S countable set

Three objects

- X Markov chain
  - right-continuous filtration
  - càdlàg paths
  - finitely many jumps in any finite interval
- P_t transition function
  - $P_t(x,y) \geq 0$
  - $\sum_y P_t(x,y) = 1$
  - $\lim_{t \to 0} P_t(x,x) = 1$
  - $P_{s+t}(x,y) = \sum_z P_s(x,z) P_t(z,y)$

    "Chapman-Kolmogorov" "CK"
- Q-matrix
  - $q(x,y) \geq 0 \quad \forall x \neq y$
  - $\sum_y q(x,y) = 0$
  - $c(x) = -q(x,x) \geq 0$

" P_t gives finite dimensional distributions of X
Q gives transition rates "

Last time: $X \longrightarrow P_t$.

Now: $P_t \longrightarrow Q$

Properties of transition functions

a) $P_t(x,x) > 0 \quad \forall x, t$

b) $P_t(x,x) = 1$ for some $t > 0 \implies P_t(x,x) = 1 \quad \forall t$

"x is an absorbing state"

c) $|P_t(x,y) - P_s(x,y)| \leq 1 - P_{|t-s|}(x,x)$

d) $c(x) := -q(x,x) := -\frac{d}{dt}\Big|_{t=0} P_t(x,x) \in [0, \infty]$

e) $P_t(x,x) \geq e^{-c(x)t}$

f) if $c(x) < \infty$ then $\forall y \neq x$

$$q(x,y) := \frac{d}{dt}\Big|_{t=0} P_t(x,y) \in [0, \infty)$$

g) if $c(x) < \infty$ then $\sum_y q(x,y) \leq 0$

h) if $\sum_y q(x,y) = 0$ then $\forall y$, $P_t(x,y)$ is continuously differentiable in t, and

$$\frac{d}{dt} P_t(x,y) = \sum_z q(x,z) P_t(z,y)$$

"Kolmogorov backward equations" "KBE"

Proof:

a) $p_t(x,x) \to 1$ as $t \to 0$ $\Rightarrow$ for all small enough t, $p_t(x,x) > 0$.

CK $\Rightarrow$ $p_t(x,x) \geq p_{t/k}^k(x,x) > 0$ for k large enough.

b)
$$p_{s+t}(x,x) = \sum_z p_s(x,z)\, p_t(z,x)$$
$$\leq p_s(x,x)\, p_t(x,x) + \sum_{z \neq x} p_s(x,z)$$
$$= p_s(x,x)\, p_t(x,x) + 1 - p_s(x,x)$$
$$= 1 - p_s(x,x)\left(1 - p_t(x,x)\right)$$

$p_s(x,x) > 0$, so $p_{s+t}(x,x) = 1 \Rightarrow p_t(x,x) = 1$.

So $\{t : p_t(x,x) = 1\}$ is an interval.
$p_{2t}(x,x) \geq p_t^2(x,x) \Rightarrow$ if the interval is non-trivial then it is all of $[0,\infty)$

c)
$$p_{t+s}(x,y) - p_t(x,y) = \underbrace{\sum_{z \neq x} p_s(x,z)\, p_t(z,y)}_{0 \leq \quad \leq 1 - p_s(x,x)} - \underbrace{p_t(x,y)\left(1 - p_s(x,x)\right)}_{0 \leq \quad \leq 1 - p_s(x,x)}$$
$$\leq 1 - p_s(x,x)$$

d) $f(t) := -\log p_t(x,x)$ is continuous (by part c) and sub-additive (by CK)

$$\Rightarrow \lim_{t \to 0} \frac{f(t)}{t} \text{ exists } \in [0,\infty]$$
$$= \lim_{t \to 0} \frac{1 - p_t(x,x)}{t} \qquad \text{(chain rule)}$$
$$=: c(x)$$

e) $f(t)$ subadditive $\Rightarrow f(t) \leq c(x)\, t$
$\Rightarrow p_t(x,x) \geq e^{-c(x)t}$

f) by (e), $1 - p_t(x,x) \leq 1 - e^{-c(x)t} \leq c(x)\, t$

$$\Rightarrow \frac{1}{t} \sum_{y \neq x} p_t(x,y) = \frac{1 - p_t(x,x)}{t} \leq c(x) < \infty$$

let $q(x,y) = \limsup_{t \to 0} \frac{p_t(x,y)}{t} \leq c(x) < \infty$.

want to show $q(x,y) = \liminf_{t \to 0} \frac{p_t(x,y)}{t}$.

Fix t. If $n\delta \leq t$ then
$$p_t(x,y) \geq \sum_{k=0}^{n-1} \underbrace{p_\delta^k(x,x)}_{\geq e^{-tc(x)}}\, p_\delta(x,y)\, \underbrace{p_{t-(k+1)\delta}(y,y)}_{\geq \inf_{0 \leq s \leq t} p_s(y,y)}$$
$$\geq e^{-tc(x)} \cdot \inf_{0 \leq s \leq t} p_s(y,y) \cdot n\, p_\delta(x,y)$$

$$\frac{p_t(x,y)}{t} \geq e^{-tc(x)} \inf_{0\le s\le t} p_s(y,y)\ \frac{n\delta}{t} \cdot \frac{p_\delta(x,y)}{\delta}$$

Now choose a sequence of $\delta \to 0$ with

$$\frac{p_\delta(x,y)}{\delta} \to q(x,y)$$

Set $n = \lfloor t/\delta \rfloor$. Then $\frac{n\delta}{t} \to 1$, so

$$\liminf_{t\to 0} \frac{p_t(x,y)}{t} \geq q(x,y).$$

g) $$\sum_{y\ne x} q(x,y) \le \lim_{t\to 0} \sum_{y\neq x} \frac{p_t(x,y)}{t} \le c(x)$$

h) $$\frac{p_{t+s}(x,y) - p_t(x,y)}{s} - \sum_z q(x,z)\, p_t(z,y)$$

$$\overset{(CK)}{=} \sum_z \left[\frac{p_s(x,z) - p_0(x,z)}{s} - q(x,z)\right] p_t(z,y)$$

Each term converges to zero. But we need some uniformity.

For $T \ni x$ finite,

$$\sum_{z\notin T} \left|\left[\frac{p_s(x,z) - p_0(x,z)}{s} - q(x,z)\right] p_t(z,y)\right|$$

$$\le \sum_{z\notin T} \left|\frac{p_s(x,z) - p_0(x,z)}{s}\right| + \sum_{z\notin T} q(x,z)$$

$$= \sum_{z\notin T} \frac{p_s(x,z)}{s} + \sum_{z\notin T} q(x,z)$$

$$= \frac{1}{s}\left[1 - \sum_{z\in T} \frac{p_s(x,z)}{s}\right] - \sum_{z\in T} q(x,z) \quad \text{(using (g))}$$

$$\to -2 \sum_{z\in T} q(x,z)$$

$$\sum q(x,z) = 0 \implies -2\sum_{z\in T} q(x,z) \to 0 \text{ as } T \text{ grows.}$$

$$\implies \lim_{s\downarrow 0} \frac{p_{t+s}(x,y) - p_t(x,y)}{s} = \sum_z q(x,z)\, p_t(z,y)$$

RHS is continuous, and functions that have continuous right derivatives are differentiable. $\square$

Blackwell's example: transition function such that $c(x) = \infty$ $\forall x$.

Take β_i, δ_i positive sequences.

$X_i(t)$ 2-state Markov chain on $\{0,1\}$ with

Q-matrix $\begin{bmatrix} -\beta_i & \beta_i \\ \delta_i & -\delta_i \end{bmatrix}$.

$X(t) := (X_1(t), X_2(t), \dots) \in \{0,1\}^\infty$.

(Careful! X is not necessarily a Markov chain)

$S := \{ x = (x_1, x_2, \dots) \in \{0,1\}^\infty : \sum x_i < \infty \}$

$\Rightarrow$ S countable.

$p_t(x,y) := P^x(X(t) = y) = \prod_{i=1}^{\infty} P^{x_i}(X_i(t) = y_i)$

Proposition: If $\sum_i \frac{\beta_i}{\beta_i + \delta_i} < \infty$ then $p_t(x,y)$ is a transition function.

Pf: Note that $P^0(X_i(t) = 1) = \frac{\beta_i}{\beta_i+\delta_i}\left(1 - e^{-t(\beta_i+\delta_i)}\right) \le \frac{\beta_i}{\beta_i+\delta_i}$

To show $\sum_{y \in S} p_t(x,y) = 1$, need $P^x(X(t) \in S) = 1$.

Let $I = \{ i : x_i = 0 \}$. Then $\{\sum_i X_i(t) < \infty\} = \{\sum_{i\in I} X_i(t) < \infty\}$

$P^x\left(\sum_i X_i(t) < \infty\right) = P^x\left(\sum_{i \in I} X_i(t) < \infty\right)$

$= 1 - P^x(X_i(t) = 1 \text{ i.o. for } i \in I)$

infinitely often

For $i \in I$, $P^x(X_i(t) = 1) \le \frac{\beta_i}{\beta_i + \delta_i}$

Borel-Cantelli + $\sum \frac{\beta_i}{\beta_i+\delta_i} < \infty$ $\Rightarrow$ $P^x(X_i(t) = 1 \text{ i.o.}) = 0$.

$\Rightarrow$ $P^x(X(t) \in S) = 1$.

Next: $p_t(x,x) \to 1$ as $t \to 0$:

Choose n large enough so $x_i = 0$ $\forall i > n$

$p_t(x,x) \ge \prod_{i=1}^{n} P^{x_i}(X_i(t) = x_i) \cdot \prod_{i>n} \frac{\delta_i}{\beta_i + \delta_i}$

$\Rightarrow \liminf_{t\to 0} p_t(x,x) \ge \prod_{i>n} \frac{\delta_i}{\beta_i+\delta_i}$

$\to 1$ as $n \to \infty$.

Finally, CK equations:
Let $p_t^{(n)}(x,y) = \prod_{i=1}^{n} P^{x_i}(X_i(t)=y_i)$

Then $p_t^{(n)}$ satisfies CK equations and $p_t^{(n)}(x,y) \downarrow p_t(x$

$$p_{t+s}(x,y) = \lim_{n\to\infty} p_{t+s}^{(n)}(x,y)$$
$$= \lim_{n\to\infty} \sum_z p_t^{(n)}(x,z)\, p_s^{(n)}(z,y)$$
$$= \sum_z p_t(x,z)\, p_s(z,y) \qquad \square$$

Thm: Suppose $\sum \frac{\beta_i}{\beta_i+\delta_i} < \infty$ and $\sum \beta_i = \infty$. Then

1) $c(x) = \infty \quad \forall x$.

2) $\forall\, \varepsilon > 0,\quad P^x(X(t)=x \ \ \forall \text{ rational } t<\varepsilon) = 0$

3) There is no Markov chain with transition function p_t.

Pf: 1) Choose $m \le n$ such that $\forall\, i \ge m,\ x_i = 0$.

$$p_t(x,x) = \prod_i P^{x_i}(X_i(t)=x_i)$$
$$\le \prod_{i=m}^{n} P^0(X_i(t)=0)$$
$$= \prod_{i=m}^{n}\left(1 - \frac{\beta_i}{\beta_i+\delta_i}\underbrace{\left(1-e^{-t(\beta_i+\delta_i)}\right)}_{\substack{t(\beta_i+\delta_i)+O(t^2)\\ \text{as } t\to 0}}\right)$$
$$\sim \prod_{i=m}^{n}(1 - t\beta_i) \quad \text{as } t\to 0$$
$$\le \exp\left(-t\sum_{i=m}^{n}\beta_i\right).$$

$$p_t(x,x) \ge e^{-tc(x)} \Rightarrow c(x) \ge \sum_{i=m}^{n}\beta_i$$
$$\Rightarrow c(x) = \infty$$

2) $\{X(t)=x \ \ \forall \text{ rational } t<\varepsilon\}$

$= \{X(t)=x \ \ \forall\, t<\varepsilon\}$ because X_i have càdlàg paths.

$$P^x(X(t)=x \ \forall\, t<\varepsilon)$$
$$= \prod_i P^{x_i}(X_i(t)=x_i \ \forall\, t<\varepsilon)$$
$$\le \prod_{i=m}^{\infty} P^0(X_i(t)=0 \ \forall\, t<\varepsilon) \qquad (\text{where } m \text{ is such that } x_i=0\ \forall\, i\ge m)$$
$$\le \prod_{i=m}^{\infty} e^{-\beta_i\varepsilon}$$
$$= 0.$$

3) Suppose there exists such a Markov chain $X(t)$. The events in (2) may be written as a limit of events only depending on finitely many $X(t_i)$, so their probability depends only on p_t.

Therefore $X(t)$ does not have càdlàg paths. $\square$.

$Q \to p_t$

Prop: Suppose $p_t(x,y) \in [0,1]$ $\forall$ t, x, y. TFAE:

a) $p_t(x,y)$ is continuously differentiable in t and

$$\frac{\partial}{\partial t} p_t(x,y) = \sum_z q(x,z)\, p_t(z,y) \qquad \text{(KBE)}$$

$$p_0(x,y) = \delta(x,y)$$

b) (*) $p_t(x,y)$ is continuous in t and

$$p_t(x,y) = \delta(x,y) e^{-c(x)t} + \int_0^t e^{-c(x)(t-s)} \sum_{z \neq x} q(x,z)\, p_s(z,y)\, ds$$

Pf: $a \Rightarrow b$: bring $c(x)p_t(x,y)$ to the other side, multiply by $e^{c(x)t}$ and integrate

$b \Rightarrow a$: RHS is continuously differentiable. Differentiate.

Def: $p_t^{(0)} \equiv 0$

$$p_t^{(n+1)}(x,y) = \delta(x,y) e^{-c(x)t} + \int_0^t e^{-c(x)(t-s)} \sum_{z \neq x} q(x,z)\, p_s^{(n)}(z,y)\, ds$$

Prop: a) $p_t^{(n)}(x,y) \geq 0$.

b) $\sum_y p_t^{(n)}(x,y) \leq 1$

c) $p_t^{(n+1)}(x,y) \geq p_t^{(n)}(x,y)$

Pf: a) by induction, since everything in the def'n of $p_t^{(n+1)}$ is non-negative

b) $$\sum_y p_t^{(n+1)}(x,y) \leq e^{-c(x)t} + \int_0^t e^{-c(x)(t-s)} \sum_{z \neq x} q(x,z)\, ds$$
$$= e^{-c(x)t} + \int_0^t c(x) e^{-c(x)(t-s)}\, ds$$
$$= 1.$$

c) $$p_t^{(n+1)}(x,y) - p_t^{(n)}(x,y) = \int_0^t e^{-c(x)(t-s)} \sum_{z \neq x} q(x,z) \left(p^{(n)}(x,y) - p^{(n-1)}(x,y)\right)$$

≥ 0 by induction. $\square$.

Define $p_t^*(x,y) = \lim_{n \to \infty} p_t^{(n)}(x,y)$.

(*) it is actually redundant to assume p_t continuous in t, since $p_t \in [0,1]$ implie

$$|p_s(x,y) - p_t(x,y)| \leq \delta(x,y)\left|e^{-c(x)t} - e^{-c(x)s}\right| + \int_{s \wedge t}^{s \vee t} e^{-c(x)(t-u)} \sum_{z \neq x} q(x,z)\, ds$$
$$\leq \left|e^{-c(x)t} - e^{-c(x)s}\right| + c(x)\,|s-t|.$$

Thm:

a) $p_t^*(x,y) \ge 0$

b) $\sum_y p_t^*(x,y) \le 1$

c) $p_{t+s}^*(x,y) = \sum_z p_t^*(x,z)\, p_s^*(z,y)$ (CK)

d) $\frac{\partial}{\partial t} p_t^*(x,y) = \sum_z q(x,z)\, p_t^*(z,y)$ (KBE)

e) If $p_t(x,y)$ is a non-negative solution of the KBE then $p_t^*(x,y) \le p_t(x,y)$

f) If $\sum p_t^*(x,y) = 1$ then p_t^* is the unique transition function satisfying KBE

Pf: a, b follow from previous proposition.

c) later

d) by monotone limits and the definition of $p_t^{(n)}$,

$$p_t^*(x,y) = e^{-c(x)t}\, \delta(x,y) + \int_0^t e^{-c(x)(t-s)} \sum_{z \ne x} q(x,z)\, p_t^*(z,y)\, d$$

which is equivalent to KBE.

e) Claim that $p_t^{(n)}(x,y) \le p_t(x,y)$ $\forall n$.
This is true for $n=0$, and follows for all n by induction, since

$$p_t^{(n+1)}(x,y) = e^{-c(x)t}\, \delta(x,y) + \int_0^t e^{-c(x)(t-s)} \sum_{z \ne x} q(x,z)\, p_t^{(n)}(z,y)\, ds$$

$$\le \quad " \qquad\qquad " \quad p_t(z,y)\, ds$$

$$= p_t(x,y)$$

f) If p_t is another solution then $p_t \ge p_t^*$ by (e).
If $p_t^* \ne p_t$ then $\exists\, x$ such that

$$1 = \sum_y p_t^*(x,y) < \sum_y p_t(x,y) = 1,$$

a contradiction $\square$

$Q \longrightarrow X$

Prop: Let X be Markov chain.
$\tau := \inf\{t \ge 0 : X(t) \ne X(0)\}$
Then $\exists\; c(x) \in [0,\infty]$ such that $P^x(\tau > t) = e^{-c(x)t}$

Pf: Fix $0 \le s \le t$, apply Markov property to $Y = 1_{\{X(r) = x\ \forall\ 0 \le r \le t-s\}}$
Then

$$P^x(\underbrace{X(r) = x \ \forall\ s \le r \le t}_{Y \circ \theta_s} \mid \mathcal{F}_s) = P^{X(s)}(X(r) = x \ \forall\ 0 \le r \le t-s) \quad P^x\text{-a.s.}$$

$$= 1_{\{X(s) = x\}}\, P^x(\tau > t-s)$$

Multiply both sides by $\{\tau > s\}$ and integrate:

$$\text{LHS} = E^x\left(1_{\{\tau > s\}}\, P^x(X(r) = x \ \forall s \le r \le t \mid \mathcal{F}_s)\right)$$

$$= P^x(\tau > s + t)$$

$$\text{RHS} = P^x(\tau > s)\, P^x(\tau > t)$$

So $f(t) := P^x(\tau > t)$ satisfies
- f non-increasing
- $f(s+t) = f(s)f(t)$

$\Rightarrow f(t) = e^{-ct}$ for some $c \in [0,\infty]$ $\square$.

The embedded discrete-time chain

Given q, c, define

$$p(x,y) = \begin{cases} \delta(x,y) & \text{if } c(x) = 0 \\ \frac{q(x,y)}{c(x)} & \text{if } c(x) > 0,\ x \neq y \\ 0 & \text{if } c(x) > 0,\ x = y \end{cases}$$

Then $\sum_y p(x,y) = 1 \quad \forall x$.

Let $Z_0, Z_1, \dots$ be a discrete-time Markov chain with transition matrix p. Write P^x for distribution of $Z_0, Z_1, \dots$ starting from x.

Let $\xi_0, \xi_1, \dots \sim$ i.i.d. $\mathrm{Exp}(1)$

$$\tau_i := \frac{\xi_i}{c(Z_i)} \qquad (= \infty \text{ if } c(Z_i) = 0)$$

$N(t) := \inf\{n \geq 0 : \sum_{i=0}^{n} \tau_i > t\}$ "clock"

$X(t) := Z_{N(t)}$ on $\{N(t) < \infty\}$

Remark: (Z_n, τ_{n-1}) is a Markov chain (discrete time)

Prop:
a) $p_t^{(n)}(x,y) = P^x(X(t)=y,\ N(t) < n)$
b) $p_t^*(x,y) = P^x(X(t)=y,\ N(t) < \infty)$
c) $\sum_y p_t^*(x,y) = P^x(N(t) < \infty)$.

Pf: a) $P^x(X(t)=y,\ N(t) < n+1 \mid \tau_0 = s,\ Z_1 = z)$

(if $s < t$)

$$= \sum_{m=1}^{n} P^x(Z_m = y,\ N(t) = m \mid \tau_0 = s,\ Z_1 = z)$$

$$= \sum_{m=1}^{n} P^x\Big(Z_m = y,\ \sum_{i=1}^{m-1} \tau_i \leq t-s < \sum_{i=1}^{m} \tau_i \,\Big|\, \tau_0 = s,\ Z_1 = z\Big)$$

$$= \sum_{m=1}^{n} P^z\Big(Z_{m-1} = y,\ \sum_{i=0}^{m-2} \tau_i \leq t-s < \sum_{i=1}^{m} \tau_i\Big)$$

$$= \sum_{m=0}^{n-1} P^z(Z_m = y,\ N(t-s) = m)$$

$$= P^z(X(t-s) = y,\ N(t-s) < n)$$

(if $s \geq t$)

$$= \delta(x,y)$$

$$\Rightarrow P^x(X(t)=y,\ N(t) < n+1) = \delta(x,y)\, P^x(\tau_0 \geq t) + \int_0^t \underbrace{c(x) e^{-c(x)s}}_{\text{density of } \tau_0} \sum_{z \neq x} \overbrace{\frac{q(x,z)}{c(x)}}^{P^x(Z_1 = z)} P^x(X(t)=y,\ N(t) < n+1 \mid \tau_0 = s,\ Z_1 = z)\, ds$$

$$= e^{-c(x)t}\,\delta(x,y)$$

$$+ \int_0^t e^{-c(x)s} \sum_{z\neq x} q(x,z)\, P^z\big(X(t-s)=y,\ N(t-s)<n\big)\, ds$$

$$\overset{\text{(induction)}}{=} e^{-c(x)t}\,\delta(x,y) + \int_0^t e^{-c(x)s} \sum_{z\neq x} q(x,z)\, p^{(n)}_{t-s}(z,y)\, ds$$

$$= p^{(n+1)}_t(x,y)$$

b) Follows from a because $p^{(n)} \uparrow p^*$.

c) Follows from b by summing over y □.

Corollary: p^*_t satisfies CK equations.

Pf: $p^*_{t+s}(x,y) = P^x(X(s+t)=y,\ N(s+t)<\infty)$

$$= \sum_z P^x(X(s+t)=y,\ N(s+t)<\infty,\ X(s)=z,\ N(s)<\infty)$$

$$= \sum_z P^x(X(s+t)=y,\ N(s+t)<\infty \mid X(s)=z,\ N(s)<\infty) \cdot P^x(X(s)=z, N(s)<\infty)$$

$$= \sum_z P^z(X(t)=y,\ N(t)<\infty) \cdot P^x(X(s)=z,\ N(s)<\infty)$$

$$= \sum_z p^*_t(z,y)\, p^*_s(x,z)$$ □

Thm: TFAE:

a) p^*_t is stochastic $\left(\sum_y p^*_t(x,y)=1\right)$

b) $P^x(N(t)<\infty)=1 \ \forall x,t$

c) $\sum_n \tau_n = \infty$ P^x-a.s. $\forall x$.

d) $\sum_n \frac{1}{c(Z_n)} = \infty$ P^x-a.s. $\forall x$.

Pf: By previous proposition, $a \Leftrightarrow b$

Since $\{\sum \tau_n > t\} = \{N(t)<\infty\}$, $b \Leftrightarrow c$.

$c \Leftrightarrow d$:

$$E^x\left(\exp\left(-\lambda \sum_{k=0}^n \tau_k\right) \,\Big|\, Z_0, Z_1, \ldots\right)$$

$$= E^x\left(\exp\left(-\lambda \sum_{k=0}^n \frac{\xi_k}{c(Z_k)}\right) \,\Big|\, Z_0, \ldots\right)$$

$$= \prod_{k=0}^n \frac{c(Z_k)}{c(Z_k)+\lambda} \qquad \left(\text{since } E\exp(-\mu\xi) = \frac{1}{1+\mu}.\right)$$

$$\underbrace{E^x \exp\left(-\lambda \sum_{k=0}^\infty \tau_k\right)}_{\downarrow\,(\lambda\to 0)\ \ 1\{\sum \tau_k<\infty\}} = \underbrace{E^x \prod_{k=0}^\infty \frac{c(Z_k)}{c(Z_k)+\lambda}}_{\downarrow\,\lambda\to 0\ \ 1\{\sum \frac{1}{c(Z_k)}<\infty\}}.$$

$$\Rightarrow P^x\left(\sum \tau_k<\infty\right) = P^x\left(\sum \frac{1}{c(Z_k)}<\infty\right)$$ □.