Feller processes

S locally compact Polish space

$\Omega = \{$ càdlàg functions $X : [0,\infty) \to S \}$.

$\mathcal{F} = \sigma(t \mapsto X(t))$

$\mathcal{F}_t$ = right-continuous filtration such that $(t \mapsto X(t)) \in \mathcal{F}_t$.

$C(S) = \{$ continuous functions $S \to \mathbb{R}$ vanishing at infinity $\}$

$\Rightarrow \forall \varepsilon > 0 \ \exists K$ compact: $|f(x)| \leq \varepsilon \ \forall x \notin K$.

If S is compact, $C(S) = \{$ continuous functions $S \to \mathbb{R} \}$.

$\|f\| = \sup_x |f(x)|$

Then $(C(S), \|\cdot\|)$ is a Banach space.

Recall the shift operator $(\Theta_s X)(t) = X(s+t)$.

Definition: A Feller process is a collection $\{P^x\}$ of prob. measures on Ω such that

a) $P^x(X(0) = x) = 1$

"Feller property" b) $x \mapsto E^x f(X(t))$ is continuous in x $\forall f \in C(S)$, $t > 0$.

c) $E^x(Y \circ \Theta_s \mid \mathcal{F}_s) = E^{X(s)} Y$ P^x-a.s.

Remark:
- (c) is equivalent to the special case $Y(X) = f(X(t))$
- (b) + (c) $\Rightarrow$ strong Markov property.

Feller semigroup:

Def: A Feller semigroup is a set $\{T(t) : t \geq 0\}$ of bounded linear operators on $C(S)$ such that:

a) $T(0) f = f$ $\forall f$

b) $\lim_{t \to 0} T(t) f = f$ $\forall f$ "strong continuity"

c) $T(s+t) = T(s)\, T(t)$ "semigroup property"

d) $T(t) f \geq 0$ $\forall f \geq 0$.

e) $\exists f_n \in C(S)$ such that
- $\sup_n \|f_n\| < \infty$
- $\forall t$, $T(t) f_n \to 1$ pointwise.

Remarks:
- (c) $\Rightarrow$ $T(s)$ commutes with $T(t)$
- (d) + Riesz representation theorem $\Rightarrow$ $\forall x, t \ \exists$ Borel regular measure $\mu_{x,t}$ on S such that $(T(t) f)(x) = \int f(y)\, d\mu_{x,t}(y)$.
- (e) + dominated convergence $\Rightarrow$ the measure $\mu_{x,t}$ above is a probability measure
- if S compact then (e) $\Leftrightarrow$ $T(t) 1 = 1$.
- in general, the sequence f_n in (e) may be taken to be non-negative and converging to 1 uniformly on compact sets

Lemma: $\|T(t)f\| \leq \|f\| \quad \forall f, t.$

Proof: First, assume f is non-negative, and has compact support. Choose f_n as in (e). $\forall \varepsilon > 0$, $\forall$ large n, $f_n \geq (1-\varepsilon) f/\|f\|$

$$\text{Then} \quad T(t)\left((1-\varepsilon)\frac{f}{\|f\|} - f_n\right) \leq 0$$

$$\Rightarrow \forall x, \left(T(t)(1-\varepsilon)\frac{f}{\|f\|}\right)(x) \leq (T(t) f_n)(x) \to 1$$

$$\Rightarrow \forall x, (T(t)f)(x) \leq \frac{\|f\|}{1-\varepsilon}$$

$$\Rightarrow \|T(t)f\| \leq \|f\|.$$

Next, let f be any function with compact support. Let $f = f^+ - f^-$, where f^+, f^- non-negative, minimal.

$$(T(t)f)(x) = (T(t)f^+)(x) - (T(t)f^-)(x)$$

$$\Rightarrow |(T(t)f)(x)| \leq \max\{|(T(t)f^+)(x)|, |(T(t)f^-)(x)|\}.$$

$$\Rightarrow \|T(t)f\| \leq \max\{\|T(t)f^+\|, \|T(t)f^-\|\}$$

$$\leq \max\{\|f^+\|, \|f^-\|\}.$$

$$\leq \|f\| \qquad \square.$$

Example: $(T(t)f)(x) = \sum p_t(x,y) f(y), \quad \mu_{x,t} = p_t(x, \cdot)$

Resolvent

$$U(\alpha)f := \int_0^\infty e^{-\alpha t} \, T(t) f \, dt \qquad (\alpha > 0).$$

Then $\|U(\alpha)f\| \leq \int_0^\infty e^{-\alpha t} \|f\| \, dt = \frac{1}{\alpha}\|f\|$

Lemma: $\forall f$, $\alpha U(\alpha) f \to f$ as $\alpha \to \infty$.

Proof: $\|\alpha U(\alpha) f - f\| \leq \int_0^\infty \alpha e^{-\alpha t} \|T(t)f - f\| \, dt. \quad =(*)$

Choose $\delta > 0$. $\exists\, \varepsilon > 0$ such that $\|T(t)f - f\| \leq \delta \quad \forall t \leq \varepsilon$

$$\Rightarrow (*) \leq \delta \int_0^\varepsilon \alpha e^{-\alpha t} \, dt + 2\|f\| \int_\varepsilon^\infty \alpha e^{-\alpha t} \, dt$$

$$\leq \delta + 2\|f\| e^{-\varepsilon\alpha}$$

$$\Rightarrow \limsup_{\alpha \to \infty} \|\alpha U(\alpha) f - f\| \leq \delta \qquad \square.$$

Lemma: $U(\alpha) - U(\beta) = (\beta - \alpha) U(\alpha) U(\beta)$.
In particular, $U(\alpha)$ and $U(\beta)$ commute.

Proof:
$$U(\alpha)U(\beta)f = \int_0^\infty e^{-\alpha t} T(t) \int_0^\infty e^{-\beta s} T(s) f \, ds \, dt$$

$$= \int_0^\infty e^{-\alpha t} \int_0^\infty e^{-\beta s} \, T(s+t) f \, ds \, dt$$

$$= \int_0^\infty T(r) f \int_0^r e^{-\alpha t - \beta(r-t)} \, dt \, dr$$

$$= \frac{1}{\beta - \alpha} \int_0^\infty T(r) f \cdot \left(e^{-\alpha r} - e^{-\beta r}\right) dr$$

$$= \frac{U(\alpha) - U(\beta)}{\beta - \alpha} f \qquad \square$$

Example: Lévy process

Let $\beta \in \mathbb{R}$, $\sigma \geq 0$, ν a measure on $\mathbb{R}$ satisfying

$$\int_{-\infty}^{\infty} \frac{x^2}{1+x^2} \, d\nu(x) < \infty.$$

Define $\psi(u) = i\beta u - \frac{\sigma^2 u^2}{2} + \int_{-\infty}^{\infty} \left(e^{iux} - 1 - \frac{iux}{1+x^2} \right) d\nu(x)$

Lemma: $e^{\psi(u)}$ is the characteristic function of a probability distribution. (i.e. $\exists \mu$: $\psi(u) = \log \int_{-\infty}^{\infty} e^{iux} \, d\mu(x)$)

Proof: First, we do the case where ν is finite. Let $\lambda = \nu(\mathbb{R})$, $\mu = \nu/\lambda$. Take $Y_1, Y_2, \ldots \overset{iid}{\sim} \mu$ and $N \sim \text{Poisson}(\lambda)$. Then the characteristic function of $\sum_{j=1}^{N} Y_i$ is

$$\exp\left(\lambda \int_{-\infty}^{\infty} (e^{iux} - 1) \, d\mu(x) \right),$$

which is of the required form with $\sigma = 0$, $\beta = \int_{-\infty}^{\infty} \frac{x}{1+x^2} \, d\nu(x)$.

A Gaussian with mean β, variance σ^2 takes the required form with $\nu = 0$.

Combining these two examples, we get every function ψ with ν finite.

Now let ν satisfy $\int \frac{x^2}{1+x^2} \, d\nu(x) < \infty$. Let

$$\nu_\varepsilon(A) = \nu(A \setminus [-\varepsilon, \varepsilon]).$$

Then ν_ε finite. Let

$$\psi_\varepsilon(u) = i\beta u - \frac{\sigma^2 u^2}{2} + \int_{-\infty}^{\infty} \left(e^{iux} - 1 - \frac{iux}{1+x^2} \right) d\nu_\varepsilon(x)$$

Then each e^{ψ_ε} is a characterisic function, and $\psi_\varepsilon \to \psi$ pointwise $\Rightarrow e^{\psi}$ is a characteristic function. $\square$

Def: Let ξ_t be a random variable with characteristic function $e^{t\psi}$. Define

$$(T(t)f)(x) = \mathbb{E}(f(x + \xi_t))$$

Prop: This defines a Feller semigroup.

Proof:

a) $\xi_0 = 0$ a.s. $\Rightarrow T(0)f = f$.

b) $e^{t\psi} \to 1$ pointwise as $t \to 0$

$\Rightarrow \xi_t \to 0$ in distribution as $t \to 0$

$\Rightarrow (T(t)f)(x) \to f(x)$ as $t \to 0$

$f \in C(S) \Rightarrow f$ uniformly continuous

$\Rightarrow$ above convergence uniform in x.

c) $\xi_s + \xi_t \overset{d}{=} \xi_{s+t}$, then use Fubini.

d) obvious

e) take $f_n(x) = 1_{\{|x| \leq n\}}$ (but made continuous) $\square$

Remark:
- Lévy processes include Brownian motion and compound Poisson.
- Feller process + stationary, independent increments $\Leftrightarrow$ Lévy process

Generator

$\mathcal{D}$ and $\mathcal{R}$ denote domain and range.

Def: An operator L on $C(S)$ is a generator if
a) $\mathcal{D}(L)$ is dense in $C(S)$
b) $f \in \mathcal{D}(L)$, $\lambda \geq 0 \Rightarrow \inf_x f(x) \geq \inf_x ((I-\lambda L)f)(x)$
c) $\mathcal{R}(I-\lambda L) = C(S)$ for sufficiently small $\lambda > 0$
d) for sufficiently small $\lambda > 0$, $\exists f_n$, $g_n = (I-\lambda L) f_n$:
- $\sup_n \|g_n\| < \infty$.
- $f_n \to 1$ pointwise
- $g_n \to 1$ pointwise.

Lemma: $\forall \lambda > 0$, $f \in \mathcal{D}(L)$, $\|f\| \leq \|(I-\lambda L) f\|$

Proof: apply (b) to f and $-f$.

Hence, $(I-\lambda L)^{-1}$ is defined everywhere (for small λ, by (c)) and is a contraction.

Goal: Process, semigroup, generator are all equivalent.

$X \to T$:

Thm: Given a Feller process, define $T(t)$ by $(T(t)f)(x) = E^x f(X(t))$
This defines a Feller semigroup.

Proof:
a) follows from $P^x(X(0) = x) = 1$.
c) follows from Markov property:

$$\begin{aligned} T(s+t) f(x) &= E^x f(X(s+t)) \\ &= E^x E^{X(s)} f(X(t)) \\ &= E^x (T(t)f)(X(s)) \\ &= (T(s)\, T(t) f)(x). \end{aligned}$$

d) obvious
e) take a sequence of functions converging up to 1 pointwise.

b) want to show $E^x f(X(t)) \to f(x)$ uniformly in x.
pointwise convergence follows from right-continuity of X, continuity of f, and dominated convergence.

Now, we need to re-do resolvents using pointwise convergence: extend $T(t)$ to an operator $L^\infty(S) \to L^\infty(S)$, and define

$$(U(\alpha) f)(x) = \int_0^\infty e^{-\alpha t} \underbrace{T(t) f\,(x)}\, dt$$

bounded function $[0,\infty) \to \mathbb{R}$ by (c) and previous paragraph

$$U(\alpha) : L^\infty(S) \to L^\infty(S)$$

Resolvent equation still holds for $U(\alpha)$.
$\alpha U(\alpha) f \to f$ pointwise $\forall f \in C(S)$

Let $L = U(\alpha)\, C(S)$, which is independent of α by the resolvent equation. If $f \in L$, $f = U(\alpha) g$ for $g \in C(S)$, then

$$(T(t)f)(x) = \int_0^\infty e^{-\alpha s}\, T(s+t)\, g(x) ds$$
$$= \int_t^\infty e^{-\alpha(r-t)}\, (T(r)g)(x)\, ds$$

RHS converges to $\int_0^\infty e^{-\alpha r}\, T(r) g(x)\, ds$ uniformly in x, because $T(r)$ is a contraction.

$\Rightarrow \quad \forall\, f \in L, \quad T(t)f \to f$

Claim: $L \cap C(S)$ is dense in $C(S)$

Pf: Since $\alpha U(\alpha) f \to f$ pointwise, $L \cap C(S)$ is weakly dense in $C(S)$.
For a subspace of a Banach space, strong closure = weak closure.
$\Rightarrow$ $L \cap C(S)$ (strongly) dense in $C(S)$ $\square$.

$T \to L$:

Thm: Suppose $T(t)$ is a Feller semigroup. Define L by
$$Lf = \lim_{t \to 0} \frac{T(t)f - f}{t} \qquad \text{on } \mathcal{D} = \{\text{limit exists}\}$$
Then:
1) L is a probability generator
2) $\forall\, \alpha > 0$, "$\alpha U(\alpha) = (I - \frac{1}{\alpha} L)^{-1}$"

i.e. $\quad (I - \frac{1}{\alpha} L) \cdot \alpha U(\alpha) f = f \quad \forall\, f \in C(S)$
$\qquad \alpha U(\alpha) \cdot (I - \frac{1}{\alpha} L) f = f \quad \forall\, f \in \mathcal{D}(L)$.

3) If $f \in \mathcal{D}(L)$ then $T(t) f \in \mathcal{D}(L)$ $\forall t$, and
$$\frac{d}{dt} T(t) f = L\, T(t) f = T(t) L f$$

4) $\forall\, f \in C(S)$ and $t \geq 0$,
$$\lim_{n \to \infty} (I - \tfrac{t}{n} L)^{-n} f = T(t) f$$
$$\text{" } T(t) = e^{tL} \text{ "}$$

Pf: First, assume $f = \alpha U(\alpha) g$. Then
$$\frac{T(t)f - f}{t} = \frac{\alpha}{t}\int_t^\infty e^{-\alpha(s-t)}\, T(s) g\, ds - \frac{\alpha}{t}\int_0^\infty e^{-\alpha s} T(s) g\, ds$$
$$= \alpha \frac{e^{\alpha t} - 1}{t} \int_t^\infty e^{-\alpha s}\, T(s) g\, ds - \frac{\alpha}{t} \int_0^t e^{-\alpha s}\, T(s) g\, ds$$
$$\to \alpha^2 U(\alpha) g - \alpha g = \alpha f - \alpha g$$

$\Rightarrow f \in \mathcal{D}(L)$ and $Lf = \alpha f - \alpha g$
$\Rightarrow (I - \frac{1}{\alpha} L)\, \alpha U(\alpha) = I$.

1) a) $\mathcal{D}(L) \supseteq \mathcal{R}(U(\alpha))$, which is dense in $C(S)$.
b) Set $g_t = f - \lambda \frac{T(t) f - f}{t}$. Then $f \cdot (1 + \frac{\lambda}{t}) = g_t + \frac{\lambda}{t} T(t) f$.

$$(1 + \tfrac{\lambda}{t}) \inf_x f(x) \geq \inf_x g_t(x) + \frac{\lambda}{t} \inf_x T(t) f(x)$$
$$\geq \inf_x g_t(x) + \frac{\lambda}{t} \inf_x f(x).$$

$\Rightarrow \inf_x f(x) \geq \inf_x g_t(x) \quad \Rightarrow \quad \inf_x f(x) \geq \inf_x (I - \lambda L) f(x)$

c) follows from $(I - \frac{1}{\alpha} L)\, \alpha\, \mathcal{U}(\alpha) = I$ $(\forall \alpha\, !)$

d) Choose $g_n \in C(S)$ such that $\sup_n \|g_n\| < \infty$ and $T(t) g_n \to 1$ pointwise $\forall t$. Set $f_n = \frac{1}{\lambda} \mathcal{U}(\frac{1}{\lambda}) g_n \Rightarrow g_n = (I - \lambda L) f_n$, and $f_n \to 1$ by definition of $\mathcal{U}(\alpha)$ + dominated convergence.

2) Already did one direction. If $f \in \mathcal{D}(L)$, set $g = (I - \frac{1}{\alpha} L) f$, $h = \alpha\, \mathcal{U}(\alpha) g$. Then $g = \underbrace{(I - \frac{1}{\alpha} L)}_{\text{injective, by (b)}} h \Rightarrow h = f$

3)

$$\underbrace{\frac{T(t+s)f - T(t)f}{s}}_{\frac{d}{dt} T(t) f} = T(t) \underbrace{\frac{T(s)f - f}{s}}_{\text{has a limit}} = \underbrace{\frac{T(s)[T(t)f] - T(t)f}{s}}_{L\, T(t) f}$$

$\Rightarrow$ all have limits.

$T(t)\, Lf$

4) $(I - \frac{1}{\alpha} L)^{-n} f = \alpha^n\, \mathcal{U}(\alpha)^n f$

Let $\xi_1, \xi_2, \dots$ be iid $\mathrm{Exp}(1)$ variables. Then $\alpha\mathcal{U}(\alpha) = E\, T(\frac{\xi_1}{\alpha}) f$

$\alpha^n \mathcal{U}^n(\alpha) = E\, T(\frac{\xi_1 + \dots + \xi_n}{\alpha}) f$

$$(I - \tfrac{t}{n} L)^{-n} f = E\, T(\underbrace{\tfrac{\xi_1 + \dots + \xi_n}{n}}_{Y_n} t) f$$

$\forall \delta > 0$, $P(|Y_n - 1| > \delta) \to 0$ as $n \to \infty$.

$\forall \varepsilon > 0\ \exists\, \delta > 0$: $s \in [(1-\delta)t, (1+\delta)t] \Rightarrow \|T(s) f - T(t) f\| < \varepsilon$

$$\begin{aligned} \| E\, T(Y_n t) f - T(t) f \| &\le E \| T(Y_n t) f - T(t) f \| \\ &\le E \left(\varepsilon 1_{\{|Y_n - 1| < \delta\}} + 2 1_{\{|Y_n - 1| \ge \delta\}} \right) \\ &\le \varepsilon + 2 P(|Y_n - 1| > \delta) \end{aligned}$$

$\Rightarrow \limsup_n \| E\, T(Y_n t) f - T(t) f \| \le \varepsilon$

$\Rightarrow (I - \frac{t}{n} L)^{-n} f = E\, T(Y_n t) f \to T(t) f$. $\square$

Examples of generators

1) Brownian motion:

$$(T(t) f)(x) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} f(x+y)\, e^{-y^2/2t}\, dy$$

$$= E^x f(X(t))$$

$Lf = f''$, $\mathcal{D}(L) = \{ f \in C(S) : f'' \in C(S) \}$.

2) Ornstein-Uhlenbeck process:

Gaussian process with $E^x X(t) = e^{-t} x$

$E^x X(t) X(t+s) = e^{-s}(1 - e^{-2t})$

$$(T(t) f)(x) = \frac{1}{\sqrt{2\pi (1 - e^{-2t})}} \int_{-\infty}^{\infty} f(e^{-t} x + y)\, e^{-y^2/2(1 - e^{-2t})}\, dy$$

$$= E f(e^{-t} x + \sqrt{1 - e^{-2t}}\, Z)$$

$Z \sim N(0,1)$

$(Lf)(x) = f''(x) - x f'(x)$, $\mathcal{D}(L) = \{ f \in C(S) : f'' \in C(S) \}$

3) Motion to the right: $X(t) = x + t$ P^x-a.s.

$$(T(t)f)(x) = f(x+t)$$
$$Lf = f'$$

4) Cauchy process: a process with stationary, independent increments: $X(t+s) - X(s)$ has density

$$x \longmapsto \frac{1}{\pi} \frac{t}{t^2 + x^2}$$

$$(T(t)f)(x) = \frac{t}{\pi} \int_{-\infty}^{\infty} \frac{f(x+y)}{t^2+y^2}\, dy.$$

For any odd function $\varphi \in C_c^2(\mathbb{R})$ with $\varphi'(0) = 1$,

$$(Lf)(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(x+y) - f(x) - f'(x)\varphi(y)}{y^2}\, dy$$

$$\mathcal{D}(L) \supseteq C_c^2(\mathbb{R}).$$