Institute for Applied Mathematics
SS 2023
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Sheet 5, "Stochastic Analysis"

To hand in until May 08, 15:00

Problem 1 (Construction of a two-dimensional PPP, 5 Pt)

Let $\xi = \sum_i \delta_{X_i}$ be a Poisson point process on $\mathbb{R}$ with a σ-finite intensity measure μ. Let $K : \mathbb{R} \times \mathscr{B}(\mathbb{R}) \to [0,1]$ be a probability kernel and let $\{Y_i\}_{i\in\mathbb{N}}$ be $\mathbb{R}$-valued random variables, where Y_n essentially only depends only on X_n:

$$\mathbb{P}\big[Y_n \in A \,\big|\, \sigma\big(\{X_i\}_{i\in\mathbb{N}}, \{Y_i\}_{i\in\mathbb{N}, i\neq n}\big)\big] = K(X_n, A) \quad \forall n \in \mathbb{N},\, A \in \mathscr{B}(\mathbb{R}).$$

Show that

$$\zeta := \sum_i \delta_{(X_i, Y_i)}$$

is a Poisson point process on $\mathbb{R}^2$ with intensity measure

$$\nu(dx, dy) = \mu(dx) \cdot K(x, dy).$$

Hints: i) First prove that for any bounded and product-measurable function $f : \mathbb{R}^2 \to \mathbb{R}_0^+$:

$$\mathbb{E}\big[f(X_i, Y_i)\big|\{X_i\}_{i\in\mathbb{N}}\big] = \int_{\mathbb{R}} f(X_i, y) K(X_i, y)\, dy.$$

ii) Then use that the rectangle sets $[a,b] \otimes [c,d]$ are a Dynkin System of $\mathscr{B}(\mathbb{R}^2)$.

Problem 2 (A limit theorem for dependent variables, 5 Pt)

For each $n \in \mathbb{N}$, let $\{X_i^n\}_{i\in\mathbb{N}}$ be a sequence of random variables with values on $\big(\mathbb{R}_+, \mathscr{B}(\mathbb{R}_+)\big)$, with filtration $\mathcal{F}_i^n = \sigma\big(X_1^n, \ldots, X_i^n\big)$. Let μ be a finite measure on $\mathbb{R}_+$. Assume that there is a sequence $\{a_n\}_{n\in\mathbb{N}}$ (a linear time-scale), such that for all $t > 0, x > 0$, in probability:

$$\lim_{n\to\infty} \sum_{i=1}^{\lceil a_n \cdot t\rceil} \mathbb{P}\big[X_i^n > x \,\big|\, \mathcal{F}_{i-1}^n\big] = t \cdot \mu((x, \infty)),$$

$$\lim_{n\to\infty} \sum_{i=1}^{\lceil a_n \cdot t\rceil} \big(\mathbb{P}\big[X_i^n > x \,\big|\, \mathcal{F}_{i-1}^n\big]\big)^2 = 0.$$

1. Compute $\lim_{n\to\infty} \mathbb{P}\big[\max_{i\leq\lceil a_n t\rceil} X_i^n \leq x\big]$, or find it in your notes.

2. Next, we define a sequence of Point Processes $\xi_n := \sum_{i\in\mathbb{N}} \delta_{\left(i/a_n, X_i^n\right)}$. Prove that $\xi_n \to \mathscr{P}$, where $\mathscr{P}$ is a Poisson Point process on $\mathbb{R}_+^2$ with intensity $dt \times d\mu$.

Hints: i) The following identity is useful:

$$\mathbb{E}\Big[\prod_{i=1}^{k} \mathbb{1}_{\{X_i^n \in A_i\}} \exp\Big(-\ln \mathbb{P}\big[X_i^n \in A_i \,|\, \mathcal{F}_{i-1}^n\big]\Big)\Big] = 1.$$

ii) For the second part, use Kallenberg's Theorem (Theorem 3.19), or Laplace functionals.