Institute for Applied Mathematics
SS 2023
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Sheet 6, "Stochastic Analysis"

To hand in until May 15, 15:00

Problem 1 (Continuous mapping theorem, 2 Pt)

Let $\{\mu_n\}_{n\in\mathbb{N}}$ and μ be measures on a metric space $(\Omega, \mathcal{A})$. Suppose that $\mu_n \to \mu$ weakly. Let $f : \Omega \mapsto \Omega$ be a bounded and continuous function with inverse image f^{-1}. Show that

$$\mu_n \circ f^{-1} \to \mu \circ f^{-1} \quad \text{weakly.}$$

Problem 2 (Converging together lemma, 2 Pt)

Suppose that a sequence of random variables $(X_n)_{n\in\mathbb{N}}$ on a metric space S converges weakly to a limiting random variable Z. Suppose that there is a second sequence $(Y_n)_{n\in\mathbb{N}}$ such that $|X_n - Y_n| \to 0$ in probability. Show that then also $Y_n \to Z$ weakly.

Hint: You can use Proposition 9.15 from the lecture notes (also known as **Portmanteau-Theorem**).

Problem 3 (Topology on the space of measures, 3 Pt)

The space of measures $M_+(\mathbb{R}^d)$ equipped with the vague topology can be turned into a complete and separable space. This topology has as a *neighborhood basis* $\mathcal{U}$ (every open neighborhood contains a set in $\mathcal{U}$) of the sets of the form

$$U_{\epsilon,h_1,\dots,h_n}(\mu) := \left\{\nu \in M_+(\mathbb{R}^d) \; : \quad |\mu(h_i) - \nu(h_i)| < \epsilon, \quad i = 1,\dots,n\right\},$$

where $h_i \in C_0^+(\mathbb{R}^d)$. A topological space S is called *locally compact*, if for every $x \in S$, there exists an open neighborhood O and a compact set K such that $x \in O \subseteq K$.

- Is $M_+(\mathbb{R})$ equipped with the vague topology locally compact?
- On $M_+(\mathbb{R})$, find a sequence that converges vaguely but not weakly.
- Construct a metric on $M_+(\mathbb{R})$, such that $\mu_n \to \mu$ vaguely if and only if $d(\mu_n, \mu) \to 0$.
  Hint: Find a countable set of functions $\{g_i\}_{i\in\mathbb{N}}$, such that $\mu_n(g_i) \to \mu(g_i)$ for all i if and only if $\mu_n \to \mu$.

Problem 4 (Tightness of random measures, 3 Pt)

Given a polish space S, show that a sequence of point processes $(\xi_n)_{n\in\mathbb{N}}$ is tight in $M_+(S)$ if and only if for every measurable and relatively compact set $B \subset S$:

$$\lim_{t\to\infty} \limsup_{n\to\infty} \mathbb{P}[\xi_n(B) > t] = 0.$$