Institute for Applied Mathematics
Winter term 2015/16
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"Markov Processes", Problem Sheet 9.

Hand in solutions on Thursday 07.01. during the lecture.

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1. (Ergodicity for independent spin systems) (5+5+5 points)

Let

$$c(x,\eta) = \begin{cases} \beta(x), & \text{if } \eta(x) = 0, \\ \delta(x), & \text{if } \eta(x) = 1, \end{cases}$$

where $\beta(x)$ and $\delta(x)$ are strictly positive (but not necessarily bounded). Define $\mathcal{L}f$ by

$$\mathcal{L}f(\eta) = \sum_{x \in S} c(x,\eta)[f(\eta^x) - f(\eta)],$$

for functions f that only depend on finitely many coordinates.

(i) Show that $\overline{\mathcal{L}}$ is a generator of a semigroup.

(ii) Show that for the corresponding process, if the initial distribution is deterministic, then $\{\eta_t(x), x \in S\}$ are independent two state Markov chains.

Hint: Two possible ways to show this are:
(1) Proving independence by showing that $T(t)(fg) = T(t)fT(t)g$ for f,g depending on disjoint sets of coordinates with the help of uniqueness for the heat equation. (Why is that enough?)
(2) Defining a continuous time Markov process $X = (X^x)_{x \in S}$ on $\{0,1\}^S$ with independent components and deterministic initialization, and showing that its generator coincides with $\bar{\mathcal{L}}$.

(iii) Use (ii) to show that η_t is ergodic, and identify the unique stationary distribution μ explicitly.

Hint: First, show that $\lim_{t\to\infty} T(t)f(\eta) = \int f d\mu$ for all η and all f that only depend on finitely many coordinates with the help of exercise 2 from sheet 2. Then use density.

2. (Noisy voter model) (5 points)

Let $p(x,y)$ be the transition probabilities of a discrete time Markov chain on S with $p(x,x) = 0$ for all x, and let $\beta, \delta \geq 0$. Consider the spin system with flip rates

$$c(x,\eta) = \sum_{y:\eta(y)\neq\eta(x)} p(x,y) + \begin{cases} \beta, & \text{if } \eta(x) = 0, \\ \delta, & \text{if } \eta(x) = 1, \end{cases}$$

Compute M and ε, and determine necessary and sufficient conditions for the ergodicity of the system.

The Markov Processes team wishes you a merry Christmas and a happy new year!