Institute for Applied Mathematics
SS 2023
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Sheet 9, "Stochastic Analysis"

To hand in until June 12, 15:00

Problem 1 (The Skorokhod metric, 3 Pt)

Prove that the Skorokhod metric (see Definition 5.6. in the lecture notes) fulfills the triangle inequality.

Problem 2 (Precompact sets of step functions, 2 Pt)

For a metric space (E, ρ), let $\Gamma \subset E$ be a compact set. For $\delta > 0$, let $A(\Gamma, \delta)$ be the set of step functions in $D_E\big([0,\infty)\big)$, such that for all $x \in A(\Gamma, \delta)$:

1. $x(t) \in \Gamma \quad \forall t \geq 0$,
2. $s_k(x) - s_{k-1}(x) > \delta$, for all $k \in \mathbb{N}$ with $s_{k-1}(x) < \infty$,

where $s_k(x)$ is the sequence of jump times:

$$s_k(x) := \begin{cases} \inf\big\{t > s_{k-1}(x) : x(t) \neq x(t_-)\big\}, & \text{if } s_{k-1} < \infty, \\ +\infty, & \text{if } s_{k-1} = +\infty. \end{cases}$$

Prove that for any $\delta > 0$, the closure of $A(\Gamma, \delta)$ is compact w.r.t. the Skorokhod metric.

Hint: For any sequence $\{x_n\}_{n \in \mathbb{N}}$ in $A(\Gamma, \delta)$, find a converging subsequence by a diagonal argument.

Problem 3 (Precompact sets of càdlàg functions, 5 Pt)

Let (E, ρ) be a complete metric space. Prove that a set $S \subset D_E\big([0,\infty)\big)$ has compact closure if

1. For every rational $t \geq 0$, there exists a compact set $\Gamma_t \in E$, such that for all $x \in S$: $x(t) \in \Gamma$.
2. For each $T < \infty$:

$$\lim_{\delta \to 0} \sup_{x \in S} w(x, \delta, T) = 0.$$

Here, w is the 'mode of continuity' of a càdlàg function x:

$$w(x, \delta, T) := \inf_{\{t_i\}_\delta} \max_i \sup_{s,t \in [t_{i-t}, t_i)} \rho\big(x(s), x(t)\big),$$

where $\{t_i\}_\delta$ is the family of collections $0 = t_0 < t_1 < \cdots < t_{n-1} < T < t_n$ with $t_i - t_{i-1} > \delta$.

Remark: This is in fact an equivalence, see Theorem 5.10 in the lecture notes.

Hints: 1) You can use that a subset S of a complete metric space has compact closure if and only if it is totally bounded: that is, for each $\epsilon > 0$, there exists a finite collection $\{B_i\}$ of open balls of radius ϵ with center in S, such that $S \subset \{B_i\}$. In the following, we prove that for each $\epsilon > 0$, there exists a set of step functions A_ϵ with compact closure, such that

$$\forall x \in S : \exists\, y \in A_\epsilon : \, d(x, y) \leq \epsilon, \tag{1}$$

where d is the Skorokhod metric. First prove that (1) is indeed sufficient.

2) Given $\epsilon > 0$, choose a suitable finite time interval $[0, T]$, and approximate any function $x \in S$ by some step function $y \in A_\epsilon$ (in the sup-norm over $[0, T]$), for a set of step functions A_ϵ with compact closure. Use Problem 2 to construct such a set.