Example: absorbed Brownian motion

Let $X(t)$ be Brownian motion, $\tau = \inf\{t : X(t) = 0\}$

$$X_a(t) = \begin{cases} X(t) & \text{if } t < \tau \\ 0 & \text{if } t \geq \tau \end{cases}$$

- X_a is a Markov process
- X_a is a Feller process: for any $f \in C[0,\infty)$, set
$$f_o(x) = \begin{cases} f(x) & \text{if } x \geq 0 \\ 2f(0) - f(-x) & \text{if } x < 0. \end{cases}$$
Then $E^x f(X_a(t)) = E^x f_o(X(t))$, which is continuous in x.

Let L_a, $T_a(t)$ be the generator and semigroup of X_a.

$$(T_a(t) f)(x) = (T(t) f_o)(x)$$

So, $f \in \mathcal{D}(L_a) \iff f_o \in \mathcal{D}(L)$
$\iff f', f'' \in C[0,\infty)$, $f''(0) = 0$.

$$L_a f = \tfrac{1}{2} f'', \quad \mathcal{D}(L_a) = \{f \in C(S) : f', f'' \in C(S),\ f''(0) = 0\}$$

Example: reflected Brownian motion

$$X_r(t) = |X(t)|$$

- X_r is a Markov process
- X_r is a Feller process: for any $f \in C[0,\infty)$, set
$$f_e(x) = \left\{ \begin{array}{ll} f(x) & \text{if } x \geq 0 \\ f(-x) & \text{if } x < 0 \end{array} \right\} = f(|x|)$$
Then $E^x f(X_r(t)) = E^x f_e(X(t))$, which is continuous in x.

Let L_r, $T_r(t)$ be the generator and semigroup of X_r.

$$(T_r(t) f)(x) = (T(t) f_e)(x).$$

So, $f \in \mathcal{D}(L_r) \iff f_e \in \mathcal{D}(L)$
$\iff f', f'' \in C[0,\infty)$, $f'(0) = 0$.

$$L_r f = \tfrac{1}{2} f'', \quad \mathcal{D}(L_r) = \{f \in C(S) : f', f'' \in C(S),\ f'(0) = 0\}$$

Remark: as a consequence of the preceding examples,
$$D = \{f \in C(S) : f', f'' \in C(S),\ f''(0) = f'(0) = 0\}$$
is not a core for either L_a or L_r.

Example: Brownian motion with sticky boundary

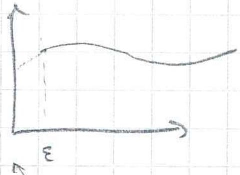
Fix $c>0$. Let $L_c f = \frac{1}{2} f''$ on

$$\mathcal{D}(L_c) = \{ f \in C[0,\infty) : f', f'' \in C[0,\infty),\ f'(0) = c f''(0) \}$$

Prop: this is a generator

Proof:

a) exercise



modify function on a small set.

b) It is enough to show that $(L_c f)(x) \geq 0$ whenever x is a local minimum of f. If x is an interior minimum then $f'' \geq 0$. If $x=0$ is a minimum then $f'(0) \geq 0$, so $f''(0) = c f'(0)$ $\Rightarrow f''(0) \geq 0$.

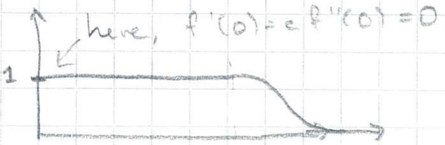
c) For any $g \in C[0,\infty)$ and $\lambda > 0$, choose $f_a \in \mathcal{D}(L_a)$ and $f_r \in \mathcal{D}(L_r)$ so that

$$(I - \lambda L_a) f_a = g$$
$$(I - \lambda L_r) f_r = g.$$

Choose $\gamma \in \mathbb{R}$ to solve $\gamma f_a'(0) = c(1-\gamma) f_r''(0)$. Then $f_c := \gamma f_a + (1-\gamma) f_r \in \mathcal{D}(L_c)$ and

$$(I - \lambda L_c) f_c = g.$$

d) Can construct functions explicitly:



□

Remark: in part (c) above, $\gamma = \frac{2c}{2c + \sqrt{2\lambda}}$

Proof: Let $h = f_a - f_r \Rightarrow (I - \lambda L) h = 0$

$$h = \frac{2}{\lambda} h''$$

h bounded, continuous $\Rightarrow h(x) = h(0)\, e^{-\sqrt{2/\lambda}\, x}$

$$h'(0) = -\sqrt{\tfrac{2}{\lambda}}\, h(0) \Rightarrow f_a'(0) = -\sqrt{\tfrac{2}{\lambda}}\, h(0)$$
$$h''(0) = \tfrac{2}{\lambda} h(0) \Rightarrow f_r''(0) = -\tfrac{2}{\lambda} h(0) \quad □$$

Prop: For every $t > 0$, $P^0(X_c(t) = 0) > 0$.

Proof: Fix g, choose $f \in \mathcal{D}(L_c)$ with $(I - \frac{1}{\alpha} L) f = g$. Then we can write

$$f(x) = \frac{2c f_a(x) + \sqrt{2/\alpha}\, f_r(x)}{2c + \sqrt{2/\alpha}}, \qquad \begin{array}{l} f_a \in \mathcal{D}(L_a) \\ f_r \in \mathcal{D}(L_r) \\ (I - \frac{1}{\alpha} L) f_a = (I - \frac{1}{\alpha} L) f_r = g \end{array}$$

Also,
$$f(x) = \alpha \int_0^\infty e^{-\alpha t}\, E^x g(X_c(t))\, dt \quad (1)$$
$$f_a(x) = \alpha \int_0^\infty e^{-\alpha t}\, E^x g(X_a(t))\, dt \quad (2)$$
$$f_r(x) = \alpha \int_0^\infty e^{-\alpha t}\, E^x g(X_r(t))\, dt \quad (3)$$

Take $g \uparrow 1_{(0,\infty)}$, $x=0$. $(1) \uparrow \alpha \int_0^\infty e^{-\alpha t} P^0(X_c(t) > 0)\, dt$

$(2) = 0$

$(3) \uparrow 1$

$$\Rightarrow \alpha \int_0^\infty e^{-\alpha t} P^0(X_c(t) > 0)\, dt = \frac{1}{1 + c\sqrt{2\alpha}}$$

$$\Rightarrow \lim_{t \to 0} P^0(X_c(t) = 0) = 1 \Rightarrow P^0(X_c(t) = 0) > 0 \ \forall t \quad □$$

Prop: Let $Y(t)$ be a continuous Feller process on $[0,\infty)$.
$\tau = \inf\{t : Y(t) > 0\}$.
Then $P^0(0 < \tau < \infty) = 0$

Proof: Let $Y_\varepsilon = 1_{\{X(t) > 0 \ \forall\, 0 < t < \varepsilon\}}$.

By SMP and the fact that $Y_\tau = 0$ on $\{\tau < \infty\}$,

$$E^0 Y_\varepsilon = \underline{E^0 (Y_\varepsilon \circ \tau \mid \mathcal{F}_\tau)} \qquad P^0\text{-a.s on } \{\tau < \infty\}$$

$\longrightarrow 1$ as $\varepsilon \to 0$.

$$\Rightarrow \lim_{\varepsilon \to 0} E^0 Y_\varepsilon = 1 \quad \text{on } \{\tau < \infty\}$$

$$\{\tau = 0\} \supseteq \{\lim_{\varepsilon \to 0} Y_\varepsilon = 1\}.$$

$$\Rightarrow \tau = 0 \quad P^0\text{-a.s. on } \{\tau < \infty\}. \qquad \square.$$

Cor: $\{t \geq 0 : X_c(t) = 0\}$ contains no intervals.

<u>Martingales</u> Let $X(t)$ be a Feller process, generator L, semigroup $T(t)$.

Thm: $\forall f \in \mathcal{D}(L)$,

$$M(t) := f(X(t)) - \int_0^t Lf(X(s))\,ds$$

is a martingale under P^x $\forall x \in S$.

Pf: First, $E^x M(t) = T(t)f(x) - \int_0^t T(s) L f(x)\,ds$

$$= T(t)f(x) - \int_0^t \frac{d}{ds} T(s) f(x)\,ds$$
$$= T(0) f(x) = f(x).$$

For $s < t$,

$$E^x(M(t) \mid \mathcal{F}_s) = E^x\Big(f(X(t-s)) \circ \Theta_s - \int_0^{t-s} Lf(X(r)) \circ \Theta_s\,dr$$
$$- \int_0^s Lf(X(r))\,dr \;\Big|\; \mathcal{F}_s \Big)$$

$$= E^{X(s)} f(X(t-s)) - E^{X(s)} \int_0^{t-s} Lf(X(r))\,dr$$
$$- \int_0^s Lf(X(r))\,dr$$

$$= E^{X(s)} M(t-s) - \int_0^s Lf(X(r))\,dr$$

$$= f(X(s)) - \int_0^s Lf(X(r))\,dr = M(s) \qquad \square.$$

Thm: If P is a probability measure on Ω such that $P(X(0) = x) = 1$ and $f(X(t)) - \int_0^t Lf(X(s))\,ds$ is a martingale $\forall f \in \mathcal{D}(L)$ then $P = P^x$

Proof: Take $g \in C(S)$, $f \in \mathcal{D}(L)$, $(I - \frac{1}{\alpha} L) f = g$, $\alpha > 0$.
Since $M(t)$ is a martingale,

$$E\left[f(X(t)) - f(X(s)) - \int_0^t Lf(X(r))dr \;\middle|\; \mathcal{F}_s \right] = 0.$$

Multiply by $\alpha e^{-\alpha t}$ and integrate t from s to ∞:

$$\int_s^\infty \int_0^t \alpha e^{-\alpha t} Lf(X(r))\,dr\,dt = \int_s^\infty Lf(X(r)) \int_r^\infty \alpha e^{-\alpha t}\,dt\,dr$$
$$= \int_s^\infty e^{-\alpha r} Lf(X(r))dr$$

$$\Rightarrow E\left[\int_s^\infty \left[\alpha e^{-\alpha t} f(X(t)) - e^{-\alpha t} Lf(X(t))\right]dt - e^{-\alpha s} f(X(s)) \;\middle|\; \mathcal{F}_s \right] = 0$$

$$\Rightarrow E\left[\int_s^\infty \alpha e^{-\alpha t} g(X(t)) \;\middle|\; \mathcal{F}_s \right] = e^{-\alpha s} f(X(s))$$

$$\Rightarrow \forall A \in \mathcal{F}_s, \quad \int_0^\infty \alpha e^{-\alpha t} E\left[g(X(\cancel{s t}))1_A\right]dt = E\left[f(X(s))1_A\right]$$

With $s=0$, $A=\Omega$, $\int_0^\infty \alpha e^{-\alpha t} Eg(X(t))\,dt = f(x)$.

By the previous theorem, same holds for E^x.
Uniqueness of Laplace transform + càdlàg paths $\Rightarrow$ distribution of $X(t)$ the same under P and P^x

Induction: suppose $(X(t_1), \dots, X(t_k))$ have the same distribution under P and P^x. Take $t_{k+1} > t_k$, $t = t_{k+1} - t_k$.

For A depending on $X(t_1), \dots X(t_k)$, $E\left[f(X(t_k))1_A\right]$
$$= E^x\left[f(X(t_k))1_A\right]$$

$$\Rightarrow \int_0^\infty \alpha e^{-\alpha t} E\left[g(X(t_{k+1}))1_A\right]dt = \text{" --- } E^x \text{ --- "}$$

$\Rightarrow$ $X(t_1), \dots, X(t_{k+1})$ have the same distribution under P and P^x

Stationary distributions

Given a prob. measure μ on S and a Feller semigroup $T(t)$, write $\mu T(t)$ for the measure given by

$$\int f \, d(\mu T(t)) = \int T(t) f \, d\mu.$$

(In functional analysis, we would write $\mu T(t) = T(t)^* \mu$)

Def: μ is stationary if $\mu T(t) = \mu \quad \forall t \geq 0$
(equivalently, $\int T(t) f \, d\mu = \int f \, d\mu \quad \forall t, \forall f \in C(S)$)

Thm: Suppose D is a core for L. Then

$$\mu \text{ stationary} \iff \int Lf \, d\mu = 0 \quad \forall f \in D$$

Pf: suppose μ stationary, $f \in D$.

$$\int Lf \, d\mu = \int \lim_{t \to 0} \frac{T(t)f - f}{t} \, d\mu = 0$$

Suppose $\int Lf \, d\mu = 0 \quad \forall f \in D$.
given $f \in \mathcal{D}(L)$, $\exists f_n \in D$: $f_n \to f$, $Lf_n \to Lf$
$\Rightarrow \int Lf \, d\mu = 0 \quad \forall f \in \mathcal{D}(L)$

$$T(t)f - f = \int_0^t \frac{d}{ds} T(s) f \, ds = \int_0^t L T(s) f \, ds$$

$$\int T(t) f - f \, d\mu = \int \int_0^t L T(s) f \, ds \, d\mu = 0 \quad \forall f \in \mathcal{D}(L)$$

because $f \in \mathcal{D}(L) \Rightarrow \int_0^t L T(s) f \, ds = \int_0^t T(s) L f \, ds$, which is a bounded functi

density $\Rightarrow \int T(t) f - f \, d\mu = 0 \quad \forall f \in C(S)$

Thm: If S is compact then $\exists$ stationary distribution.

Pf: Take any prob. measure μ on S

$$\nu_n := \frac{1}{n} \int_0^n \mu T(r) \, dr$$

S compact $\Rightarrow$ $\{$prob measures on $S\}$ weakly compact (i.e. compact under convergence in distribution)

$\Rightarrow \exists$ sequence n_k: $\nu_{n_k} \to \nu$ in distribution

$$\int T(t) f \, d\nu_n = \frac{1}{n} \int_0^n \int T(t) f \, d(\mu T(r)) \, dr$$

$$= \frac{1}{n} \int_0^n \int T(t+r) f \, d\mu \, dr$$

$$= \frac{1}{n} \int_t^{n+t} \int T(r) f \, d\mu \, dr$$

$$\int T(t) f - f \, d\nu_n = \frac{1}{n} \left[\int_n^{n+t} \int T(r) f \, d\mu \, dr - \int_0^t \int T(r) f \, d\mu \, dr \right]$$

$$\leq \frac{2t \|f\|}{n} \to 0 \text{ as } n \to \infty$$

$\Rightarrow \int T(t) f - f \, d\nu = 0$
$\Rightarrow \nu$ stationary distribution $\square$