Institute for Applied Mathematics
SS 2023
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Sheet 10, “Stochastic Analysis”

To hand in until June 19, 15:00

This is the last sheet which is relevant for the admission to the exams. If you want to participate in the exams, register on BASIS for BOTH the course and the exercise classes.

Problem 1 (Boundedness of a Lévy process, 5 Pt)

Let $V_{\alpha,c}(t)$ be a Lévy process with Lévy triple $(0, 0, v_{\alpha,c})$, where $v_{\alpha,c}(dx) = c\alpha x^{-\alpha-1}\mathbb{1}_{x>0}dx$ for some $\alpha \in (0,1), c > 0$. Show by a truncation argument that for every $T > 0$ and every $\delta > 0$, there exists a finite constant K, such that

$$\mathbb{P}\big[\sup_{t\in[0,T]} V_{\alpha,c}(t) \geq K\big] \leq \delta.$$

Hint: For $0 < \epsilon < B < \infty$, consider the process with intensity measure $v_{\alpha,c}(dx)\mathbb{1}_{[\epsilon,B]}(x)$.

Now take a sequence $\{X_i\}_{i\in\mathbb{N}}$ of random variables such that $n\mathbb{P}[X_1 > n^{1/\alpha}] \to cx^{-\alpha}$ as $n \to \infty$ and define $S_n(t) := n^{-1/\alpha}\sum_{i=1}^{[nt]} X_i$. Conclude - with the help of Theorem 6.5. - that for every $T > 0$ and every $\epsilon > 0$, there exists a finite constant K, such that for all $n \in \mathbb{N}$:

$$\mathbb{P}[\sup_{t\in[0,T]} S_n(t) \geq K] \leq \epsilon.$$

Problem 2 (Boundedness of tight càdlàg processes, 5 Pt)

Let E be a complete metric space. Show that if a sequence of stochastic processes $\{X_n\}_{n\in\mathbb{N}}$ is tight in $D_E[0,\infty)$ equipped with the Skorohod metric, then the following holds:

For every $\epsilon > 0$ and $T \geq 0$, there exists a compact set $\Gamma \subset E$, such that for all $n \in \mathbb{N}$:

$$\mathbb{P}\big[X_n(t) \in \Gamma \quad \text{for all } 0 \leq t \leq T\big] \geq 1 - \epsilon.$$