Institute for Applied Mathematics
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Sheet 12, "Stochastic Analysis"

For discussion in the tutorials

Problem 1 (Sums of exponentials of random variables)
Let $\{Z_i\}_{i\in\mathbb{N}}$ be i.i.d. random variables such that there exist sequences $c_n > 0, b_n \in \mathbb{R}$:

$$n\mathbb{P}\big[Z_1 > \frac{\ln(c_n) + z}{b_n}\big] \to e^{-z} \qquad \text{as } n \to \infty.$$

Let $X_i^n := \exp\left(\frac{b_n Z_i}{\alpha}\right)$. First, recapitulate that the previous statement implies

$$n\mathbb{P}\big[X_1^n > c_n^{1/\alpha} x\big] \to x^{-\alpha} \qquad \text{as } n \to \infty.$$

Assume that this convergence holds in a strong sense for some $\alpha \in (0, 1)$:

$$\int_{-\infty}^{0} e^z \cdot n\mathbb{P}\big[Z_1 > \frac{\ln(c_n) + z}{b_n}\big]\, dz \to \int_{-\infty}^{0} e^{(1-\alpha)x}\, dx \qquad \text{as } n \to \infty.$$

In this setting, Theorem 6.23 states that $c_n^{-1/\alpha} \sum_{i=1}^{[t\cdot n]} X_i^n \to V_\alpha(t)$, where V_α is the Lévy subordinator with intensity measure

$$v_\alpha(dx) = \alpha x^{-(1+\alpha)} dx \mathbb{1}_{x>0}.$$

The proof relies on Theorem 6.13. Finish it by showing that

$$\lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{n}{c_n^{1/\alpha}} \mathbb{E}\big[\mathbb{1}\{X_1^n \le c_n^{1/\alpha}\epsilon\} \cdot X_1^n\big] = 0.$$