facts:

- this defines a generator
- if $f(\eta, \xi) = g(\eta)$ then $Lf(\eta, \xi) = L_1 g(\eta)$
- if $f(\eta, \xi) = g(\xi)$ then $Lf(\eta, \xi) = L_2 g(\xi)$

$$\Rightarrow \begin{cases} T(t) f = T_1(t) g \\ T(t) f = T_2(t) g \end{cases}$$

Hence, if (η_t, ξ_t) are generated by L then η_t is generated by L_1 and ξ_t is generated by L_2. $P^{(\eta,\xi)}(\eta_t \le \xi_t \ \forall t) = 1$ because everything in the state space of L satisfies this. $\square$

Def: The spin system with rates c is <u>attractive</u> if $\eta \le \xi$ implies

$$\begin{aligned} c(x, \eta) &\le c(x, \xi) \quad \text{if } \eta(x) = \xi(x) = 0 \\ c(x, \eta) &\ge c(x, \xi) \quad \text{if } \eta(x) = \xi(x) = 1. \end{aligned}$$

Examples

- contact process
- Ising model
- noisy voter model (homework)

Def: $f \in C\{0,1\}^S$ is increasing if $f(\eta) \le f(\xi)$ whenever $\eta \le \xi$.

Def: $\mu \le \nu$ if $\int f \, d\mu \le \int f \, d\nu$ $\forall$ increasing f.

Remark: the above two definitions make sense on any partially ordered set. On $\mathbb{R}$, $\mu \le \nu \iff \forall t, \ \mu(-\infty, t] \ge \nu(-\infty, t]$ "stochastic domination"

Prop: If $T(t)$ is the semigroup of an attractive spin system then
a) f increasing $\Rightarrow T(t) f$ increasing
b) $\mu \le \nu \Rightarrow \mu T(t) \le \nu T(t)$

Proof

a) Suppose $\eta \le \xi$. Let η_t, ξ_t be a coupling with $\eta_t \le \xi_t \ \forall t$, $\eta_0 = \eta$, $\xi_0 = \xi$. Then

$$T(t) f(\eta) = E f(\eta_t) \le E f(\xi_t) = T(t) f(\xi).$$

b) For any increasing f,

$$\begin{aligned} \int f \, d\mu T(t) &= \int T(t) f \, d\mu \\ &\le \int T(t) f \, d\nu \quad \text{(by (a))} \\ &= \int f \, d\nu T(t) \end{aligned}$$

$\square$

Thm: Suppose $T(t)$ is the semigroup of an attractive spin system.
δ_0 = unit mass on $(0, 0, \ldots)$
δ_1 = " " $(1, 1, \ldots)$.

a) $\forall\, s \le t$, $\delta_0 T(s) \le \delta_0 T(t)$
$\delta_1 T(s) \ge \delta_1 T(t)$

b) $\underline{\nu} = \lim_{t\to\infty} \delta_0 T(t)$
$\overline{\nu} = \lim_{t\to\infty} \delta_1 T(t)$
} exist, are stationary.

c) $\forall\, \mu, t$, $\delta_0 T(t) \le \mu T(t) \le \delta_1 T(t)$

d) if $\mu T(t_k) \to \nu$ as $k \to \infty$ then $\underline{\nu} \le \nu \le \overline{\nu}$
(in particular, all stationary ν satisfy $\underline{\nu} \le \nu \le \overline{\nu}$.

Cor: $T(t)$ is ergodic iff $\underline{\nu} = \overline{\nu}$.