Markov processes and PDE

Thm: Suppose L is a generator with semigroup $T(t)$. For $f \in \mathcal{D}(L)$, the unique $u(t,x)$ satisfying

$$* \begin{cases} \frac{\partial u}{\partial t}(t,x) = (L\,u(t,\cdot))(x) & \forall\, x,t \\ u(0,x) = f(x) \\ \sup_{x,t} |u(t,x)| < \infty \\ \sup\{|\frac{\partial u}{\partial t}(t,x)| : x \in S,\ t \in [0,T]\} < \infty & \forall\, T. \end{cases}$$

implicit here is the assumption $u(t,\cdot) \in \mathcal{D}(L)$ $\forall\, t$.

is $u(t,x) = T(t)f(x)$

Proof: Instead of writing things like $(T(t)u(t,\cdot))(x)$, we will write $T(t)u(t,$
It is easy to check that $T(t)f(x)$ is a solution to (*).

To prove uniqueness, it suffices to consider the case $f(x) = 0$.
Let u be a solution to (*) with $f(x) = 0$.
Set $v(t,x) = e^{-\alpha t}\, u(t,x)$ for some $\alpha > 0$. Then

$$\begin{aligned} \frac{\partial v}{\partial t}(t,x) &= -\alpha\, v(t,x) + L\,v(t,x) \\ &= -\alpha\,(I - \tfrac{1}{\alpha}L)\,v. \\ \Rightarrow \quad v &= -\tfrac{1}{\alpha}(I - \tfrac{1}{\alpha}L)^{-1}\frac{\partial v}{\partial t}. \end{aligned}$$

$$v(t,x) = -\frac{1}{\alpha}\int_0^\infty e^{-\alpha s}\, T(s)\frac{\partial v}{\partial t}(t,x)\, ds.$$

$$\begin{aligned} \int_0^r v(t,x)\,dt &= -\frac{1}{\alpha}\int_0^r \int_0^\infty e^{-\alpha s}\, T(s)\frac{\partial v}{\partial t}(t,x)\,ds\,dt \\ &= -\frac{1}{\alpha}\int_0^\infty e^{-\alpha s}\, T(s) \int_0^r \frac{\partial v}{\partial t}(t,x)\,dt\,ds \\ &= -\frac{1}{\alpha}\int_0^\infty e^{-\alpha s}\, T(s)\, v(r,x)\,ds. \end{aligned}$$

Now, $\sup_x |T(s)\,v(r,x)| \le \sup_x |v(r,x)| \le e^{-\alpha r} \sup_{x,t} |u(t,x)|$

$$\Rightarrow \left|\int_0^r v(t,x)\,dt\right| \le \frac{1}{\alpha}\, e^{-\alpha r} \sup_{x,t}|u(t,x)| \int_0^\infty e^{-\alpha s}\,ds$$

$$\longrightarrow 0 \text{ as } r \to \infty$$

$$\Rightarrow \quad \int_0^\infty e^{-\alpha t}\, u(t,x)\,dt = \int_0^\infty v(t,x)\,dt = 0$$

$$\Rightarrow \quad u \equiv 0$$

□

Duality

Suppose $X_1(t)$ and $X_2(t)$ are Feller processes on S_1, S_2
$T_1(t)$ $T_2(t)$
L_1 L_2

$H : S_1 \times S_2 \to \mathbb{R}$ bounded, jointly measurable.

Def: X_1 and X_2 are dual with respect to H if

$$E^{x_1} H(X_1(t), x_2) = E^{x_2} H(x_1, X_2(t)) \qquad \forall x_1, x_2, t$$

Note that we write $E^{x_1} H(X_1(t), x_2)$ instead of $T_1(t)$, since $H(\cdot, x$
may not be in $C(S_1)$.

Thm: Suppose $(T_1(t) H)(x_1, \cdot) \in \mathcal{D}(L_2)$ and $(T_2(t) H)(\cdot, x_2) \in \mathcal{D}(L_1)$
$\forall x_1, x_2, t$. If
$$(L_1 H(\cdot, x_2))(x_1) = (L_2 H(x_1, \cdot))(x_2) \qquad \forall x_1, x_2$$
then $X_1(t)$ and $X_2(t)$ are dual with respect to H

Proof: First, note that $T_1(t) T_2(s) H = T_2(s) T_1(t) H$ by Fubini.
Hence, $T_1(t) L_2 H = L_2 T_1(t) H$.

Let $u(t, x_1, x_2) = (T_1(t) H)(x_1, x_2)$
Then
$$\begin{aligned} \frac{\partial u}{\partial t}(t, x_1, x_2) &= (T_1(t) L_1 H)(x_1, x_2) \\ &= (T_1(t) L_2 H)(x_1, x_2) \\ &= (L_2 T_1(t) H)(x_1, x_2) \\ &= L_2 u(t, x_1, x_2) \\ \text{(more precisely)} \quad &= L_2 (u(t, x_1, \cdot))(x_2). \end{aligned}$$
Let $v(t, x_2) = u(t, x_1, x_2)$. Then $\frac{\partial v}{\partial t}(t, x_2) = L_2 v(t, x_2)$
and $\sup_{x_2, t} |v(t, x_2)| < \infty$ because $T_1(t)$ is a contraction.

Hence, $v(t, x_2) = (T_2(t) v(0, \cdot))(x_2)$
$= (T_2(t) H(x_1, \cdot))(x_2)$.
$\Rightarrow$ $E^{x_1} H(X_1(t), x_2) = u(t, x_1, x_2) = E^{x_2} H(x_1, X_2(t))$. $\square$

Actually, duality will be more useful when one of the two processes is a Markov chain: Let X_1 be a Markov chain and define
$$(T_1(t) f)(x) = \sum_y p_t(x, y) f(y) = E^x f(X_1(t))$$
$$(L_1 f)(x) = \sum_y q(x, y) f(y)$$
$$\mathcal{D}(L_1) = \{f : \text{above converges } \forall x\}$$

KBE $\Rightarrow$ $(T_1(t) f)(x)$ is continuously differentiable in t, and
$\frac{\partial}{\partial t}(T_1(t) f)(x) = (T_1(t) L_1 f)(x)$

Using this, the previous proof still goes through, even if X_1 is not a Feller process.