Institute for Applied Mathematics
SS 2021
Prof. Dr. Anton Bovier,
Florian Kreten

Sheet 9, "Stochastic Analysis"

To be discussed on June 23, 2021

Information from the Student Council: This year the Maths Summer Party will take place virtually on Friday, June 25, starting at 18 c.t.. Latest information can be found **here**. Come by!

Problem 1 (The Skorokhod metric, part 1)

Prove that the Skorokhod metric (see Definition 6.6. in the lecture notes) fulfills the triangle inequality.

Problem 2 (The Skorokhod metric, part 2)

Prove that on $D_E\big([0,\infty),\mathbb{R}\big)$ equipped with the Skorokhod metric:

$$x_n(t) := \mathbb{1}_{[1+1/n,\infty]}(t) \;\rightarrow\; x(t) := \mathbb{1}_{[1,\infty]}(t) \qquad \text{as } n \to \infty.$$

Does it also hold that $\mathbb{1}_{[1/n,\infty]}(t) \to \mathbb{1}_{[0,\infty]}(t)$?

Problem 3 (A simpler tightness criterion)

Show that a sequence of probability measures $\{\mu_n\}_{n\in\mathbb{N}}$ on $W = C\big([0,\infty),\mathbb{R}\big)$ is tight if

1. for every $\epsilon > 0$, there exist $A \in \mathbb{R}$ and $n_0 \in \mathbb{N}$, such that

$$\mu_n\big[|\omega(0) > A|\big] < \epsilon \qquad \text{for all } n \geq n_0,$$

2. for any $N > 0$ and $\epsilon, \eta > 0$, there exist $\delta \in (0,1)$ and $n_0 \in \mathbb{N}$, such that

$$\delta^{-1}\mu_n\big[\sup_{s\in[t,t+\delta]} |\omega(s) - \omega(t)| \geq \epsilon\big] \leq \eta \qquad \text{for all } n \geq n_0 \text{ and } t \in [0,N].$$

Problem 4 (Donsker's invariance principle)

Let $\{Y_i\}_{i\in\mathbb{N}}$ be i.i.d. random variables with mean 0 and variance $\sigma > 0$. Additionally assume that $E[Y_i^4] < \infty$. Let $S_n := \sum_{i=1}^n Y_i$ and define the piecewise linear function

$$X_t^n(\omega) := \frac{1}{\sigma\sqrt{n}} S_{\lfloor nt \rfloor} + (nt - \lfloor nt \rfloor)\frac{1}{\sigma\sqrt{n}} Y_{\lfloor nt \rfloor + 1}.$$

We denote the law of X^n as μ_n. We show that as $n \to \infty$, the series μ_n converges weakly to the law μ of a Brownian motion. If we assume that μ exists in $\mathcal{M}_1(W)$ and if we lean on the theory which is developed in chapter 6 of the lecture, then it suffices to check the following two statements:

1. All finite dimensional distributions of μ_n converge to those of μ.
2. The family $\{\mu_n\}_{n \in \mathbb{N}}$ is tight in $\mathcal{M}_1(W)$.

For part 1, use the **central limit theorem**. For part 2, first prove the following lemma with the help of **Doob's maximal inequality**:

There exists a finite constant $K \in \mathbb{R}$ and some $n_1 \in \mathbb{N}$, such that for every $k \in \mathbb{N}$:

$$\mathbb{P}\Big[\max_{i \leq n} |S_{i+k} - S_k| > \lambda \sigma \sqrt{n}\Big] \leq K \lambda^{-4} \qquad \text{for all } n \geq n_1 \text{ and } \lambda > 0.$$

Then, given $\epsilon, \eta \in (0,1)$, choose $\lambda \gg 1$ large enough such that $K\lambda^{-2} < \eta \epsilon^2$ holds. Set $\delta := \epsilon^2 \lambda^{-2}$. Show that there exists $n_0 \in \mathbb{N}$, such that

$$\delta^{-1} \mathbb{P}\Big[\max_{i \leq \lceil n\delta \rceil} |S_{i+k} - S_k| > \epsilon \sigma \sqrt{n}\Big] \leq \eta \qquad \text{for all } n \geq n_0.$$

Finally, verify the tightness condition given in Problem 3.

Remark: You can use without proof that $\mathbb{E}[S_n^4] = n\mathbb{E}[Y_i^4] + 3n(n-1)\sigma^4$. Moreover, the assumption $E[Y_i^4] < \infty$ is not necessary, one can use a truncation argument and approximate the Y_i's by bounded functions.