Institute for Applied Mathematics
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Sheet 8, "Stochastic Analysis"

To be discussed on June 16, 2021

Problem 1 (Construction of a two-dimensional PPP)

Let $\xi = \sum_i \delta_{X_i}$ be a Poisson point process on $\mathbb{R}$ with a σ-finite intensity measure μ. Let $K : \mathbb{R}\times\mathscr{B}(\mathbb{R}) \to [0,1]$ be a probability kernel and let $\{Y_i\}_{i\in\mathbb{N}}$ be $\mathbb{R}$-valued random variables which are conditionally independent given $\{X_i\}_{i\in\mathbb{N}}$, such that

$$\mathbb{P}\big[Y_n \in A \,\big|\, \{X_i\}_{i\in\mathbb{N}}\big] = K(X_n, A),$$

for any n and $A \in \mathscr{B}(\mathbb{R})$. Show that

$$\zeta := \sum_i \delta_{(X_i,Y_i)}$$

is a Poisson point process on $\mathbb{R}^2$ and compute its Laplace functional.

Problem 2 (A limit theorem for dependent variables)

For each $n \in \mathbb{N}$, let $\{X_i^n\}_{i\in\mathbb{N}}$ be a sequence of random variables with values on $\big(\mathbb{R}_+, \mathscr{B}(\mathbb{R}_+)\big)$, with filtration $\mathcal{F}_i^n = \sigma\big(X_1^n, \dots, X_i^n\big)$. Let μ be a finite measure on $\mathbb{R}_+$. Assume that there is a sequence $\{a_n\}_{n\in\mathbb{N}}$ (a linear time-scale), such that for all $t > 0, x > 0$, in probability:

$$\lim_{n\to\infty} \sum_{i=1}^{\lceil a_n\cdot t\rceil} \mathbb{P}\big[X_i^n > x \,\big|\, \mathcal{F}_{i-1}^n\big] = t\cdot\mu((x,\infty)),$$

$$\lim_{n\to\infty} \sum_{i=1}^{\lceil a_n\cdot t\rceil} \big(\mathbb{P}\big[X_i^n > x \,\big|\, \mathcal{F}_{i-1}^n\big]\big)^2 = 0.$$

1. Compute $\lim_{n\to\infty} \mathbb{P}\big[\max_{i\le\lceil a_n\cdot t\rceil} X_i^n \le x\big]$.
2. Next, we define a sequence of Point Processes $\xi_n := \sum_{i\in\mathbb{N}} \delta_{\left(i/a_n, X_i^n\right)}$. Prove that $\xi_n \to \mathscr{P}$, where $\mathscr{P}$ is a Poisson Point process on $\mathbb{R}_+^2$ with intensity $dt \times d\mu$.

Hints: The following identity helps:

$$\mathbb{P}\big[\forall_{i=1}^k X_i^n \in A_i\big] = \exp\Big(\sum_{i=1}^{k} \ln \mathbb{P}\big[X_i^n \in A_i \,|\, \mathcal{F}_{i-1}^n\big]\Big).$$

For the second part, use Kallenberg's Theorem (Theorem 4.19).