Irreducibility

<u>Def</u> p_t irreducible if $p_t(x,y) > 0 \quad \forall x,y; \quad t > 0$.

Recurrence / transience

<u>Def:</u> $x \in S$ recurrent if
$$P^x(\exists \text{ arbitrarily large } t : X(t) = x) = 1.$$
otherwise, x is transient.

<u>Def:</u> Green's Function:
$$G(x,y) = E^x \int_0^\infty 1_{\{X(t)=y\}} \, dt = \int_0^\infty p_t(x,y)\, dt.$$

<u>Thm:</u> x transient $\iff G(x,x) < \infty$.

<u>Pf:</u> $\tau_0 = 0, \quad s_0 = \inf\{t \geq 0 : X(t) \neq x\}$.
$\tau_1 = \inf\{t > s_0 : X(t) = x\} \qquad s_1 = \inf\{t > \tau_1 : X(t) \neq x\}$

$\tau_{i+1} = \inf\{t > s_i : X(t) = x\} \qquad s_{i+1} = \inf\{t > \tau_{i+1} : X(t) \neq x\}$

$$G(x,x) = E^x \sum_{i : \tau_i < \infty} s_i - \tau_i.$$
$$= \sum_i E^x \left(1_{\{\tau_i < \infty\}} (s_i - \tau_i)\right)$$

SMP: $E^x\left[1_{\{\tau_i<\infty\}} (s_i - \tau_i) \mid \mathcal{F}_{\tau_i}\right]$
$$= 1_{\{\tau_i < \infty\}} E^x s_0$$
$$= 1_{\{\tau_i < \infty\}} e^{-c(x)}$$
$$\Rightarrow E^x 1_{\{\tau_i < \infty\}} (s_i - \tau_i) = P^x(\tau_i < \infty) \cdot e^{-c(x)}$$

SMP: $P^x[\tau_{i+1} < \infty \mid \mathcal{F}_{\tau_i}]$
$$= P^x[\tau_i < \infty, \tau_{i+1} < \infty \mid \mathcal{F}_{\tau_i}]$$
$$= 1_{\{\tau_i < \infty\}} P^x[\tau_0 < \infty]$$
$$\Rightarrow P^x[\tau_{i+1} < \infty] = P^x[\tau_i < \infty] \, P^x[\tau_0 < \infty]$$
$$\Rightarrow P^x[\tau_i < \infty] = P^x[\tau_0 < \infty]^k.$$
$$\Rightarrow G(x,x) = \sum_i e^{-c(x)} \cdot P^x[\tau_0 < \infty]^k.$$

So, TFAE

a) $P^x(\text{arbitrarily large } t : X(t) = x) = 1$
b) $P^x(\tau_i < \infty \ \forall i) = 1.$
c) $P^x(\tau_1 < \infty) = 1$
d) $G(x,x) = \infty$

$\square$

Thm: For an irreducible chain, either
- all states are recurrent, or
- all states are transient, and $G(x,y) < \infty \ \forall x,y$.

Pf: By CK, $p_{2t+s}(x,x) \geq p_t(x,y)\, p_s(y,y)\, p_t(y,x)$

$$\Rightarrow G(x,x) \geq \int_0^\infty p_{2t+s}(x,x)\, ds$$

$$\geq \underbrace{p_t(x,y)\, p_t(y,x)}_{>0}\ G(y,y)$$

So, $G(x,x) < \infty \Rightarrow G(y,y) < \infty$.

Let $\tau_y = \inf\{t > 0 : X(t) = y\}$.

SMP: $$G(x,y) = E^x\left[1_{\{\tau_y < \infty\}} \int_{\tau_y}^\infty 1_{\{X(t)=y\}}\, dt\right]$$

$$= P^x(\tau_y < \infty)\ G(y,y)$$

$$\leq G(y,y) < \infty \qquad \square.$$

Remark: X is recurrent iff the embedded discrete-time chain is recurrent.

Def: $f : S \to \mathbb{R}$ is superharmonic if

- $E^x f(X(t)) \leq f(x) \quad \forall x,t$
- $E^x |f(X(t))| < \infty \quad \forall x,t$

Prop: f superharmonic $\iff$ $f(X(t))$ is a super-martingale under $P^x\ \forall x$.

Pf: $E^x(f(X(t+s)) \mid \mathcal{F}_s) = E^{X(s)} f(X(t))$

f superharmonic $\Rightarrow E^{X(s)} f(X(t)) \leq f(X(s)) \Rightarrow f(X(t))$ super-martingale.

f super-martingale $\Rightarrow E^{X(s)} f(X(t)) \leq f(X(s))\quad P^x$-a.s.

$P^x(X(s) = x) > 0 \Rightarrow E^x f(X(t)) \leq f(x) \qquad \square$

Thm: An irred. MC is transient $\iff \exists$ non-constant, bounded, superharmonic function.

Pf: Suppose transient: fix y, set $f(x) = G(x,y)$.

$$E^x f(X(t)) = \sum_z p_t(x,z)\, f(z)$$

$$= \int_0^\infty \sum_z p_t(x,z)\, p_s(z,y)\, ds$$

$$= \int_0^\infty p_{t+s}(x,y)\, ds$$

$$< G(x,y) = f(x) < \infty$$

because $p_t(x,y) > 0\ \forall t$

Strict inequality $\Rightarrow f(x)$ non-constant.

f non-negative $\Rightarrow$ integrability condition.

Suppose f bounded, superharmonic.
Then $f(X(t))$ bounded supermartingale $\Rightarrow \lim_{t\to\infty} f(X(t)) = f_\infty$ exists a.s.

Claim: X recurrent $\Rightarrow f$ constant
irreducible $\Rightarrow \forall z$, $P^x(X(t) = z$ for arbitrarily large $t) = 1$.

$\Rightarrow f(z) = f_\infty \quad \forall z$
$\Rightarrow f$ constant.

Cor: Suppose $0 \in S$ is absorbing, but p_t is otherwise irreducible. Then

$$\underbrace{\forall x \neq 0, P^x(X(t) \neq 0 \;\; \forall t) > 0}_{\text{"X survives"}} \iff \exists \text{ non-constant superharmonic } f \text{ such that } f(0) \geq f(x) \;\forall x$$

Pf: Suppose chain survives. Set

$$f(x) = P^x(X(t) = 0 \text{ for some } t \geq 0).$$

$$\begin{aligned} E^x f(X(t)) &= E^x P^{X(t)}(X(s) = 0 \text{ for some } s \geq 0) \\ &= P^x(X(s+t) = 0 \text{ for some } s \geq 0) \\ &\geq f(x). \end{aligned}$$

f is bounded and non-constant, since X survives $\Rightarrow f(x) < 1$ for $x \neq 0$

Choose another state (1, say). Define $\bar{q}$ by

$$\bar{q}(0,0) = -1, \quad \bar{q}(0,1) = 1$$
$$\bar{q}(x,y) = q(x,y) \text{ otherwise.}$$

Let Y be the Markov chain with Q-matrix $\bar{q}$.
Y irreducible.
Take f bounded, non-const. superharmonic for X, $f(0) \geq f(x) \;\forall x$.
Let $\tau = \inf\{t : X(t) = 0\}$, and couple X with Y so that $\tau = \inf\{t : Y(t) = 0\}$ and $X(t) = Y(t) \;\forall t \leq \tau$.

$$\begin{aligned} E^x f(Y(t)) &= E^x f(Y(t)) 1_{\{t \leq \tau\}} + E^x f(Y(t)) 1_{\{t > \tau\}} \\ &\leq E^x f(X(t)) 1_{\{t \leq \tau\}} + f(0) \\ &= E^x f(X(t)) \\ &\leq f(x). \end{aligned}$$

$\Rightarrow Y$ transient
$\Rightarrow P^x(\tau < \infty) < 1 \quad \forall x$
$\Rightarrow X$ survives. $\square$

Thm: Suppose X irreducible. Then
$\exists$ stationary distribution $\Rightarrow$ recurrent $\Rightarrow$ $\exists$ stationary measure

Pf: Suppose $\exists$ stationary distribution π

$$\pi(y) = \sum_x \pi(x) \underbrace{\frac{1}{t}\int_0^t p_s(x,y)\,ds}_{\le 1,\ \text{converges to zero if } X \text{ transient}}$$

so X transient $\Rightarrow$ $\pi \equiv 0$.

Suppose recurrent. Fix z, let $s = \inf\{t \ge 0 : X(t) \ne z\}$
$\tau = \inf\{t \ge s : X(t) = z\}$.

Set $\pi(x) = E^z \int_0^\tau 1_{\{X(t)=x\}}\,dt$

Note: $\pi(z) = \frac{1}{c(z)} < \infty$

SMP: $E^z \int_0^s 1_{\{X(t)=x\}}\,dt = E^z \int_\tau^{\tau+s} 1_{\{X(t)=x\}}\,dt$

$$\Rightarrow \pi(x) = E^z \int_s^{\tau+s} 1_{\{X(t)=x\}}\,dt$$
$$= \int_s^\infty P^z(X(t)=x,\ t \le \tau+s)\,dt$$
$$= \int_0^\infty P^z(X(t+s)=x,\ t \le \tau)\,dt$$
$$= \int_0^\infty \sum_y P^z(X(t)=y,\ \tau \ge t,\ X(t+s)=x)\,dt$$
$$= \int_0^\infty \sum_y P^z(X(t)=y, \tau \ge t)\, p_s(y,x)\,dt$$
$$= \sum_y p_s(y,x) \underbrace{\int_0^\infty P^z(X(t)=y, \tau \ge t)\,dt}_{\pi(y)}$$

$\Rightarrow$ π stationary measure.

$\pi(z) \in (0,\infty)$ + irreducible $\Rightarrow$ all $\pi(x) \in (0,\infty)$ $\square$

Remark. the π we just constructed is finite iff $E^z \tau < \infty$.

Prop: Irreducible, recurrent $\Rightarrow$ stationary measure is unique up to constant multiples.

Cor: Suppose irreducible, recurrent. Either
- $E^z \tau = \infty$ $\forall z$ and every stationary measure is infinite, or
- $E^z \tau < \infty$ $\forall z$ and $\exists!$ stationary distribution

Proof: Suppose π_1, π_2 stationary. Irreducible $\Rightarrow$ $\pi_1, \pi_2 > 0$

$$\bar{p}_t(x,y) = \frac{\pi_1(y)}{\pi_1(x)} p_t(y,x)$$

π_1 stationary $\Rightarrow \sum_y \bar{p}_t(x,y) = 1$.

Using Green's function, $\bar{p}_t$ is recurrent.

Set $\alpha(x) = \frac{\pi_2(x)}{\pi_1(x)}$. Then $\underbrace{\sum_y \bar{p}_t(x,y)\alpha(y)}_{E^x \alpha(Y_1)} = \underbrace{\sum_y \frac{\pi_2(y)}{\pi_1(x)} p_t(y,x)}_{\alpha(x)}$

$Y_0, Y_1, \ldots$ discrete time MC with transitions $\bar{p}_t$

$\Rightarrow \alpha(Y_k)$, $k \geq 0$ is a discrete-time martingale

nonnegative $\Rightarrow$ converges a.s.

recurrent $\Rightarrow$ α constant

$\Rightarrow$ π_1 proportional to π_2. $\square$

Warning: some transient chains have stationary measures. e.g. random walk on $\mathbb{Z}^d$, $d \geq 3$.

Convergence

Thm: Suppose irreducible, $\exists$ stationary distribution. Then

$$\lim_{t\to\infty} p_t(x,y) = \pi(y) \qquad \forall y.$$

Pf: Let X, Y be independent copies. Then $Z(t) = (X(t), Y(t))$ is recurrent, irreducible MC with stationary distribution $\bar{\pi}(x,y) = \pi(x)\pi(y)$

$\tau = \inf\{t : X(t) = Y(t)\} < \infty$ a.s. because Z recurrent.

$$W(t) = \begin{cases} Y(t), & t < \tau \\ X(t), & t \geq \tau. \end{cases}$$

$W(t)$ is MC, transition function p_t

Consider $P^{\delta_x \times \pi}$. $P^{\delta_x \times \pi}(Y(t) = y) = \pi(y) \quad \forall t.$

$P^{\delta_x \times \pi}(X(t) = y) = p_t(x,y) \quad \forall t.$

$P^{\delta_x \times \pi}(W(t) = y) = \pi(y) \quad \forall t$

$$|p_t(x,y) - \pi(y)| = |P(X(t) = y) - P(W(t) = y)|$$

$$\leq 2\, P(X(t) \neq W(t))$$

$$= 2\, P(\tau > t) \longrightarrow 0 \text{ as } t \to \infty \qquad \square.$$

Example: Branching chain

$\{p_k, k \geq 0\}$ probability distribution on $\{0, 1, \ldots\}$

$p_0 \in (0,1)$, $p_1 = 0$.

$$\sum_{k \geq 0} k p_k = m < \infty.$$

$X(t)$ = # individuals in population

at rate 1, each individual dies and is replaced by a random number of individuals $\sim \{p_k\}$

$$q(k,l) = \begin{cases} k p_{l-k+1} & \text{if } k \neq l \\ -k & \text{if } k = l \end{cases}$$

Not irreducible: 0 is absorbing.

Claim: This defines a Markov chain.
Pf: Need to check it is non-explosive.

Let $\xi_1, \xi_2, \ldots$ iid, $P(\xi_i = k-1) = p_k$

$Z_n = Z_0 + \sum_{k=1}^{n} \xi_k$ is the embedded, discrete-time chain.

Strong law of large numbers $\Rightarrow \frac{1}{n} Z_n \to m-1$ a.s.

$\Rightarrow \exists\ N^*$ such that $Z_n \le mn \quad \forall\ n \ge N^*$.

$$\Rightarrow \sum_{n=0}^{\infty} \frac{1}{c(Z_n)} = \sum_{n=0}^{\infty} \frac{1}{Z_n} \ge \frac{1}{m} \sum_{n=N^*}^{\infty} \frac{1}{n} = \infty.$$

$\Rightarrow$ non-explosive $\square$

<u>Kolmogorov forward equations</u>

Thm: Suppose KBE hold, and $\sum_y p_t(x,y)\, c(y) < \infty \quad \forall t$. Then

$$\frac{d}{dt} p_t(x,y) = \sum_z p_t(x,z)\, q(z,y) \qquad \text{(KFE)}$$

Pf:
$$\frac{p_{t+h}(x,y) - p_t(x,y)}{h} = \sum_z p_t(x,z) \underbrace{\frac{p_h(z,y) - \delta(z,y)}{h}}_{\to\, q(z,y) \text{ as } h \to 0}$$

$$\begin{aligned} |p_h(z,y) - \delta(z,y)| &\le 1 - p_h(z,z) \\ &\le 1 - e^{-c(z)h} \\ &\le c(z)h \end{aligned}$$

$\Rightarrow$ summand $\le p_t(x,z)\, c(z)$, which is summable $\square$

Back to example:

Claim: $E^k X(t) = k e^{(m-1)t}$.

Pf: $E^k X(t) = \sum_l p_t(k,l) \cdot l$.

$$\begin{aligned} \sum_{l \ne k} q(k,l)\, l &= k \sum p_{l-k+1}\, l \\ &= k \sum_l p_l\, (k+l-1) \\ &= k^2 + km - k. \end{aligned}$$

$$\Rightarrow \sum_l q(k,l)\, l = k(m-1)$$

Therefore,
$$\begin{aligned} \frac{d}{dt} E^k X(t) &= \sum_l l \sum_y p_t(k,y)\, q(l,y) \qquad \text{(KFE)} \\ &= (m-1) \sum_y y\, p_t(k,y) \\ &= (m-1)\, E^k(X(t)) \end{aligned}$$

$$\Rightarrow E^k X(t) = \exp((m-1)t)\, k.$$

To justify use of KFE, need to show

$$\sum p_t(k,\ell)\, c(\ell) < \infty$$
$$= \sum p_t(k,\ell) \cdot \ell$$
$$= E^k X(t).$$

Define $X_n(t)$ by $\quad q_n(k,\ell) = \begin{cases} q(k,\ell) & \text{if } k < n \\ 0 & \text{otherwise.} \end{cases}$

$$\sum_\ell \ell\, q_n(k,\ell) \le km$$

$$\Rightarrow E^k X_n(t) \le k e^{mt}$$

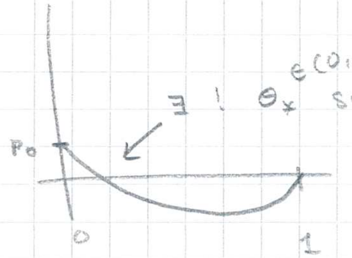
(Fatou)
$$\Rightarrow E^k X(t) \le k e^{mt} < \infty. \qquad \square.$$

<u>Survival</u>

- $m < 1 \Rightarrow E^k X(t) \to 0 \Rightarrow$ no survival
- $m = 1 \Rightarrow X(t)$ is a non-negative martingale
  $\Rightarrow X(t)$ converges a.s.
  $\Rightarrow X(t) \to 0$ a.s (no survival)

- $m > 1$:
  Let $\psi(\theta) = \sum_j p_j \theta^j = p_0 - \theta + p_2\theta^2 + \dots$

  - convex
  - $\psi(0) = p_0 > 0$
  - $\psi(1) = 0$
  - $\psi'(1) = m - 1 > 0$



$\exists !\ \theta_* \in (0,1)$ such that $\psi(\theta_*) = 0$

By KFE and the fact that $\sum_\ell q(k,\ell)\,\theta_*^\ell = k\,\theta_*^{k-1}\psi(\theta_*) = 0$,

$$E^k \theta_*^{X(t)} = \theta_*^k \quad \forall k, t$$

$\Rightarrow \theta_*^{X(t)}$ bounded martingale
$\Rightarrow \theta_*^{X(t)}$ converges a.s.

$$\theta_*^{X(t)} \to 1 \quad \text{if} \quad X(t) \to 0$$
$$\theta_*^{X(t)} \to 0 \quad \text{if} \quad X(t) \to \infty$$

$$\Rightarrow P^k(X(t) \to 0) = E^k \theta_*^{X(t)} = \theta_*^k \in (0,1)$$

$\Rightarrow X(t)$ survives