Institute for Applied Mathematics
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Sheet 8, "Stochastic Analysis"

To hand in until June 05, 15:00

Problem 1 (A criterion for tightness in the Wiener Space, 5 Pt)
Show that a sequence of probability measures $\{\mu_n\}_{n\in\mathbb{N}}$ on $W = C\big([0,\infty),\mathbb{R}\big)$ is tight if

1. for every $\epsilon > 0$, there exist $A \in \mathbb{R}$ and $n_0 \in \mathbb{N}$, such that

$$\mu_n\big[|\omega(0) > A|\big] < \epsilon \qquad \text{for all } n \geq n_0,$$

2. for any $N > 0$ and $\epsilon, \eta > 0$, there exist $\delta \in (0,1)$ and $n_0 \in \mathbb{N}$, such that

$$\delta^{-1}\mu_n\big[\sup_{s\in[t,t+\delta]} |\omega(s) - \omega(t)| \geq \epsilon\big] \leq \eta \qquad \text{for all } n \geq n_0 \text{ and } t \in [0,N].$$

Problem 2 (Donsker's invariance principle, 5 Pt)
Let $\{Y_i\}_{i\in\mathbb{N}}$ be i.i.d. random variables with mean 0 and variance $\sigma > 0$. Additionally assume that $E[Y_i^4] < \infty$. Let $S_n := \sum_{i=1}^n Y_i$ and define the piecewise linear functions

$$X_t^n(\omega) := \frac{1}{\sigma\sqrt{n}} S_{\lfloor nt\rfloor} + (nt - \lfloor nt\rfloor)\frac{1}{\sigma\sqrt{n}} Y_{\lfloor nt\rfloor+1},$$

and denote the law of X^n as μ_n. Show that as $n \to \infty$, the series μ_n converges weakly to the law μ of Brownian motion - which we assume to exists in $\mathcal{M}_1(W)$. If you lean on Lemma 5.5 from the lecture notes, it is enough to check the following two conditions:

1. All finite dimensional distributions of μ_n converge to those of μ.
2. The family $\{\mu_n\}_{n\in\mathbb{N}}$ is tight in $\mathcal{M}_1(W)$.

For part 1, you can use the **central limit theorem**. For part 2, first prove the following lemma with the help of **Doob's maximal inequality** (here use that $E[Y_i^4] < \infty$):

There exists a finite constant $K \in \mathbb{R}$ and some $n_1 \in \mathbb{N}$, such that for every $k \in \mathbb{N}$:

$$\mathbb{P}\Big[\max_{i\leq n} |S_{i+k} - S_k| > \lambda\sigma\sqrt{n}\Big] \leq K\lambda^{-4} \qquad \text{for all } n \geq n_1 \text{ and } \lambda > 0.$$

Then, given $\epsilon, \eta \in (0,1)$, choose $\lambda \gg 1$ large enough such that $K\lambda^{-2} < \eta\epsilon^2$ holds. Set $\delta := \epsilon^2\lambda^{-2}$. Show that there exists $n_0 \in \mathbb{N}$, such that

$$\delta^{-1}\mathbb{P}\Big[\max_{i \le \lceil n\delta \rceil} |S_{i+k} - S_k| > \epsilon\sigma\sqrt{n}\Big] \le \eta \qquad \text{for all } n \ge n_0.$$

Given this, verify the tightness condition given in Problem 1.

Remarks: You can use that $\mathbb{E}[S_n^4] = n\mathbb{E}[Y_i^4] + 3n(n-1)\sigma^4$. Moreover, the assumption $E[Y_i^4] < \infty$ is not necessary, one can use a truncation argument and approximate the Y_i's by bounded functions.

Problem 3 (Convergence with respect to the Skorokhod metric, 2 Pt)

Prove that on $D_E\big([0,\infty),\mathbb{R}\big)$ equipped with the Skorokhod metric:

$$x_n(t) := \mathbb{1}_{[1+1/n,\infty]}(t) \;\to\; x(t) := \mathbb{1}_{[1,\infty]}(t) \qquad \text{as } n \to \infty.$$

Does it also hold that $\mathbb{1}_{[1/n,\infty]}(t) \to \mathbb{1}_{[0,\infty]}(t)$?