Institute for Applied Mathematics
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“Markov Processes”, Problem Sheet 2.

Hand in solutions on Thursday 05.11. during the lecture.

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1. (Strong Markov property) (3+4+2 points)

Let Ω be the set of right-continuous functions from $[0,\infty)$ to a locally compact Polish space S with only finitely many jumps in any finite time interval. Let $X(s)$ be a Markov process on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ that is adapted to the right-continuous filtration $(\mathcal{F}_s)_{s\geq 0}$. Suppose that $Y(s,\omega) = Y_s(\omega)$ is a bounded and jointly measurable random variable on $[0,\infty)\times\Omega$ and τ is a stopping time. We will prove the strong Markov property:

$$\mathbb{E}^x[Y_\tau \circ \theta_\tau | \mathcal{F}_\tau] = \mathbb{E}^{X(\tau)}[Y_\tau] \quad \mathbb{P}^x\text{-a.s. on } \{\tau < \infty\} \quad \text{for any } x.$$

(i) Prove the statement for stopping times τ that take only countably many values.

Assumption: You may use that $(x,t) \mapsto \mathbb{E}^x[Y_t]$ is measurable on $S \times [0,\infty)$.

(ii) Prove the statement for arbitrary stopping times τ and random variables Y of the form $Y_s(\omega) = f(s)\Pi_{i=1}^k f_i(\omega(t_i)) =: f(s)Z(\omega)$ for some finite sequence $t_1 \leq \cdots \leq t_k$, f bounded continuous and the f_i continuous and compactly supported.

Assumption: You can use the fact that for such f and Z as above the function $(y,t) \mapsto f(t)\mathbb{E}^y[Z]$ is jointly continuous.

(iii) Conclude the proof of the general statement, using the monotone class theorem (i.e. check that you can apply the theorem!)

2. (The two-states Markov chain) (5 points)

Suppose that $S = \{0,1\}$. Consider a Q-matrix that is given by

$$\begin{pmatrix} -\beta & +\beta \\ +\delta & -\delta \end{pmatrix},$$

where $\beta, \delta \geq 0$ and $\beta + \delta > 0$. Show that the corresponding transition probabilities are

$$\begin{aligned} p_t(0,0) &= \tfrac{\delta}{\beta+\delta} + \tfrac{\beta}{\beta+\delta}e^{-t(\beta+\delta)}, \quad & p_t(0,1) &= \tfrac{\beta}{\beta+\delta}\left[1 - e^{-t(\beta+\delta)}\right], \\ p_t(1,1) &= \tfrac{\beta}{\beta+\delta} + \tfrac{\delta}{\beta+\delta}e^{-t(\beta+\delta)}, \quad & p_t(1,0) &= \tfrac{\delta}{\beta+\delta}\left[1 - e^{-t(\beta+\delta)}\right]. \end{aligned} \tag{1}$$

Show directly that the transition probabilities given in (1) satisfy the Chapman-Kolmogorov equations and

$$q(x,y) = \frac{d}{dt}p_t(x,y)|_{t=0}.$$

3. (From discrete to continuous time Markov chains) (2+2+2 points)
Suppose that $P = (p(x,y))_{x,y\in S}$ is the transition matrix of a discrete time Markov chain. The most natural way to make it into a continuous time Markov chain is to have it take steps at the event times of a Poisson process of intensity 1 independent of the discrete chain. In other words, the times between transitions are independent random variables with a unit exponential distribution. The forgetfulness property of the exponential distribution is needed in order to ensure that the process has the Markov property.

(i) Show that the transition function of the continuous time chain described above is given by

$$p_t(x,y) = e^{-t}\sum_{k=0}^{\infty}\frac{t^k}{k!}p_k(x,y), \tag{2}$$

where $p_k(x,y)$ are the k-step transition probabilities for the discrete time chain.

(ii) Show that (2) satisfies the Chapman-Kolmogorov equations.

(iii) Show that the Q-matrix is given by

$$Q = P - I.$$