Institute for Applied Mathematics
SS 2023
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Sheet 7, “Stochastic Analysis”

To hand in until May 22, 15:00

Problem 1 (Normal comparison, 2 Pt)

Let $\{\xi_i\}_{i=1,\dots,n}$ be centred normal variables with covariance matrix Λ, and let $\{\nu_i\}_{i=1,\dots,n}$ be i.i.d. centred normal variables. Show that

$$\mathbb{P}\big(\xi_1 \le u_1, \dots, \xi_n \le u_n\big) - \mathbb{P}\big(\nu_1 \le u_1, \dots, \nu_n \le u_n\big) \\ \le \frac{1}{2\pi} \sum_{i<j} \frac{(\Lambda_{ij})_+}{\sqrt{1-\Lambda_{ij}^2}} \exp\Big(-\frac{u_i^2+u_j^2}{2(1+|\Lambda_{ij}|)}\Big). \tag{1}$$

Similarly, show that if $|\Lambda_{ij}| \le \delta < 1$:

$$|\mathbb{P}\big(\xi_1 \le u, \dots, \xi_n \le u\big) - \Phi(u)^n| \le \frac{1}{2\pi} \sum_{i<j} \frac{|\Lambda_{ij}|}{\sqrt{1-\delta^2}} \exp\Big(-\frac{u^2}{1+|\Lambda_{ij}|}\Big). \tag{2}$$

Problem 2 (Extremes of a weakly correlated normal sequence, 2 Pt)

Let $\{\xi_n\}_{n\in\mathbb{N}}$ be a stationary normal sequence with covariance $\Lambda_{ij} = r_{|i-j|}$, and let M_n be its extremal process. Assume that $\sup_n r_n \le \delta < 1$, and that for a sequence $\{u_n\}_{n\in\mathbb{N}}$:

$$\lim_{n\to\infty} n \sum_{i=1}^{n} |r_i| \exp\Big(-\frac{u_n^2}{1+|r_i|}\Big) = 0. \tag{3}$$

Prove that

$$n(1-\Phi(u_n)) \to \tau \Leftrightarrow \mathbb{P}(M_n \le u_n) \to e^{-\tau}. \tag{4}$$

Problem 3 (The Berman condition, 6 Pt)

For $\tau \in (0,\infty)$, let

$$u_n = \sqrt{2\log n} - \frac{\log\log n + \log 4\pi}{2\sqrt{2\log n}} - \frac{\tau}{\sqrt{2\log n}}, \tag{5}$$

such that $n(1-\Phi(u_n)) \to \tau$. As before, let $\{\xi_n\}_{n\in\mathbb{N}}$ be a stationary normal sequence with covariance $\Lambda_{ij} = r_{|i-j|}$, where $|r_n| \leq \delta < 1$. Prove that Condition (3) holds if

$$r_n \log(n) \to 0. \tag{6}$$

Hints: Write, for some $\alpha \in (0,1)$:

$$n\sum_{i=1}^{n} |r_i| \exp\Big(-\frac{u_n^2}{1+|r_i|}\Big) = n\sum_{i\leq n^\alpha} |r_i| \exp\Big(-\frac{u_n^2}{1+|r_i|}\Big) + n\sum_{i>n^\alpha}^{n} |r_i| \exp\Big(-\frac{u_n^2}{1+|r_i|}\Big).$$

1. For the second summand, where $i > n^\alpha$, use that

   $$-\frac{u_n^2}{1+|r_i|} = -u_n^2 + \frac{u_n|r_i|}{1+|r_i|},$$

   then estimate $n|r_i|\exp(-u_n)^2$ and the remaining expression separately.

2. For the first summand, where $i < n^\alpha$, choose α in dependency of δ, such that the sum contains sufficiently few terms.

For both parts, focus on the leading terms of u_n and u_n^2 (5).