Fisher-Wright diffusion

Continuous model of population change.

Heuristic derivation:

Consider a population of N individuals, of which Z_0 are the number of type A, and $N - Z_0$ have type a.

Draw a new population of size N, by choosing each individual to be of type A with probability Z_0/N. Let Z_1 be the number of type - A's in the new population. i.e. $Z_1 \sim \text{Binom}(N, Z_0/N)$. Repeat.

Suppose $Z_0 = k$ a.s, set $x = k/N$. For $f \in C^2[0,1]$

$$E\left[f\left(\frac{Z_1}{N}\right) - f(x)\right] = E\left[\left(\frac{Z_1}{N} - x\right) f'(x) + \left(\frac{Z_1}{N} - x\right)^2 \frac{f''(x)}{2} + o\left(\left(\frac{Z_1}{N} - x\right)^2\right)\right]$$

$$= \frac{f''(x)}{2} \cdot \frac{x(1-x)}{N} + o\left(\frac{1}{N}\right)$$

$$\Rightarrow \lim_{N\to\infty} \frac{E f\left(\frac{Z_1}{N}\right) - f(x)}{1/N} = x(1-x) \frac{f''(x)}{2}$$

Let $(Lf)(x) = x(1-x) \dfrac{f''(x)}{2}$ on $D = \{ f : [0,1] \to \mathbb{R} \text{ polynomials}\}$

Thm: 1) $\bar{L}$ is a generator
2) $X(t)$ has continuous paths
3) If $\tau = \inf\{t \geq 0 : X(t) = 0 \text{ or } 1\}$ then
a) $P^x(X(\tau) = 1) = x$
b) $E^x \int_0^\infty X(t)(1 - X(t))\,dt = x(1-x)$
c) $E^x \tau = 2x \log \frac{1}{x} + 2(1-x) \log \frac{1}{1-x}$

Pf: 1) a) obvious
b) if $x_0 \in (0,1)$ is a minimum of f then $(Lf)(x_0) = x_0(1-x_0) \dfrac{f''(x_0)}{2} \geq 0$.
if $x_0 \in \{0,1\}$ is a minimum of f then $(Lf)(x_0) = 0$.
c) want to show $\mathcal{R}(I - \lambda L)$ is dense in $C[0,1]$. Claim it contains all polynomials.

Take $g = \sum_{k=0}^n a_k x^k$. Set $f = \sum_{k=0}^n b_k x^k$, so

$$(I - \lambda L) f = \sum_{k=0}^n x^k \cdot \left[b_k - \frac{\lambda}{2}\left[k(k+1) b_{k+1} - k(k-1) b_k\right]\right]$$

(where $b_{n+1} = 0$).

Can solve $b_k - \frac{\lambda}{2}\left(k(k+1) b_{k+1} - k(k-1) b_k\right) = a_k$ recursively for k by starting with $b_{n+1} = 0$

d) obvious.

2) Claim: $E^x (X(t) - X(s))^4 \leq \frac{3}{16}(t-s)^2$. Then continuity follows from our earlier theorem.

Pf of claim: Fix x, set $f(y) = (x-y)^2$.
$(Lf)(y) = y(1-y)$

$$\Rightarrow (X(t) - x)^2 - \int_0^t X(s)(1 - X(s))\,ds \quad \text{is a martingale}$$

$$\left(= f(X(t)) - \int_0^t Lf(X(s))\,ds\right)$$

$$E^x (X(t)-x)^2 = \int_0^t E^x X(s)(1-X(s))\,ds \leq \frac{t}{4}.$$

Now take $f(y) = (y-x)^4$, $(Lf)(y) = 6(y-x)^2 \cdot y(1-y)$

$\Rightarrow \quad (X(t)-x)^4 - 6\int_0^t (X(s)-x)^2 \, X(s)(1-X(s))\,ds$ martingale.

$$\Rightarrow \quad E^x(X(t)-x)^4 = 6\int_0^t E^x(X(s)-x)^2 \, X(s)(1-X(s))\,ds$$

$$\leq \frac{3}{2}\int_0^t E^x(X(s)-x)^2\,ds$$

$$\leq \frac{3}{2}\int_0^t \frac{s}{4}\,ds$$

$$= \frac{3}{16}t^2.$$

Finally, $s<t \Rightarrow E^x(X(t)-X(s))^4 = E^x E^{X(s)}(X(t-s)-X(0))^4$

$$\leq \frac{3}{16}(t-s)^2$$

3) a) $f(x)=x$, $Lf(x)=0 \Rightarrow X(t)$ is a martingale

$\Rightarrow \lim_{t\to\infty} X(t)$ exists a.s.

$f(x)=x(1-x)$, $Lf(x)=-x(1-x) \Rightarrow X(t)(1-X(t)) + \int_0^t X(s)(1-X(s))$ martingale

non-negative $\Rightarrow$ has a finite limit a.s.

$X(t)(1-X(t))$ bounded $\Rightarrow \int_0^t X(s)(1-X(s))ds$ bounded as $t\to\infty$

$\Rightarrow X(t) \to 0$ or 1

$$x = \lim_{t\to\infty} E^x X(t) = P^x(X(t)\to 1)$$

It remains to show that $T<\infty$ a.s. This will come later.

b) $f(x)=x(1-x) \Rightarrow X(t)(1-X(t)) + \int_0^t X(s)(1-X(s))$ martingale.

$$\Rightarrow x(1-x) = \underbrace{E^x X(t)(1-X(t))}_{\to 0 \text{ since } X(t)\to 0 \text{ or } 1} + \int_0^t E^x X(s)(1-X(s))ds$$

$$\Rightarrow x(1-x) = \int_0^\infty E^x X(s)(1-X(s))\,ds.$$

c) Take $f(x) = 2x\log\frac{1}{x} + 2(1-x)\log\frac{1}{1-x}$. Careful: $f\notin \mathcal{D}(L)$!

For $\varepsilon>0$, let $f_\varepsilon = f$ on $[\varepsilon, 1-\varepsilon]$, $f_\varepsilon \in \mathcal{D}(L)$.

$\tau_\varepsilon = \inf\{t : X(t)\notin[\varepsilon,1-\varepsilon]\}$.

on $[\varepsilon,1-\varepsilon]$, $Lf_\varepsilon = Lf = -1$.

$f_\varepsilon(X(t)) + \int_0^t Lf_\varepsilon(X(s))\,ds$ is a martingale

$\Rightarrow f_\varepsilon(X(t\wedge\tau_\varepsilon)) + \int_0^t Lf_\varepsilon(X(s\wedge\tau_\varepsilon))\,ds$ is a martingale

$= f(X(t\wedge\tau_\varepsilon)) + t\wedge\tau_\varepsilon$ $\quad P^x$-a.s. for $x\in[\varepsilon,1-\varepsilon]$

$x\in[\varepsilon,1-\varepsilon] \Rightarrow f(x) = E^x[f(X(t\wedge\tau_\varepsilon)) + t\wedge\tau_\varepsilon] \quad \forall t$

$$t\to\infty \Rightarrow f(x) = \underbrace{E^x f(\tau_\varepsilon)}_{\to 0 \text{ as } \varepsilon\to 0} + \underbrace{E^x\tau_\varepsilon}_{\to E^x\tau \text{ as } \varepsilon\to 0}$$

$$\Rightarrow f(x) = E^x\tau$$

Finally, $P^x(\tau<\infty)=1$, which completes part (a). $\square$

Remark: If $X(t)$ is Brownian motion and $Y(t) = X(ct)$ then Y moves c times as fast as X.

$$\lim_{t\to 0} \frac{E^x f(X(t)) - f(x)}{t} = c \lim_{t\to 0} \frac{E^x f(X(ct)) - f(x)}{ct}$$

$$\Rightarrow \quad L_Y f = c L_X f = \frac{c}{2} f''$$

Intuitively, a generator of the form $Lf(x) = c(x) \frac{f''}{2}$ is like Brownian motion, sped up by $c(x)$ when it's near x

Another way to construct such a process:

$dX_t = \sqrt{c(X_t)}\, dB_t$ (using Brownian scaling $\sqrt{c}B(t) \overset{d}{=} B(ct)$).

The previous result about continuity extends to other generators of the form $c(x) f''(x)$:

Thm: Suppose $c \in C(\mathbb{R})$, $0 \le c(x) \le K$ $\forall x$. Suppose L is a generator such that

$$Lf(x) = c(x) f''(x) \qquad \forall f \in C(\mathbb{R}),\ f'' \in C(\mathbb{R}).$$

Then $X(t)$ is continuous.

Pf: First, we will pretend that $f(y) = (y-x)^2$ and $f(y) = (y-x)^4$ are in $\mathcal{D}(L)$

Then
$$\left.\begin{aligned} &(X(t)-x)^2 - \int_0^t 2\, c(X(s))\, ds \\ &(X(t)-x)^4 - \int_0^t 12\, (X(s)-x)^2 c(X(s))\, ds \end{aligned}\right\} \text{martingales.}$$

$$\Rightarrow \quad E^x (X(t)-x)^2 = 2\int_0^t E^x c(X(s))\, ds \le 2tK.$$

$$\begin{aligned} E^x (X(t)-x)^4 &= 12\int_0^t E^x\left((X(s)-x)^2 c(X(s))\right) ds \\ &\le 12K \int_0^t E^x (X(s)-x)^2\, ds \\ &\le 12K^2 \int_0^t 2s\, ds \\ &= 12K^2 t^2. \end{aligned}$$

The above argument may be made rigorous by choosing f_n, g_n so that
$f_n(y) \to (y-x)^2$ pointwise, $f_n''(y) \le 2$, $f_n, f_n'' \in C(S)$
$g_n(y) \to (y-x)^4$ pointwise, $g_n''(y) \le 12(y-x)^2$, $g_n, g_n'' \in C(S)$

Finally, $$E^x (X(t)-X(s))^4 = E^x E^{X(s)} (X(t-s)-X(0))^4 \le 12K^2(t-s)^2 \quad \forall s<t$$

$\Rightarrow$ $X(t)$ continuous $\square$