Institute for Applied Mathematics
SS 2023
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Sheet 1, "Stochastic Analysis"

For discussion in the first tutorials, starting on April 11

Problem 1 (Exit time of one-dimensional Brownian motion)

Let $(B_t)_{t\geq 0}$ be a one-dimensional standard Brownian motion and let $a > 0$. For the exit-time $T := \inf\{t \geq 0 : B_t \notin (-a, a)\}$, we know that $T < \infty$ almost surely (why does this hold?). Show that

1. $\mathbb{E}[T] = a^2$.
2. $\mathbb{E}[T^2] = \frac{5}{3}a^4$.
3. $\mathbb{E}[T^3] = -\frac{14}{15}a^6 + 5a^4$.

Hint: Find polynomials $p(x,t) = x^4 - bx^2t + ct^2$ and $q(x,t) = x^6 - lx^4t + mx^2t^2 - nt^3$ such that $p(B_t, \langle B\rangle_t)$ and $q(B_t, \langle B\rangle_t)$ are (local) martingales.
Remark: Suitable polynomials of arbitrarily large degree can be constructed via a recursion.

Problem 2 (Brownian motion with drift part 1)

Let $W_t := B_t - \mu t$, $\mu \neq 0$ be a one-dimensional Brownian motion with drift starting in $x \in (a, b)$, for $a < b$ finite. Let $\mathscr{L} := \frac{1}{2}\partial_{xx} - \mu \cdot \partial_x$ be the generator of W. For the hitting times $T_a^W := \inf\{t \geq 0 : W_t = a\}$ and T_b^W, compute $\mathbb{P}_x[T_b^W < T_a^W]$ by solving

$$\mathscr{L}f(x) = 0 \quad \forall x \in (a, b), \qquad f(a) = 0, f(b) = 1.$$

Then let $a \to -\infty$ and compute $\mathbb{P}_x[T_b^W < \infty]$.

Problem 3 (Brownian motion with drift part 2)

Let $(B_t)_{t\geq 0}$ be a one-dimensional standard Brownian motion and let $b > 0$. Define the hitting time $T_b := \inf\{t \geq 0 : B_t = b\}$. Using the reflection-principle $\mathbb{P}[T_b < t] = 2\mathbb{P}[B_t > a]$, one can proof that the distribution of T_b has a density $\rho_{T_b}(s)$ with respect to the Lebesgue-measure, which is given by

$$\rho_{T_b}(s) = \frac{b}{\sqrt{2\pi s^3}} \exp(-\frac{b^2}{2s}), \qquad s \geq 0.$$

This is equivalent to the fact that for $\lambda > 0$, the moment generating function of T_b is given by $\mathbb{E}[e^{-\lambda T_b}] = e^{-b\sqrt{2\lambda}}$.

For a Brownian motion with drift $W_t := B_t - \mu t$, $\mu \neq 0$, use Girsanov's Theorem and the above formula for ρ_{T_b} to find a formula for $\mathbb{P}[T_b^W \leq t]$, where $T_b^W := \inf\{t \geq 0 : W_t = b\}$. What does this imply for $\mathbb{P}[T_b^W < \infty]$? Compare your result to the one from Problem 2.

Hint: To check your intermediate result, the density of the distribution of T_b^W is given by

$$\rho^{\mu}_{T_b^W}(s) = \frac{b}{\sqrt{2\pi s^3}} \exp\left(-\frac{(b-\mu s)^2}{2s}\right).$$